UNIT-I

DEFLECTIONS

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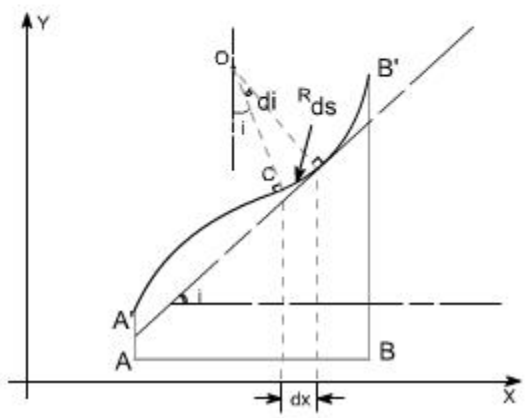
Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

2. The curvature is always small.

3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or } \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force,bending moment and deflection of the beam may be obtained as

Differentiating the equation as derived

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx} = F$$

Thus,

$$F = EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam,
then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods.The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

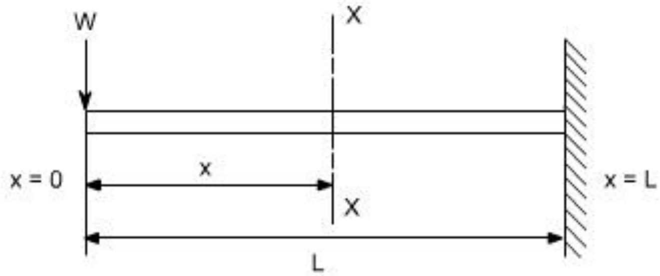
Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Case 1: Cantilever Beam with Concentrated Load at the end:-

A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

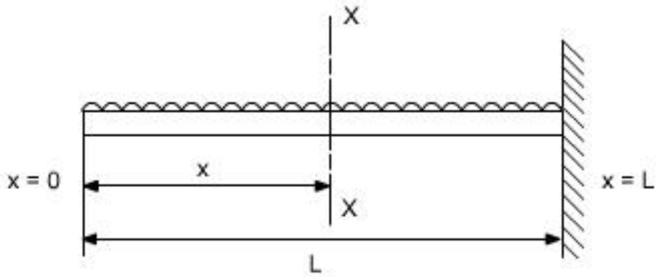
The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max} m = +\frac{WL^2}{2EI}}$$

Case 2: A Cantilever with Uniformly distributed Loads:-

In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int\frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0

2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

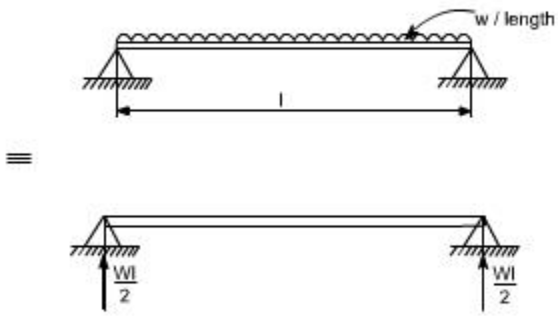
So $y_{max}m$ will be at $x = 0$

$$\boxed{y_{max}m = -\frac{wL^4}{8EI}}$$

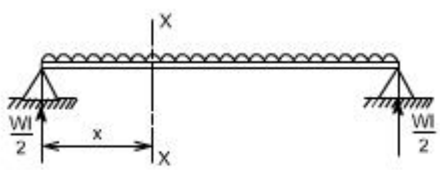
$$\boxed{\left(\frac{dy}{dx}\right)_{max}m = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads:-

In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----(1)}$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then} \quad y_{maxm} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{maxm} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2.

