# UNIT-IV

# CONTINUOUS BEAMS

## INTRODUCTION

A beam carried over more than two supports is known as a *continuous beam*. Railway bridges are common examples of continuous beams. But the beams in railway bridges are subjected to travelling loads in addition to static loads. We will only consider the effect of static, concentrated, and distributed loads for the analysis of reactions and support moments. Figure 1 shows a beam *ABCD*, carried over three spans of lengths $L_1$, $L_2$, and $L_3$, respectively. End *A* of the beam is fixed, while end *D* is simply supported. At the end *D* support moment will be zero, but at end *A*, supports *B* and *C* there will be support moments in the beam, to be determined.

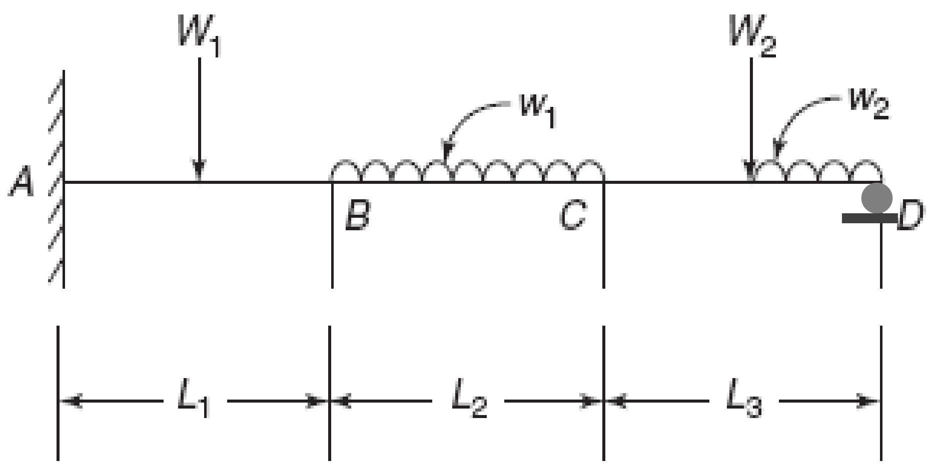



**Figure 1** Continuous beam

Prof. Clapeyron has provided a theorem showing the relationship between three support moments of any two consecutive spans of a continuous beam and the loads applied on these two spans. This theorem is generally known as the “**Clapeyron’s theorem of three moments**”.

Engineers are generally responsible with the job of analyzing support moments, support reactions, and bending moments in a continuous beam for the optimum design of beam sections.

## CLAPEYRON'S THEOREM OF THREE MOMENTS

This theorem provides a relationship between three moments of two consecutive spans of a continuous beam with the loading arrangement on these spans.

Let us consider a continuous beam *A′ ABCC′* supported over five supports, and there are four spans of the beam with lengths $L_1'$ $L_1$, $L_2$, and $L_2'$, respectively, as shown in Figure 2. In this beam let's consider consecutive spans *AB* and *BC* carrying a uniformly distributed loading *(udl)* of intensities $w_1$ and $w_2$, respectively, as shown in Figure 2.

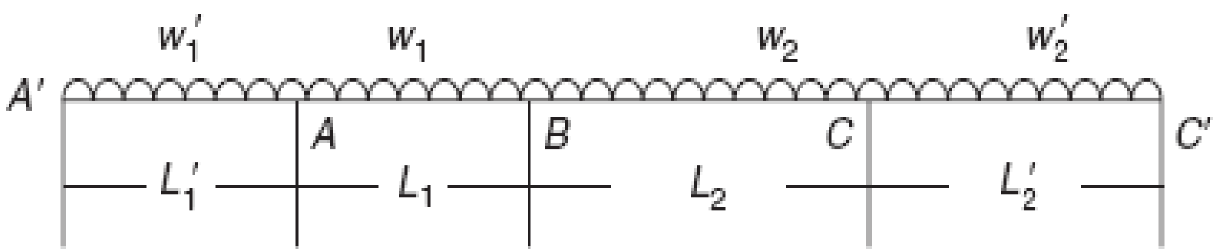



**Figure 2** Continuous beam

## CLAPEYRON'S THEOREM OF THREE MOMENTS

Let's say that support moments at *A*, *B,* and *C* are $M_A$, $M_B$, $M_C$, respectively. If the bending moment on spans *AB* and *BC* is positive, then support moments will be negative.

Now let's take two spans *AB* and *BC* independently and draw the bending moment (BM) diagram of each considering simple supports at ends. Maximum bending moment of *AB* will occur at its centre and be equal to: (Fixed beams, distributed loading, SS case, page 24)

$$\frac{w_1 L_1^2}{8}$$

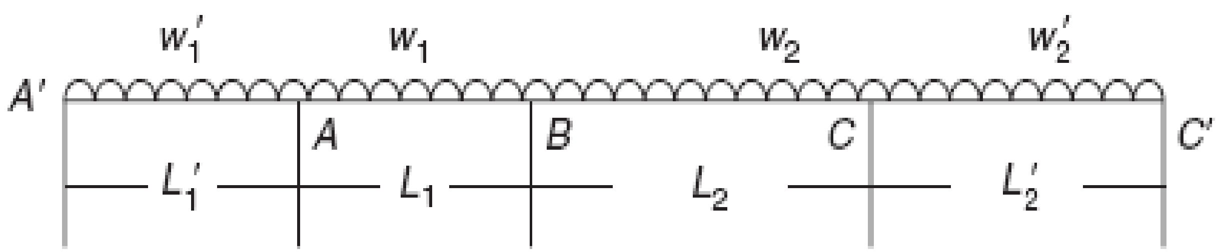



**Figure 2** Continuous beam

Similarly, maximum bending moment at the centre of span *BC* will be $\frac{w_2 L_2^2}{8}$ as shown in Figure 3.

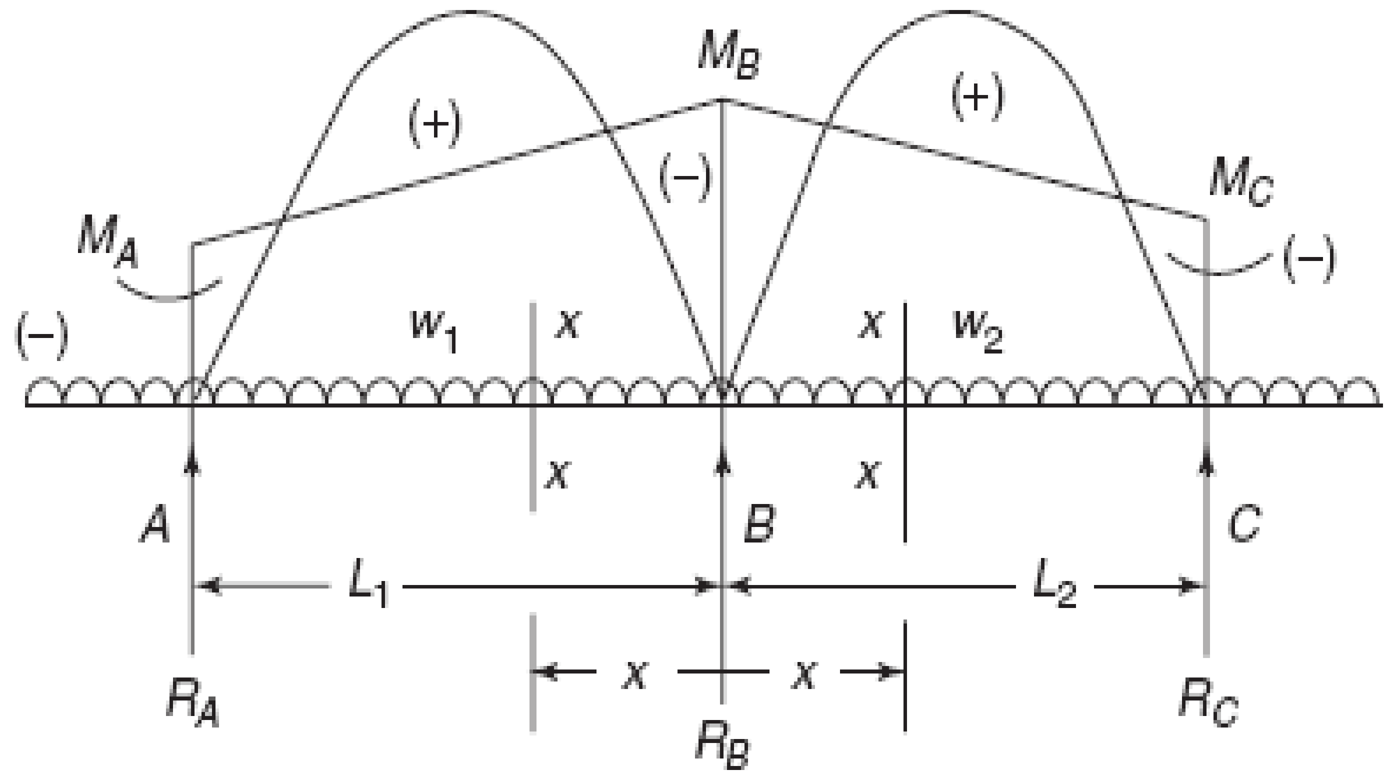



**Figure 3** BM diagrams over two supports

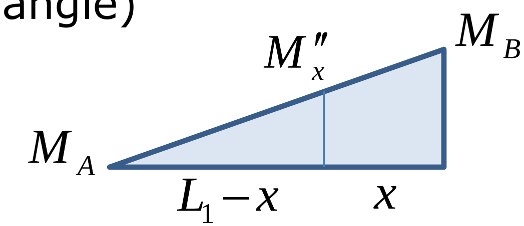
**Span AB (Independently)**

Origin at *B*, *x* positive towards left, bending moment at any section *X*-*X* is

$$M'_x = w_1\left(L_1 - x\right)\frac{x}{2} = \frac{w_1 L_1}{2}x - \frac{w_1 x^2}{2}$$

Support moments $M_A$, $M_B$, $M_C$ are shown in the diagram. Bending moment at this section due to support moments (using the top triangle)