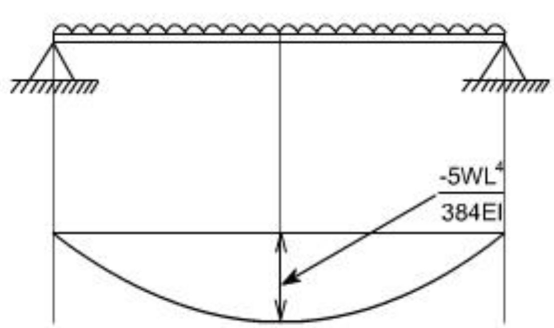
The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

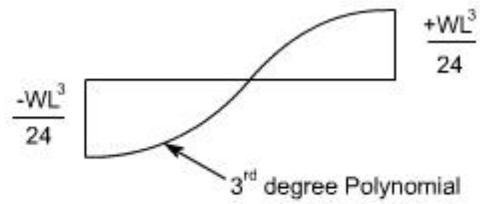
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



Slope (dy/dx)

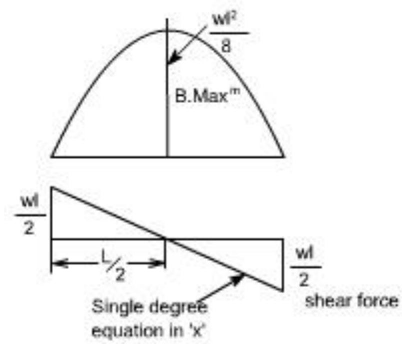
$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



So the bending moment diagram would be

Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$



Shear Force

Shear force is obtained by taking

third derivative.

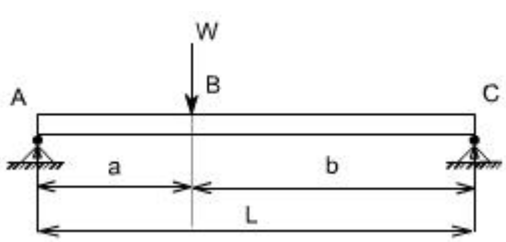
$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

Rate of intensity of loading

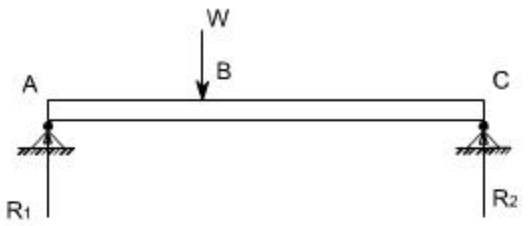
$$EI\frac{d^4y}{dx^4} = -w$$

Case 4:

The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \le x \le a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \le x \le l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1\,x$$

$$EI\frac{d^2y}{dx^2} = R_1\,x - W\,(x - a)$$

These two equations can be integrated in the usual way to find 'y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \le x \le a$

(b) at x = l; y = 0 in the portion BC i.e. $a \le x \le l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{ ------- (4)}$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{ ------(5)}$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{ ------(6)}$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a+b=l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that, At x = a; y; the deflection is the same for both portion

Therefore $y\big|_{\text{from equation 5}} = y\big|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{- - - -}\mathbf{for\ 0 \le x \le a}\text{- - - - - (7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{For } \mathbf{for\ a \le x \le l}\text{- - - - - (8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a
hence substituting x = a in either of the equation we get

$$Y\big|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

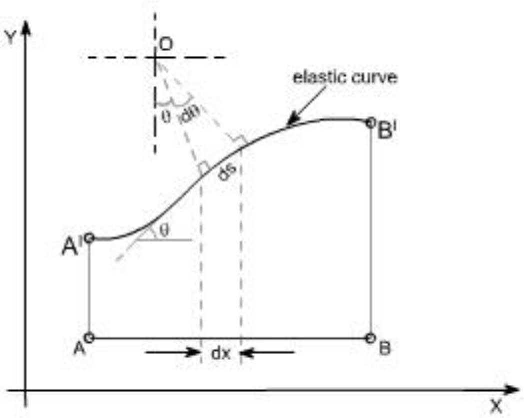
OR if a = b = l/2

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

MOMENT-AREA METHOD:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that dq is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.