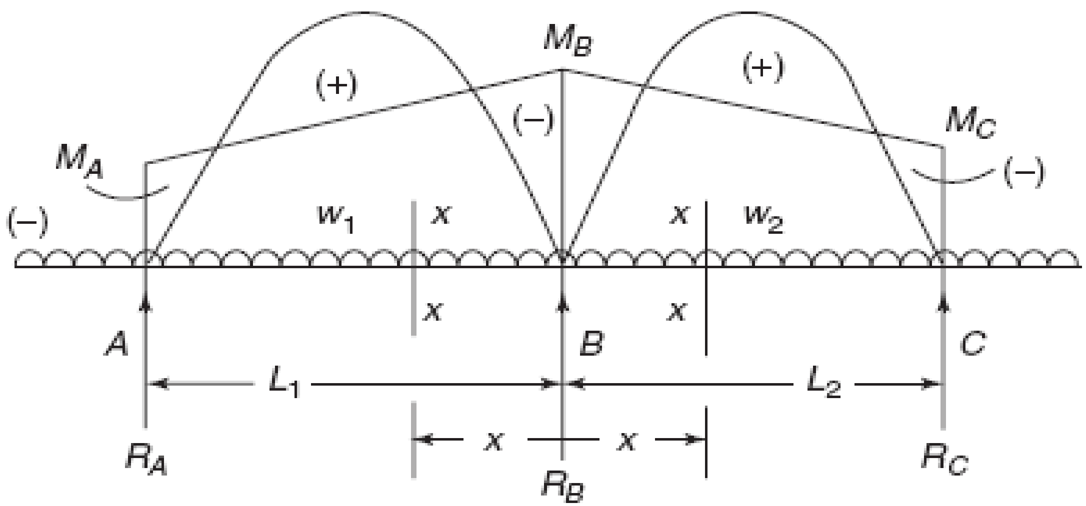
$$M''_x = M_B + \frac{x}{L_1}\left(M_A - M_B\right)$$




$$\frac{M''_x - M_A}{L_1 - x} = \frac{M_B - M_A}{L_1}$$

$$M''_x = M_A + \frac{L_1 - x}{L_1}\left(M_B - M_A\right)$$

$$M''_x = M_A + M_B - \frac{x}{L_1}M_B + M_A + \frac{x}{L_1}M_A$$



Resultant bending moment at the section (when *AB* is a part of a continuous beam)

$$M_x = M'_x + M''_x = \frac{w_1 L_1 x}{2} - \frac{w_1 x^2}{2} + M_B + \frac{x}{L_1}\left(M_A - M_B\right)$$

or

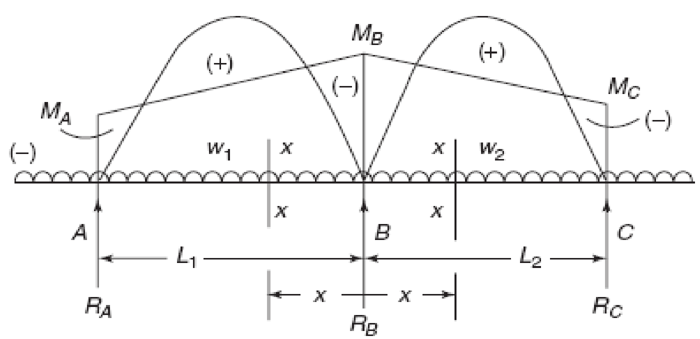
$$EI\frac{d^2 y}{dx^2} = \frac{w_1 L_1 x}{2} - \frac{w_1 x^2}{2} + M_B + \frac{x}{L_1}\left(M_A - M_B\right)$$

Integrating this equation, we get

$$EI\,\frac{dy}{dx} = \frac{w_1 L_1 x^2}{4} - \frac{w_1 x^3}{6} + M_B\,x + (M_A - M_B)\,\frac{x^2}{2L_1} + C_1,$$

where $C_1$ is a constant of integration.

At support *B*, where *x* = 0, slope $\frac{dy}{dx} = i_B$ (say)




so,

$EIi_B$ = 0–0+0+0+$C_1$

or constant of integration, $C_1 = +EIi_B$

Now

$$EI\frac{dy}{dx}=\frac{w_1L_1x^2}{4}-\frac{w_1x^3}{6}+M_Bx+(M_A-M_B)\frac{x^2}{2L_1}+EIi_B$$

Integrating this, we get

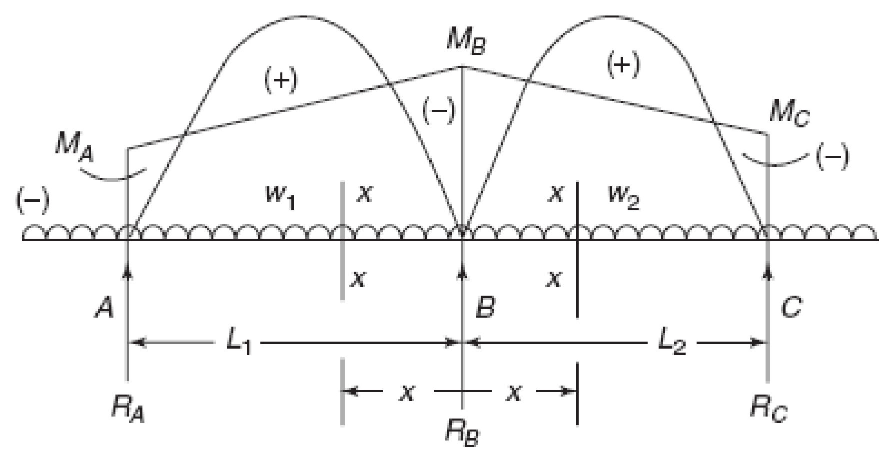
$$EIy=\frac{w_1L_1x^3}{12}-\frac{w_1x^4}{24}+M_B\frac{x^2}{2}+(M_A-M_B)\frac{x^3}{6L_1}+EIi_Bx+C_2$$

where $C_2$ is another constant of integration.

At support $B$, $x = 0$, deflection $y = 0$,

so, $0 = 0 - 0 + 0 + 0 + 0 + C_2$

Constant $C_2 = 0$



Finally

$$EIy = \frac{w_1 L_1 x^3}{12} - \frac{w_1 x^4}{24} + M_B \frac{x^2}{2} + (M_A - M_B) \frac{x^3}{6L_1} + EIi_B x$$

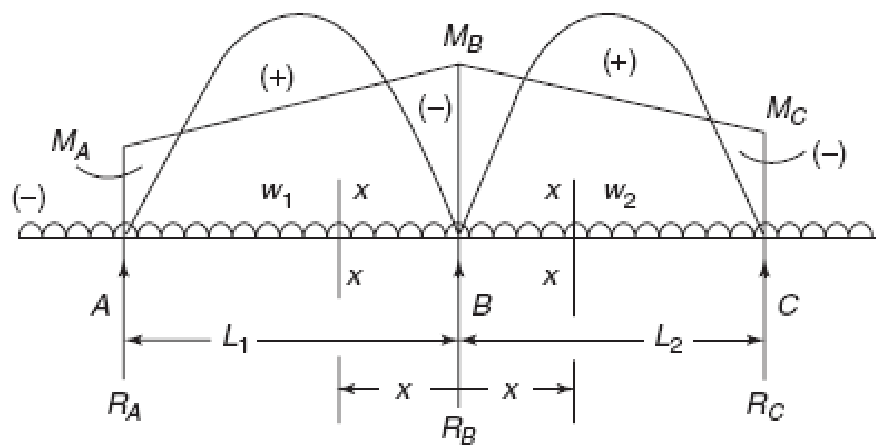
At the support $A$, $x = L_1$, deflection $y = 0$, so,

$$0 = \frac{w_1 L_1^4}{12} - \frac{w_1 L_1^4}{24} + M_B \frac{L_1^2}{2} + (M_A - M_B) \frac{L_1^2}{6} + EIi_B\ L_1$$

$$0 = \frac{w_1 L_1^4}{24} + \frac{M_B L_1^2}{3} + \frac{M_A L_1^2}{6} + EIi_B L_1$$

or

$$6EIi_B + (2M_B + M_A) L_1 = -\frac{w_1 L_1^3}{4}$$



Similarly considering span *BC*, origin at *B*, *x* positive towards right and proceeding in the same manner as before, we can write the equation

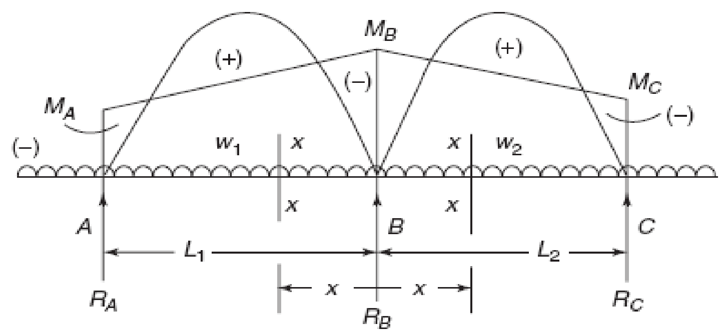
$$6EIi'_B + (2M_B + M_C)\,L_2 = -\frac{w_2 L_2^3}{4}$$

The slope $i'_B = -i'_B$ because in portion *AB*, *x* is taken positive towards left and in portion *BC*, *x* is taken positive towards right.

Adding $6EIi_B + (2M_B + M_A)\,L_1 = -\dfrac{w_1 L_1^3}{4}$

and $6EIi'_B + (2M_B + M_C)\,L_2 = -\dfrac{w_2 L_2^3}{4}$

we get




$$6EI\,(i_B + i'_B) + 2M_B\,(L_1 + L_2) + M_A L_1 + M_C L_2$$

$$= -\frac{w_1 L_1^3}{4} - \frac{w_2 L_2^3}{4}$$

Using the equation of three moments for spans *BC* and *DC*

$$4M_B + 2M_C\,(4+4) + 4M_D$$

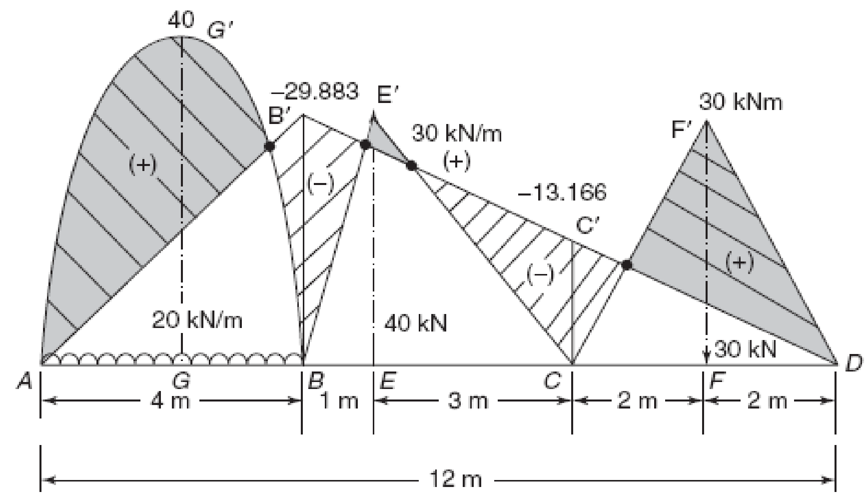
$$= -\frac{6\times100}{4} - \frac{6\times120}{4}$$

But

$M_D = 0$

$4M_B + 16M_C = -150 - 180 = -330$

$4M_B + 16M_C = -330$ (iii)




From Equations (ii) and (iii)

$15M_C = -197.5$

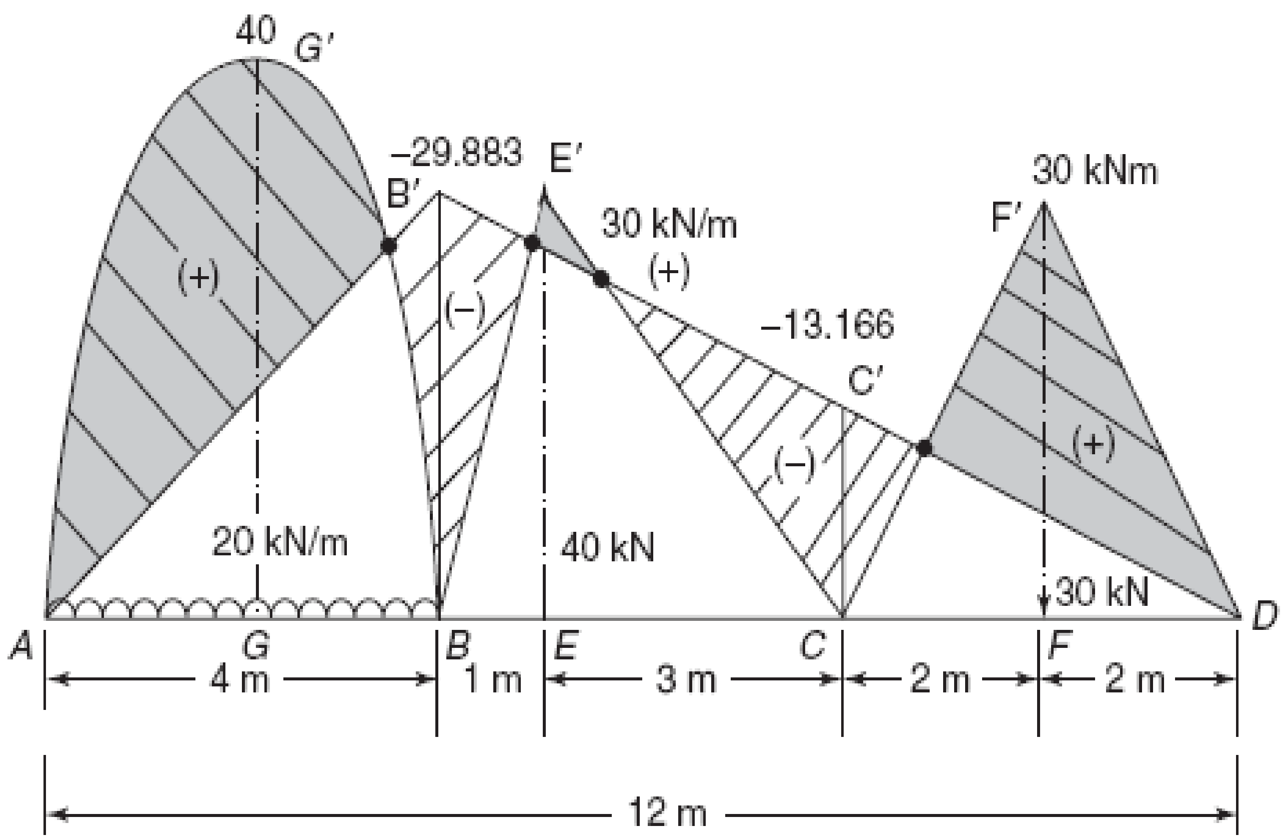
$M_C = -13.166$ kNm $M_B = -29.8335$ kNm

Taking

$BB' = -29.8335$ kNm

$CC' = -13.166$ kNm

Drawing lines *AB′*, *B′C′*, *C′D*, the diagram *AB′C′DA*, is the BM diagram due to support moments. There are four points of contraflexure in the BM diagram for continuous beam when *a′* diagrams are superimposed over *a″* diagram, the BM diagram due to support moments. Positive and negative areas of the BM diagram are also marked.



## SUPPORTS NOT AT SAME LEVEL

There are two spans *AB* and *BC* of lengths $L_1$ and $L_2$ of a continuous beam. Supports *A*, *B,* and *C* are not at same one level. Support *B* is below support *A* by $\delta_1$ and below support *C* by $\delta_2$ as shown in Figure 5 (a). These level differences are very small in comparison to span length, say of the order of 0.1% of span length. For spans *AB* and *BC*, bending moment diagrams are plotted considering the spans independently as shown by the shaded diagrams.

Say

$a_1' =$ BM diagram of span $AB$ independently

$\overline{x_1'} =$ Distance of $CG$ of $a_1'$ from $A$

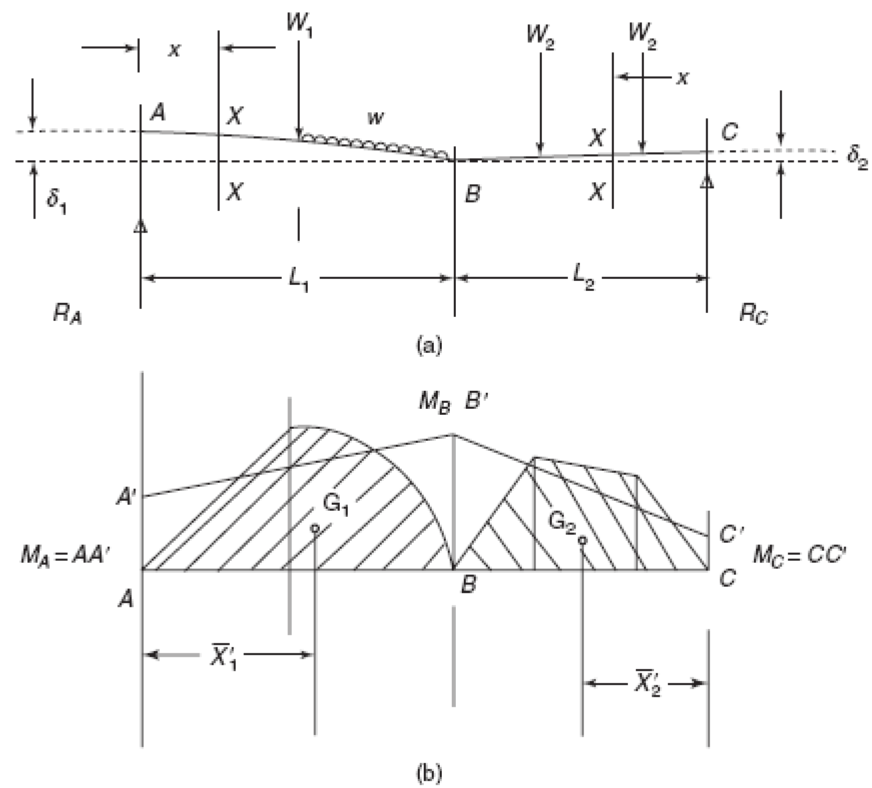



**Figure 5** Continuous beam

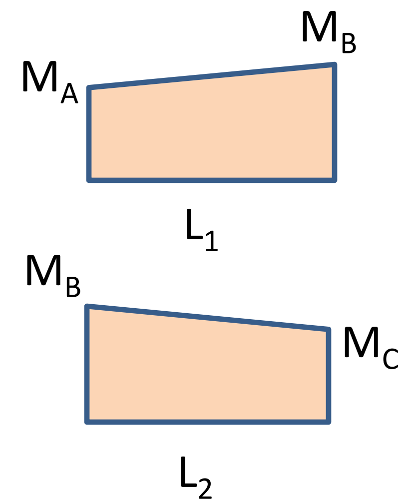
$a_2' =$ BM diagram of span $BC$ independently