We can assume,

ds = dx [since the curvature is small]

hence, R dq = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

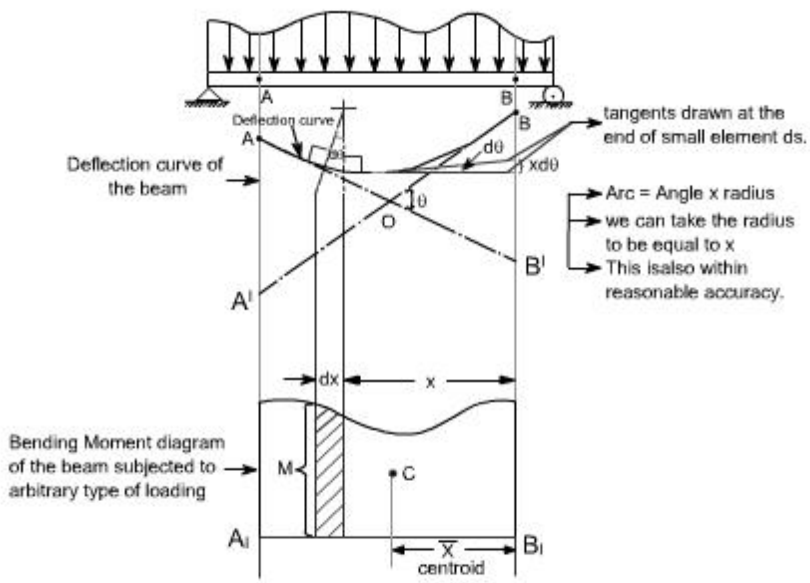
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle ,slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$,hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \text{ - - - - - (1)}$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be dq. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle q between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or } \theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x dq [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle q]

Hence the total distance B'B becomes $\delta = \int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of dq = M dx / EI as derived earlier

$\delta = \int_A^B x\frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$ [This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore,we are in a position to state the above conclusion in the form of theorem as follows:

Theorem II:

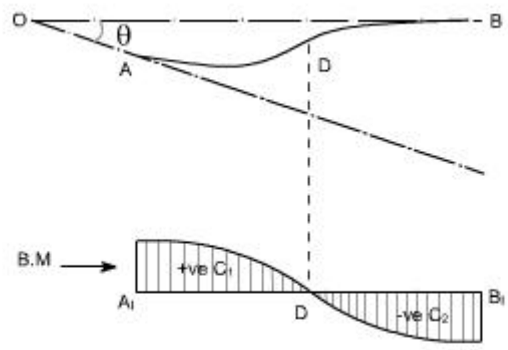
Deflection of point 'B' relative to point A $= \frac{1}{EI} \times \left\{\begin{matrix}\text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram}\end{matrix}\right\}$

Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $\boxed{\delta_A = \frac{1}{EI}A\bar{X}}$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids $C_1$and C_2. Then to find an angle q between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

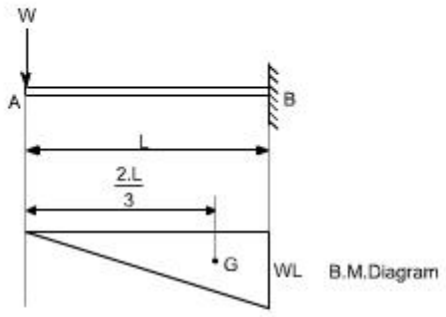
And similarly for the deflection of B away from the tangent at A becomes

$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Cantilever carrying point load at its free end

1. A cantilever is subjected to a concentrated load at the free end.It is required to find out the deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at A} = \frac{1}{EI}[\text{Area of B.M diagram between the points A and B}]$$
$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$
$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

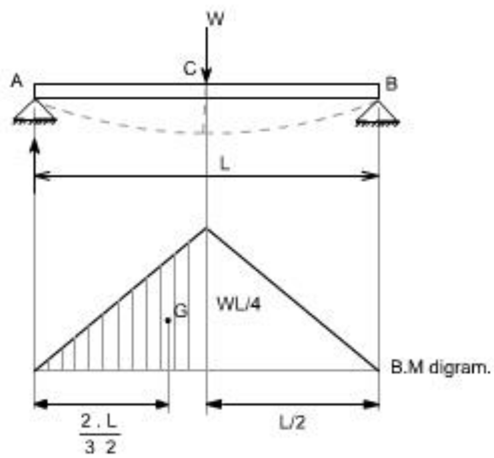
NOTE: In this case the point B is at zero slope.