$\overline{x}_2' =$ Distance of $CG$ of $a_2'$ from $C$

$a_1'' =$ BM diagram due to support moments span $AB$

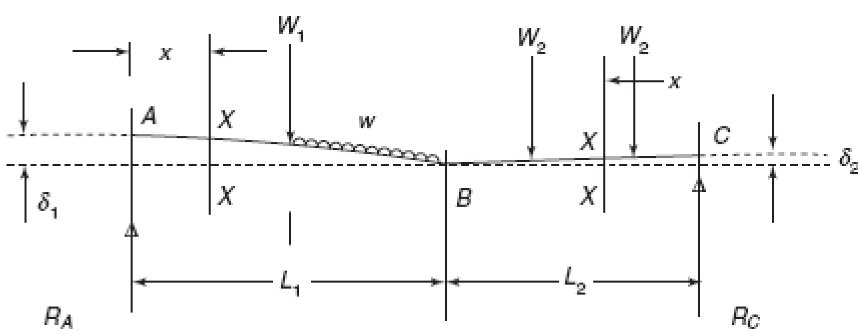
$=$ Diagram $AA'B'B$

$a_2'' =$ BM diagram due to support moments span $CB$

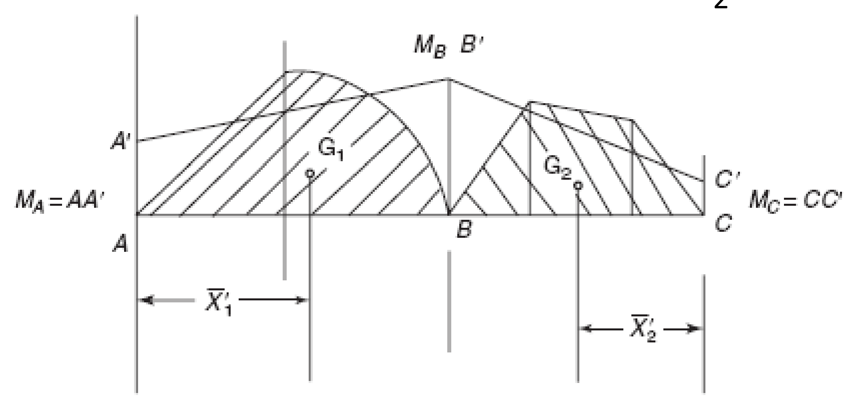



Moreover $$a_1''\overline{x}_1'' = (M_A + 2M_B)\frac{L_1^2}{6}, \text{ first moment of area about } A$$

$$a_2''\overline{x}_2'' = (M_C + 2M_B)\frac{L_2^2}{6}, \text{ first moment of area about } C$$




(a)




(b)

**Span AB:** Consider a section *X*-*X*, at a distance of *x* from *A*

$$EI\,\frac{d^2y}{dx^2} = M'_x + M''_x$$

$$= \text{BM due to span } AB \text{ as } SS + \text{BM due to support moments}$$

or

$$\int_0^{L_1} EI\,\frac{d^2y}{dx^2}x\,dx = \int_0^{L_1} M'_x\,x\,dx + \int_0^{L_1} M''_x\,x\,dx \quad \text{(multiplying both the sides by } x\,dx \text{ and integrating)}$$

$$EI\left[x\,\frac{dy}{dx} - y\right]_0^{L_1} = a'_1\,\bar{x}'_1 + a''_1\,\bar{x}''_1$$

$$EI\,[(L_1 \times i_B + \delta_1) - (0 \times i_A - 0)]$$

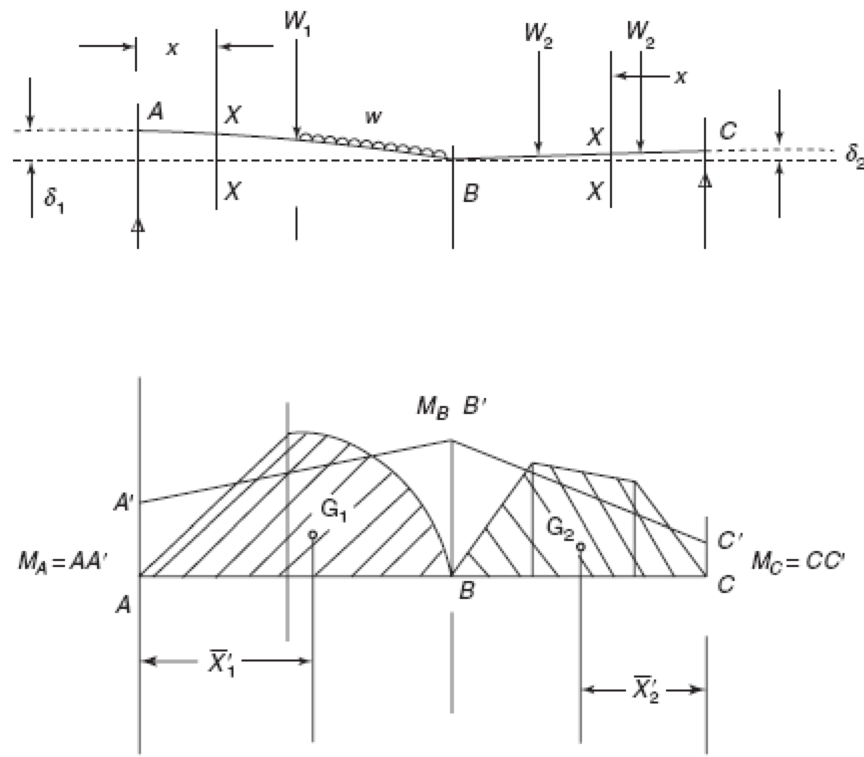
$$= a'_1\,\bar{x}'_1 + (M_A + 2M_B)\,\frac{L_1^2}{6}$$

(Note that downward deflection is negative)

or

$$EIi_B = \frac{a'_1\,\bar{x}'_1}{L_1} + (M_A + 2M_B)\,\frac{L_1}{6} - \frac{EI\,\delta_1}{L_1}$$

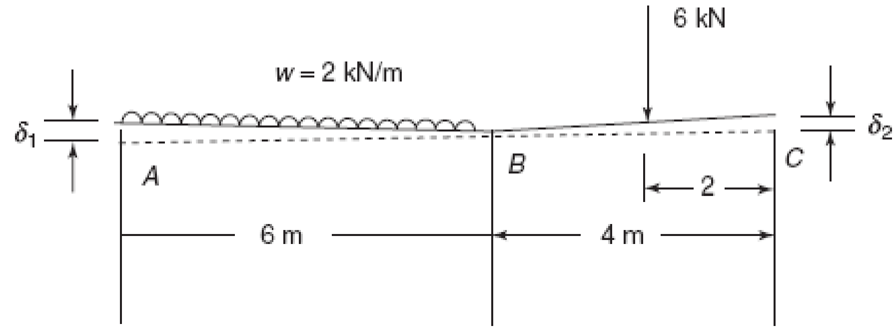
$$6EIi_B = \frac{6a'_1\,\bar{x}'_1}{L_1} + (M_A + 2M_B)\,L_1 - \frac{6EI\,\delta_1}{L_1}$$



(b)

Support moments $M_A = M_C = 0$

Using the equation derived in Fixed Beams, Support Moments:



(a)

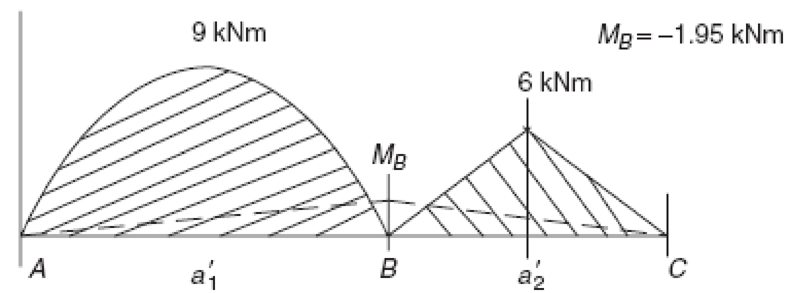
$$2M_B\,(L_1 + L_2) = \frac{6a_1'\,\bar{x}_1'}{L_1} - \frac{6a_2\,\bar{x}_2'}{L_2} + \frac{6EI\,\delta_1}{L_1} + \frac{6EI\,\delta_2}{L_2}$$

$$2M_B\,(6+4) = -\frac{6\times108}{6} - \frac{6\times24}{4} + 60 + 45$$

$$20M_B = -108 - 36 + 105$$

$$= -39$$

$$M_B = -1.95\ \text{kNm}$$



(b)

| | | | |
|---|---|---|---|
| Total load on beam | | = | 6 × 2 + 6 = 18kN |
| Moreover | $M_B$ | = | $R_A \times 6 - 6 \times 2 \times 3$ |
| | −1.95 | = | $6R_A - 36$ |
| Reaction, | $R_A$ | = | + 5.675 kN |
| also | $M_B$ | = | $4R_C - 6 \times 2$ |
| | −1.95 | = | $4R_C - 12$ |
| Reaction, | $R_C$ | = | +2.5125 kN |
| Reaction, | $R_B$ | = | 18 − 5.675 − 2.5125 = 9.8125 kN. |

## CONTINUOUS BEAM WITH FIXED END

For a continuous beam with a fixed end, the equations for support moments can be derived considering the slope and deflection at fixed end to be zero. Figure 6 (a) shows two consecutive spans AB and BC of a continuous beam. End A of the beam is fixed. Bending moment diagrams $a'_1$ and $a'_2$ are plotted considering spans AB and BC supported independently as shown in Figure 6. At any section if:

$M'_x$ = BM due load on span, considering span independently
$M''_x$ = BM due to support moments, then

$$EI\frac{d^2y}{dx^2} = M'_x + M''_x$$

$$EI\int_0^{L_1}\frac{d^2y}{dx^2}x\,dx = \int_0^{L_1}M'_x\,x\,dx + \int_0^{L_1}M''_x\,x\,dx$$

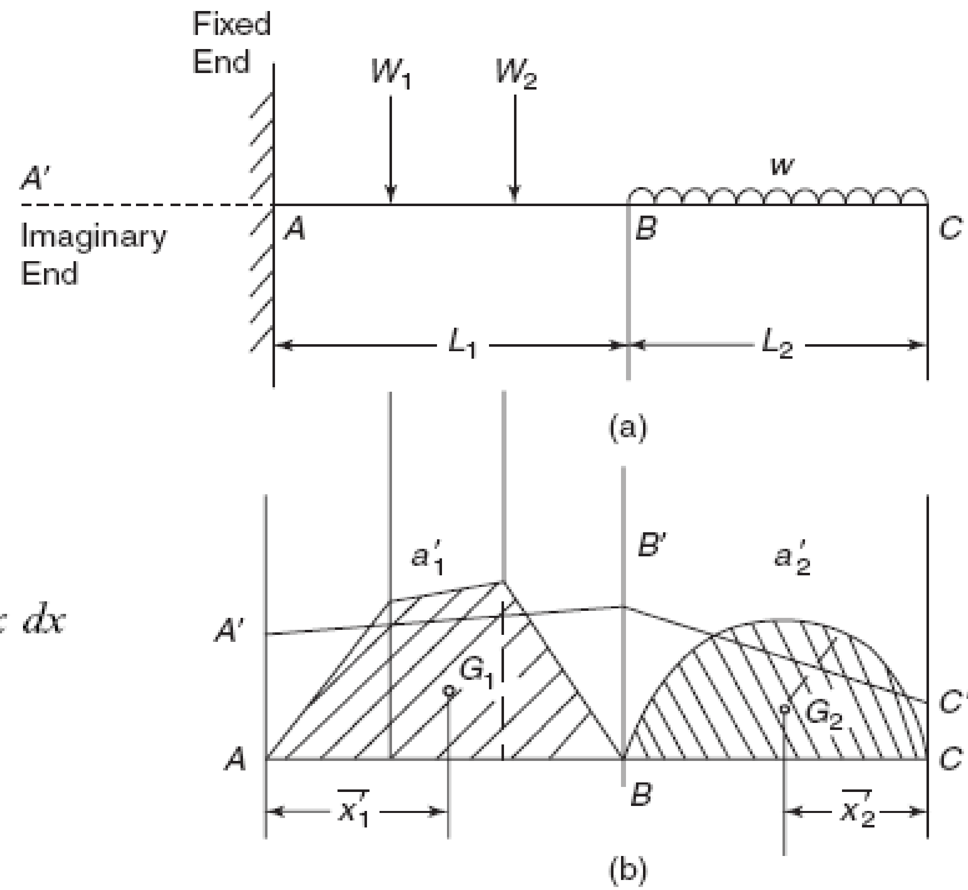



**Figure 6** Support moments

Considering origin as B and x positive towards left:

$$EI\left| x\frac{dy}{dx} - y \right|_0^{L_1} = a_1'(L_1 - \overline{x}_1') + \frac{L_1^2}{6}(M_B + 2M_A)$$

But at fixed end:

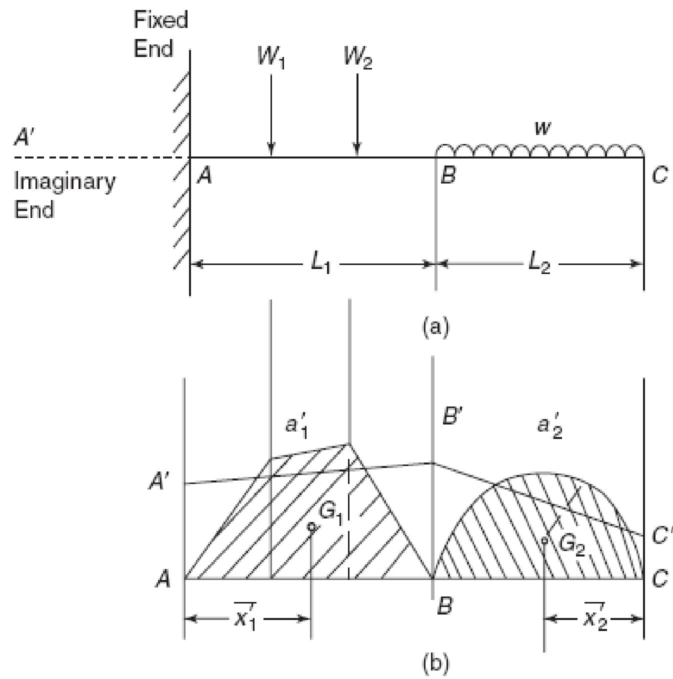
$$x = L_1, \quad y = 0, \frac{dy}{dx} = 0$$

$$EI\,[(L_1 \times 0 - 0) - (0 \times i_B - 0)]$$

$$= a_1'\,(L_1 - \overline{x}_1') + \frac{L_1^2}{6}(M_B + 2M_A)$$

or

$$2M_A L_1 + M_B L_1 + \frac{6a_1'\,(L_1 - \overline{x}_1')}{L_1} = 0$$




where $M_A$ is the fixing couple at fixed end A.

This relationship can also be obtained by considering an imaginary span A′A of zero length and bending moment at A′, $M'_A = 0$ using Clapeyron's theorem for two spans A′A and AB.

Considering origin as B and x positive towards left:

$$EI\left|x\frac{dy}{dx}-y\right|_0^{L_1}=a_1'(L_1-\overline{x}_1')+\frac{L_1^2}{6}(M_B+2M_A)$$

But at fixed end:

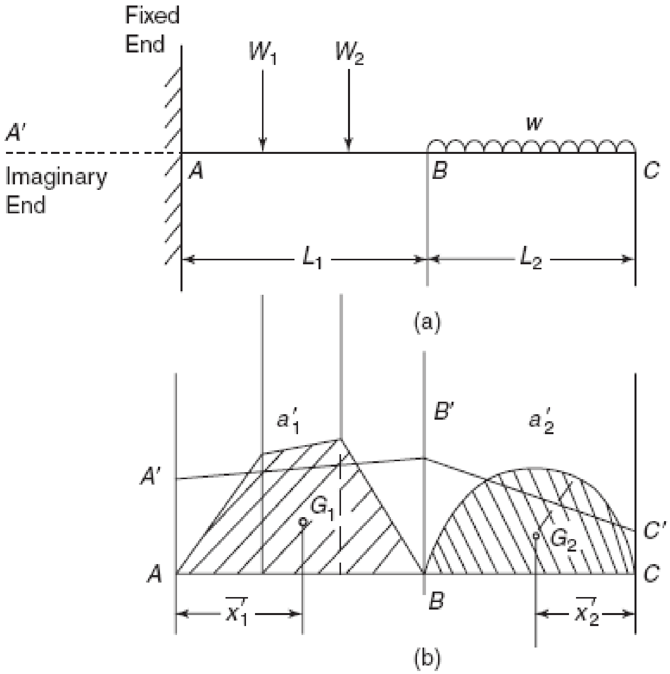
$$x=L_1,\quad y=0,\frac{dy}{dx}=0$$

$$EI\,[(L_1\times 0-0)-(0\times i_B-0)]$$

$$=a_1'\,(L_1-\overline{x}_1')+\frac{L_1^2}{6}(M_B+2M_A)$$

or

$$2M_A\,L_1+M_B\,L_1+\frac{6a_1'\,(L_1-\overline{x}_1')}{L_1}=0$$



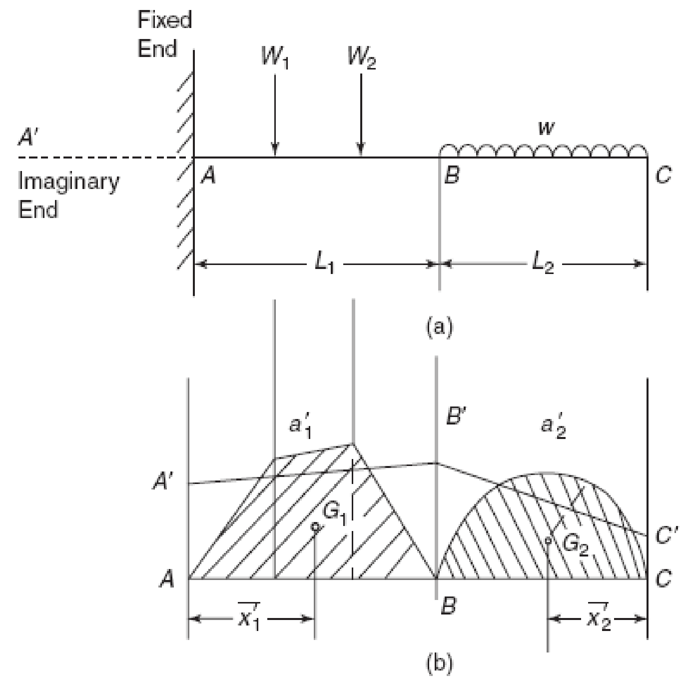

where $M_A$ is the fixing couple at fixed end A.

This relationship can also be obtained by considering an imaginary span A’A of zero length and bending moment at A′, $M'_A = 0$ using Clapeyron’s theorem for two spans A’A and AB.

$$M'_A \times 0 + 2M_A(0 + L_1) + M_B L_1$$
$$= 0 - \frac{6a'_1(L_1 - \overline{x}'_1)}{L_1}$$

or

$$2M_A L_1 + M_B L_1 + \frac{6a'_1(L_1 - \overline{x}'_1)}{L_1} = 0$$

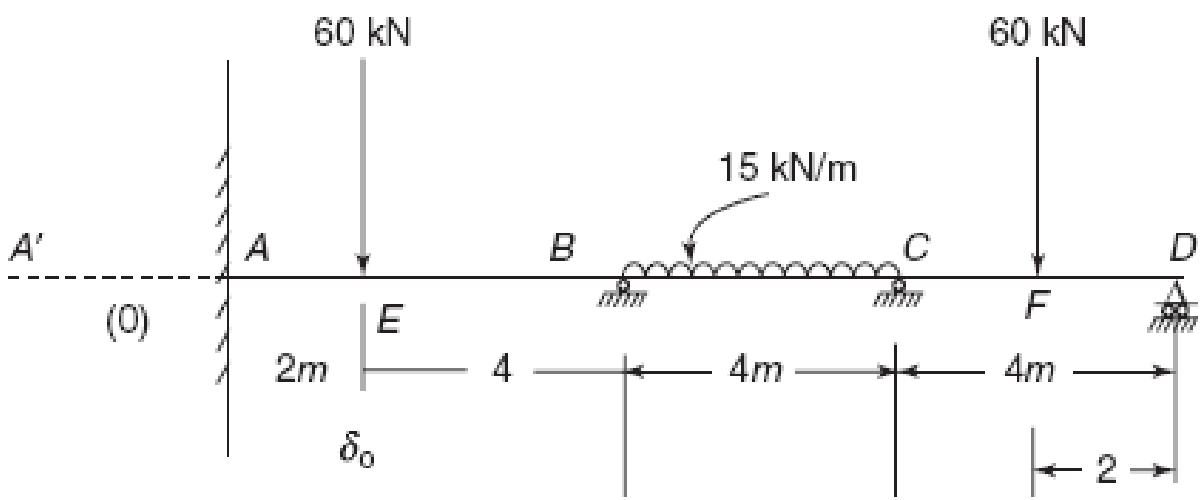



If the other end of the continuous beam is also fixed, a similar equation can be made by considering an imaginary span to the right of the other fixed end, and then applying the theorem of three moments.