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$
$$= \frac{1}{EI}[A\bar{y}]$$
$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$
$$= \frac{WL^3}{3EI}$$

Simply supported beam carrying point load at its mid span:

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

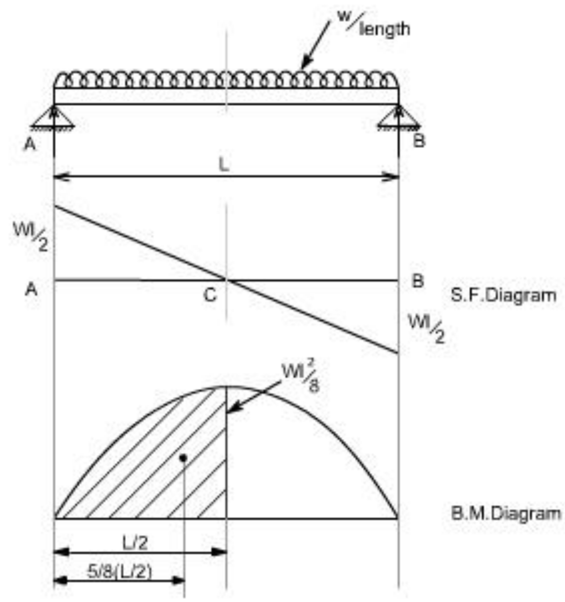
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

SIMPLY SUPPORTED BEAM CARRYING UDL OF ENTIRE SPAN:

The bending moment diagram is drawn, below the loaded beam, the value of maximum B.M is equal to $Wl^2 / 8$



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}\left[\text{Area of B.M diagram between point A and C}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}\left[A\,\bar{y}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

CONJUGATE BEAM METHOD (or) Funicular Analogy method

Slope & deflections of beams & cantilevers may be obtained by various methods like Double Integration, moment area method, Macaulay's method. But these methods become laborious when applied to beams whose flexural rigidity is not uniform throughout the length of beam. The slope & deflections of such beams can be easily obtained by conjugate beam method.

Concept of Conjugate beam method (or) Method of elastic weights

Conjugate beam is an imaginary beam of length equal to that of original beam but for which the load diagram is the $\frac{M}{EI}$ diagram. " Method Proposed by Mohr. and two theorems

The slopes & deflection @ any section of a beam by conjugate beam method is given by

1) slope @ any section of the given beam is equal to the shear force @ the corresponding section of the conjugate beam

2) The deflection @ any section for the given beam is equal to the BM @ the corresponding section of the conjugate beam.

Deflection & slope of a SSB with a point load

A SSB AB of length l carrying a point load 'w' @ centre 'c'.

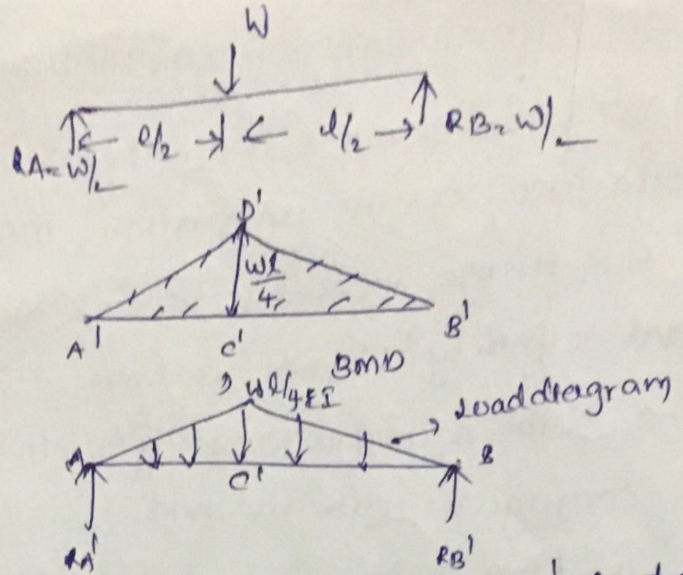
The B.M @ A and @ B is zero & @ centre BM will be $\frac{Wl}{4}$.