**Example**

A continuous beam ABCD 14 m long rests on supports A, B, C, and D all at the same level. AB = 6 m, BC = 4 m, CD = 4 m. Support A is a fixed support. It carries two concentrated loads of 60 kN each, at a distance of 2 m from end A and end D as shown in the figure. There is a udl of 15 kN/m over span BC. Find the moments and reactions at the supports.

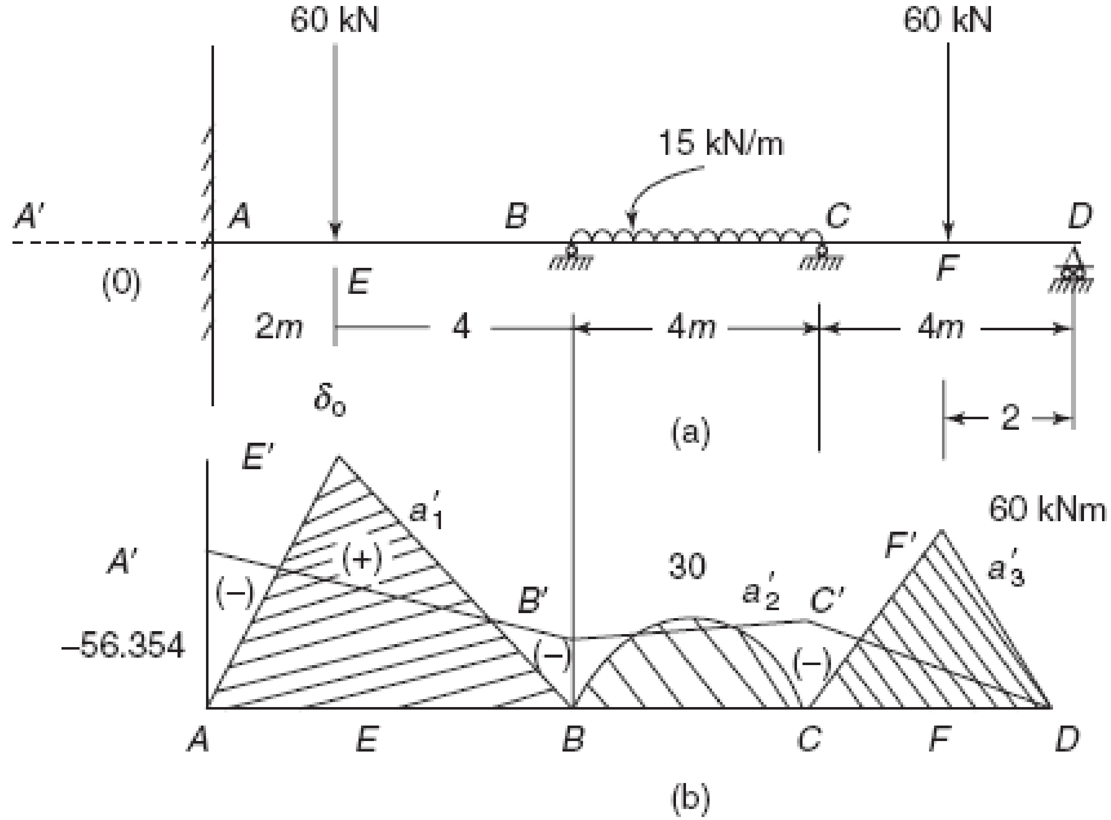
The figure shows the continuous beam ABCD, with fixed end at A. Let us first draw BM diagrams considering each span to be simply supported.



Span AB

Maximum BM,

$$EE' = \frac{60 \times 4 \times 2}{6} = 80 \text{ kNm}$$




Origin at A,

$$a'_1 \, \overline{x}'_1 = \frac{80 \times 2}{2} \times \left(\frac{4}{3}\right) + \frac{80 \times 4}{2}\left(2 + \frac{4}{3}\right)$$

$$= 640 \text{ kNm}^3$$

Origin at B,

$$a'_1 \, \overline{x}'_1 = \frac{80 \times 4}{2} \times \left(\frac{8}{3}\right) + \frac{80 \times 2}{2} \left(4 + \frac{2}{3}\right)$$

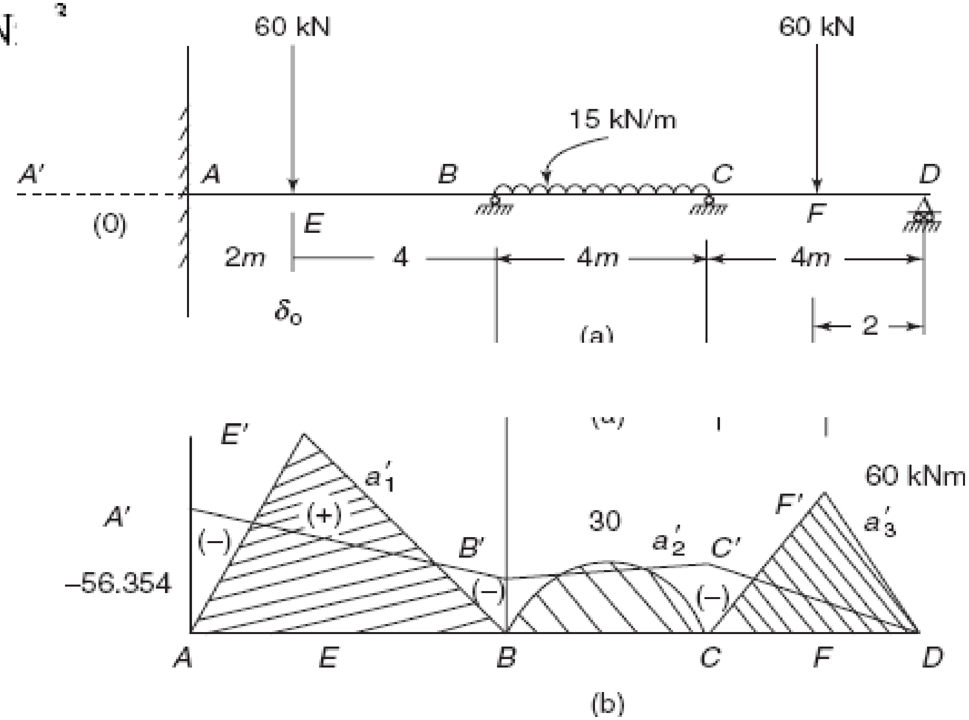
$$= \frac{1,280}{3} + \frac{1,120}{3} = 800 \text{ kNm}^3$$

Span BC

$$M_{\max} = \frac{wL^2}{8} = \frac{15 \times 4^2}{8} = 30\ \text{kNm}$$

$$a_2'\,\overline{x}_2' = \frac{2}{3} \times 30 \times 4 \times 2 = 160\ \text{kN}$$

(origin at $B$ or $C$)




Span CD

$$M_{\max} = \frac{WL}{4} = \frac{60 \times 4}{4} = 60\ \text{kNm}$$

$$a_3'\,\overline{x}_3' = \frac{60 \times 4}{2} \times 2 = 240\ \text{kNm}^3$$

(origin at $C$ or $D$)

Considering imaginary span AA′ of zero length and using Clapeyron’s theorem for two spans A’A and AB,

$$M_A'' \times 0 + 2M_A\,(0+6) + 6M_B$$

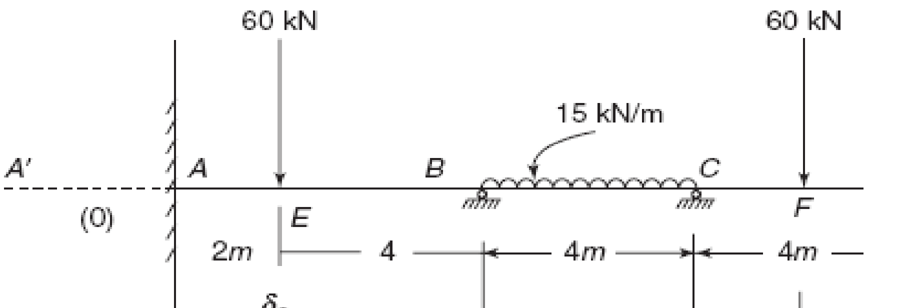
$$= 0 - \frac{6a_1'\,\overline{x}_1'}{L_1} \qquad \text{(origin at } B\text{)}$$

$$12M_A + 6M_B = -\frac{800\times 6}{6}$$

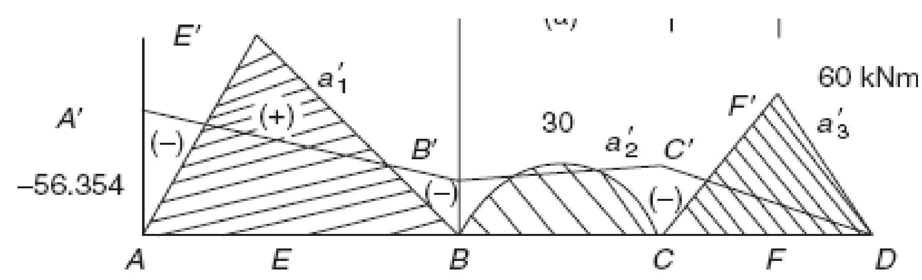
$$12M_A + 6M_B = -800 \qquad \text{(i)}$$

or

$$M_A = \frac{-800 - 6M_B}{12} \qquad \text{(ii)}$$



(a)



(b)

Spans AB and BC

$$6M_A + 2M_B\,(6+4) + 4M_C = -\frac{6\times 640}{6} - \frac{6\times 160}{4}$$

$$6M_A + 20M_B + 4M_C = -880 \qquad \text{(iii)}$$

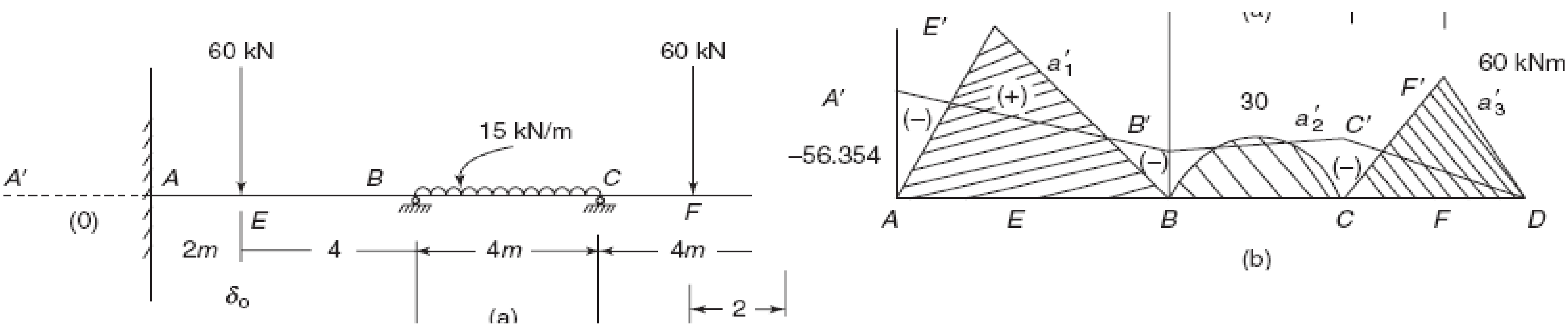
Spans BC and CD

$$4M_B + 2M_C\,(4+4) + 4M_D = -\frac{6\times 160}{4} - \frac{6\times 240}{4} = -240 - 360 = -600$$

$$M_D = 0$$

But

$4M_B + 16M_C = -600$ (iv)



Putting the value of $M_A$ in equation (iii)

$$6\left(\frac{-800-60M_B}{12}\right)+20M_B+4M_C=-880$$

$$-400-3M_B+20M_B+4M_C=-880$$

$$17M_B+4M_C=-480 \qquad \text{(v)}$$

$M_B + 4M_C = -150$ From eq. (iv) (vi)

Solving eqs. (v) and (vi), we get

$16\ M_B = -330$

Moment, $M_B = -20.625$ kNm, Putting the value of $M_B$ in eq. (vi)

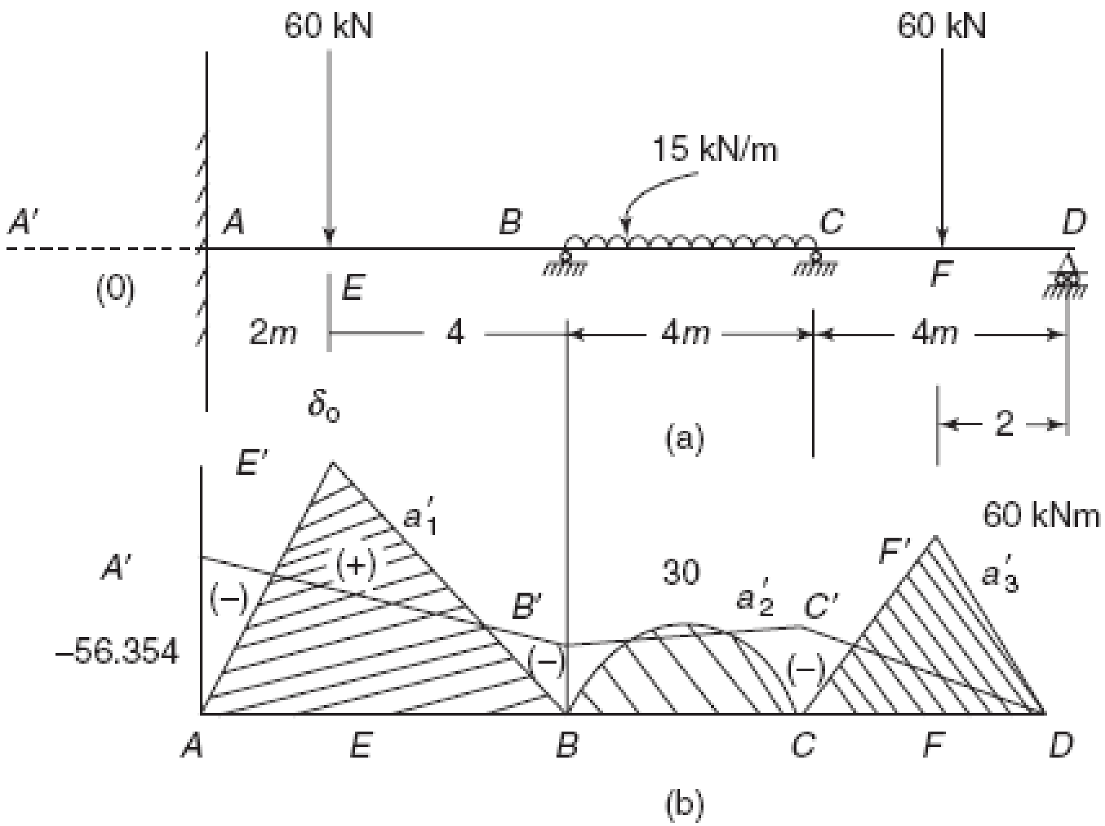
$4\ M_C = -150 + 20.625$

Moments,

$$M_C = -32.344 \text{ kNm}$$

$$M_A = -\frac{800-6M_B}{12} = \frac{-800+6\times 20.625}{12} = -56.354 \text{ kNm}$$

The figure shows shaded diagrams are a’1, a’2, and a’3 BM diagrams. Bending moment diagram due to support moments is superimposed on these diagrams to get resultant BM at any section.



## Support Reactions

| | | | |
|---|---|---|---|
| | $M_C$ | = | $4R_D - 60 \times 2 = -32.334$ |
| Reaction, | $R_D$ | = | 21.9 kN |
| | $M_B$ | = | $8R_D + 4R_C - 6 \times 60 - 60 \times 2 = -20.625$ |
| | $8R_D + 4R_C$ | = | 459.375 |
| | $8 \times 21.9 + 4R_C$ | = | 459.375 |
| Reaction, | $R_C$ | = | 71.0 kN |

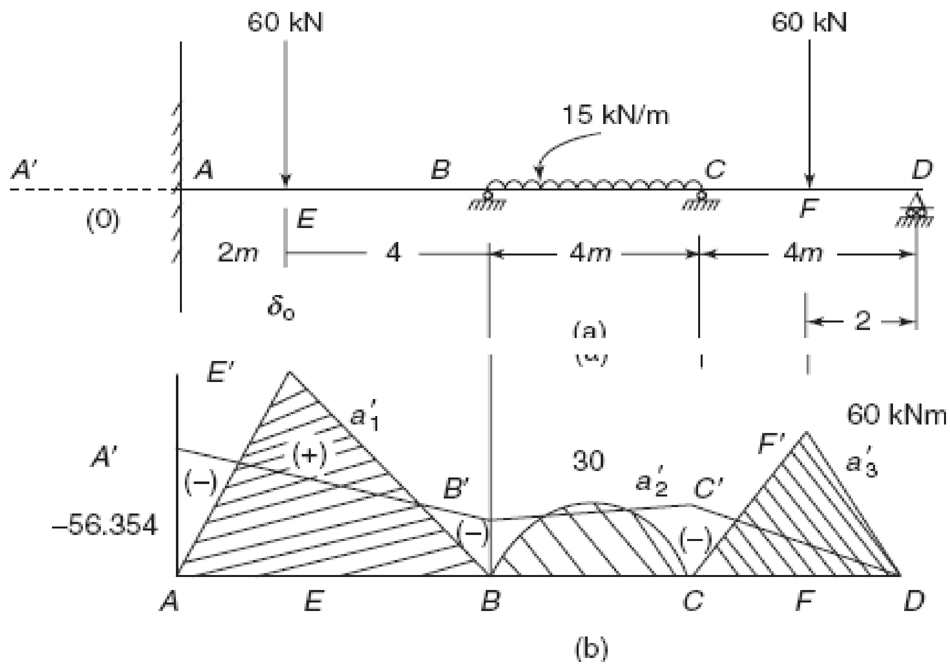
## Moments about A

$$14R_D + 10R_C + 6R_B - 12 \times 60 - 4 \times 15 \times 8 - 60 \times 2$$

$$= M_A = -56.354 \text{ kNm}$$

Putting the values in the solution:

| | | | |
|---|---|---|---|
| Reaction, | $R_B$ | = | 41.1 kN |
| Total load on beam | | = | 60 + 4 × 15 + 60 = 180 kN |
| Reaction | $R_A$ | = | 180 − 41.1 − 71.0 − 21.9 |
| | | = | 46 kN |



**Example**

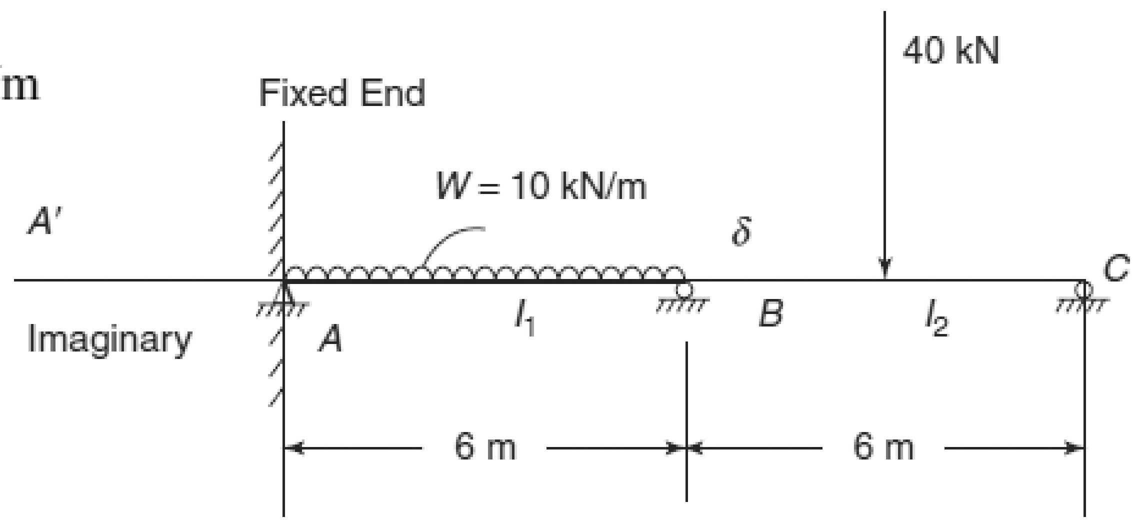
A continuous beam ABC, fixed at end A, supported over spans AB = BC = 6 m each. There is a udl of 10 kN/m over AB and a concentrated load of 40 kN at centre of BC as shown in the figure. While the supports A and C remain at the same level, the level of support B is 1 mm below due to sinking. Moment of inertia of beam from A to B is 18,000 $cm^4$ and from BC it is 12,000 $cm^4$. If E = 210 $kN/mm^2$, determine support moments and draw BM diagram.