Now conjugate AB is constructed. The load on the conjugate beam will be obtained by dividing the B.M @ that point by EI.

The shape of the loading on the conjugate beam will be same as B.M diagram. The ordinate of loading on conjugate beam will be equal to $\frac{M}{EI} = \frac{(Wl/4)}{EI} = \frac{Wl}{4EI}$.

Hence ordinate @ centre will be $\frac{Wl}{4EI}$.

The load diag. for conjugate beam is shown in fig;



Let R_A' = Reaction @ A for conjugate beam; R_B' = Reaction @ B for conj Be

Total Load on conjugate beam = Area of load diagram

$$= \frac{1}{2} \times AB \times C'D'$$

$$= \frac{1}{2} \times l \times \frac{Wl}{4EI} = \frac{Wl^2}{8EI}$$

Reactions @ each support for the conjugate beam will be half total load.

$$R_A' = R_B' = \frac{1}{2}\left(\frac{Wl^2}{8EI}\right) = \frac{Wl^2}{16EI}$$

Let θ_A = slope @ A for the given beam i.e; $\left(\frac{dy}{dx}\right)$ @ A

y_c = Deflection @ C for given beam

Then according to conjugate beam

$\theta_A = R_A'$ [SF @ A for conjugate beam $= R_A'$]

$$\boxed{\theta_A = \frac{Wl^2}{16EI}}$$

y_c = Beam @ C for conjugate beam

$= R_A' \times \frac{l}{2}$ - Load corresponding to $AC'D'$ × Distance of of $AC'D'$ from

$$= \frac{Wl^2}{16EI}\left(\frac{l}{2}\right) - \frac{1}{2}\left(\frac{l}{2} \times \frac{Wl}{4EI}\right) \times \left(\frac{1}{3} \times \frac{l}{2}\right)$$

$$\boxed{y_c = \frac{Wl^3}{48EI}}$$

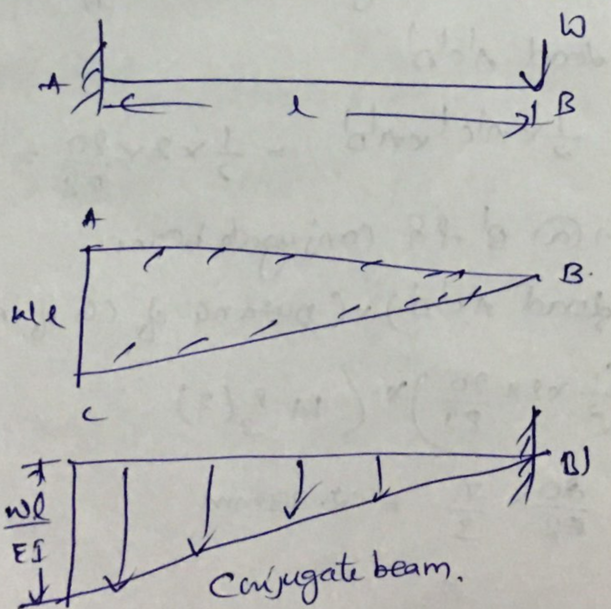
Illy Eccentric Load

$$\boxed{y_c = \frac{Wab}{6EIl}\left[l^2 - b^2 - a^2\right]}$$

~ference b/w a real beam & a conjugate beam

	Actual beam	Conjugate beam
1.	SS (or) Roller supported end (Deflection = 0 but slope exists) $\theta \neq 0$; $\delta = 0$	Simply supported end BM = 0 but SF exists $F \neq 0$ BM = 0
2)	free end (slope & deflection exists) $\theta \neq 0$, $\delta \neq 0$	fixed end (SF & BM exists) $F \neq 0$, $M \neq 0$
3)	Fixed end (~~slope~~ Deflection $\theta = 0$; $\delta = 0$)	Free end (SF & BM = 0) $F = 0$, $M = 0$
4)	Slope @ any section	Shear force @ corresponding section
5)	Deflection @ any section	BM @ the corresponding section
6)	Given system of loading	Loading diagram is $\frac{M}{EI}$ diagram.
7)	B.M diagram positive (sagging)	$\frac{M}{EI}$ Load diag is positive (ie, loading is downward)
8)	B.M " negative (hogging)	$\frac{M}{EI}$ load diag is negative (ie; loading is upward)

Deflection & slope of a cantilever with a point load @ free end



Let θ_B = slope @ B.

According to conjugate beam θ_B = SF @ B' for conjugate beam

= Load B'A'C'

$= \frac{1}{2} \times A'B' \times A'C' = \frac{1}{2} \times l \times \frac{Wl}{EI} = \frac{Wl^2}{2EI}$

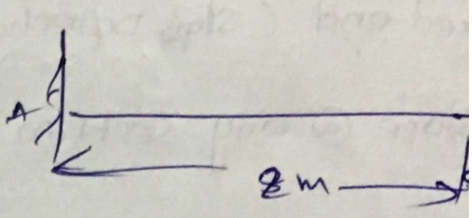
y_D = BM @ B' for conjugate beam

= Load B'A'C' × Distance of CG of B'A'C' from B'

$= \left(\frac{1}{2} l \cdot \frac{Wl}{EI}\right) \times \left(\frac{2}{3} l\right) = \frac{Wl^3}{3EI}$.

No

1) A cantilever of length 3m carries a point load of 10KN @ a dista 2m from fixed end. If $E = 2 \times 10^5 N/mm^2$ & $I = 10^8 mm^4$. find slope deflection @ free end using conjugate beam method?



Sol Given $l = 3m$; $W = 10KN$; $AC = 2m$;

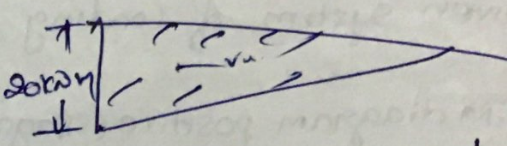
$E = 2 \times 10^5 N/mm^2$ $I = 10^8 mm^4$

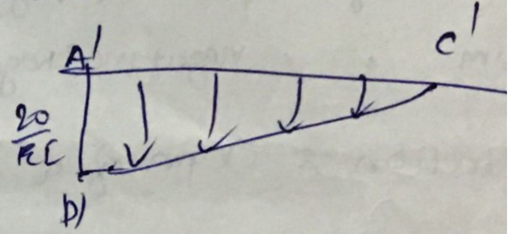
$= 2 \times 10^5 \times 10^6 N/m^2$ $= 10^8 \times \frac{1}{10^{12}} m^4 = 10^{-4} m^4$

$= 2 \times 10^8 kN/m^2$

BM @ B = 0 @ C = 0

BM @ A = $-10 \times 2 = -20 kNm$.





Let θ_B = slope @ free end

y_B = deflection @ free end.

θ_B = slope @ B' for conjugate beam

= Load A'C'D'

$= \frac{1}{2} \times A'C' \times A'D' = \frac{1}{2} \times 2 \times \frac{20}{EI} = \frac{20}{EI} = \frac{20}{2 \times 10^8 \times 10^{-4}} =$

y_B = BM @ B' for conjugate beam

= (Load A'C'D') × (Distance of CG of A'C'D' from B')

$= \left(\frac{1}{2} \times 2 \times \frac{20}{EI}\right) \times \left(1 + \frac{2}{3}(2)\right)$

$= \frac{20}{EI} \times \frac{7}{3} = 2.33 mm$

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