Let us first draw a diagram for both spans

area,

$a_1' =$ A parabola with

$$M_{max} = \frac{wL^2}{8} = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

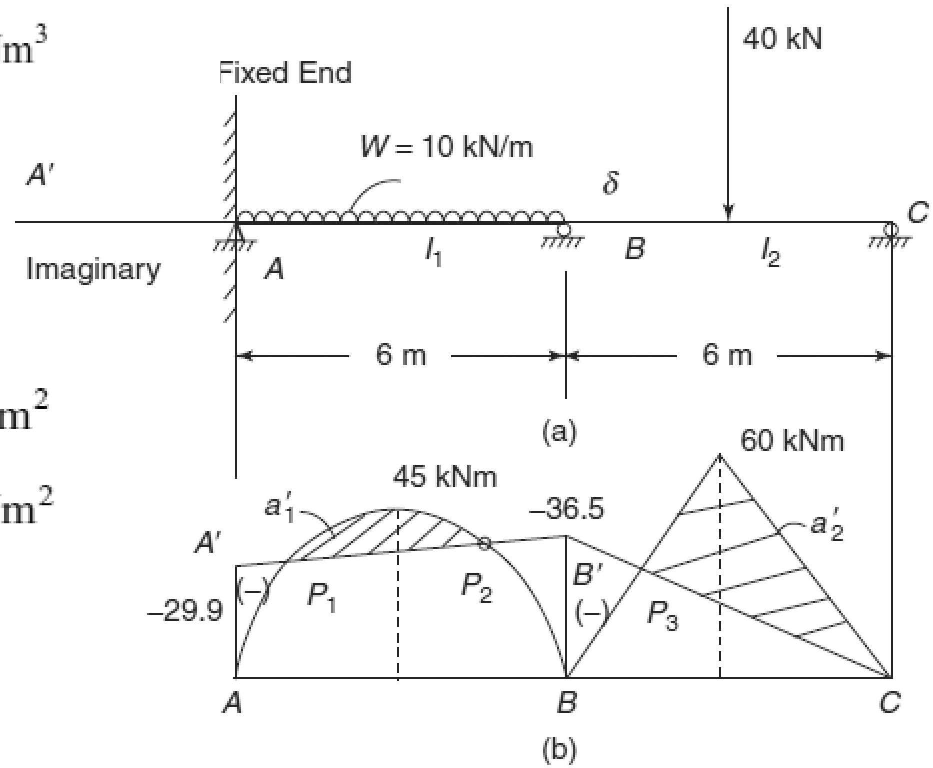


a′$_2$ is a triangle with

$$M_{max} = \frac{WL}{4} = \frac{40\times 6}{4} = 60 \text{ kNm as shown}$$

$$a'_1\,\bar{x}'_1 \text{ (about } A \text{ or } B) = \frac{2}{3}\times 45\times 6\times 3 = 540 \text{ kNm}^3$$

$$a'_2\,\bar{x}'_2 \text{ (about } B \text{ or } C) = \frac{60\times 6}{2}\times 3 = 540 \text{ kNm}^3$$




Moment, $M_C$ = 0, because end C is a simple support.

$$EI_1 = 210\times 10^6\times 18{,}000\times 10^{-8} = 37{,}800 \text{ kNm}^2$$

$$EI_2 = 210\times 10^6\times 12{,}000\times 10^{-8} = 25{,}200 \text{ kNm}^2$$

$$\delta = 0.001 \text{ m as given}$$

$$L_1 = L_2 = 6 \text{ m}$$

$$\frac{6EI_1\,\delta}{L_1} = \frac{6}{6}\times 37{,}800\times 0.001 = 37.8$$

Span AA´B

Imaginary span AA′ and AB equation of three moments.

$$\frac{2M_A\,(0+6)}{I_1}+\frac{6M_B}{I_1}=-\frac{6a_1'\,\overline{x}_1'}{L_1 I_1}+\frac{6EI_1\,(-\delta)}{I_1 L_1}$$

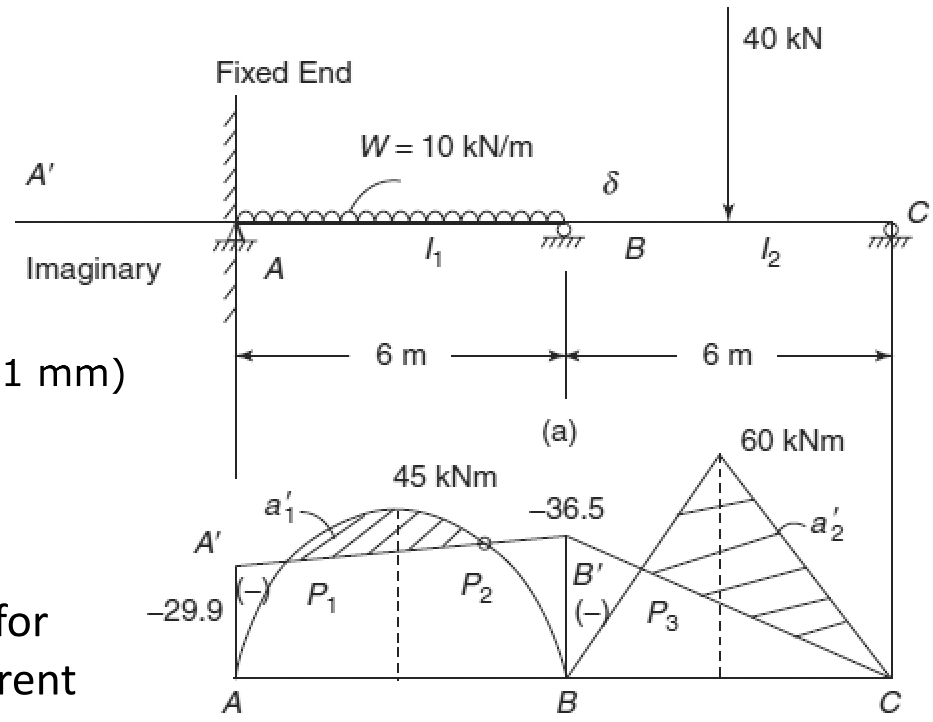
Taking $I_1$ common throughout

$$12M_A+6M_B=-\frac{6\times540}{6}-37.8=-540-37.8$$

(because level of A is higher than level of B by 1 mm)

12MA + 6MB = − 577.8 (i)

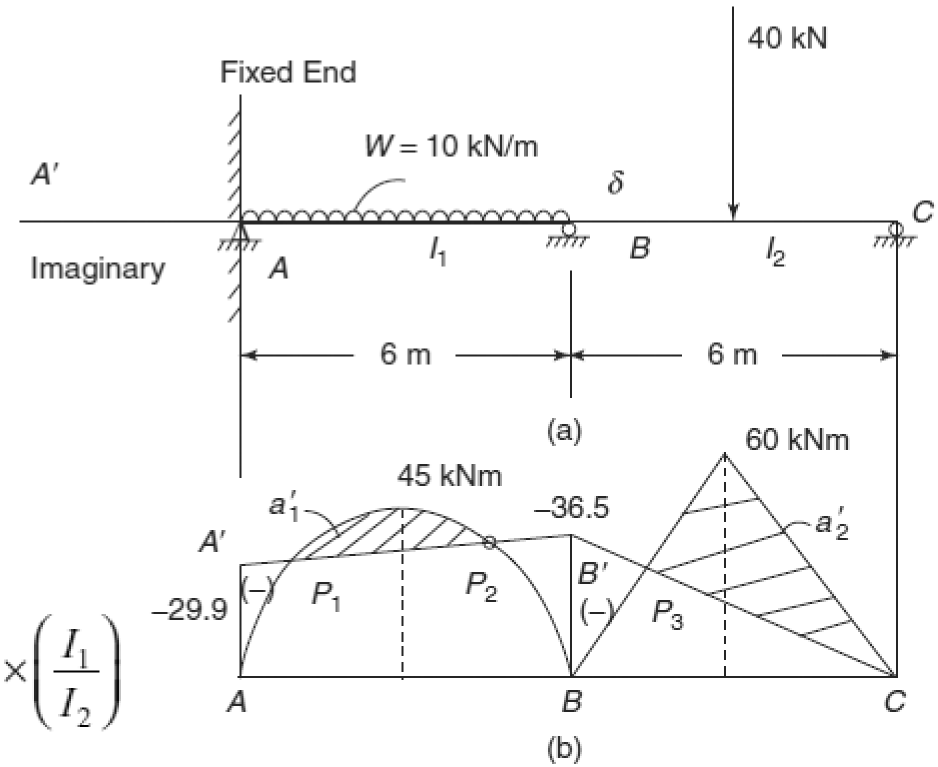
Now using the theorem of three moments for spans AB and BC, and noting that $I_1$ is different than $I_2$, equation can be modified as



(a)

(b)

$$\frac{6M_A}{I_1}+\frac{2M_B\times 6}{I_1}+\frac{2M_B\times 6}{I_2}+\frac{6M_C}{I_2} = -\frac{6a_1'\,\overline{x}_1'}{I_1 L_1}-\frac{6a_2'\,\overline{x}_2'}{I_2 L_2}+\frac{6EI_1\,\delta}{I_1 L_1}+\frac{6EI_2\,\delta}{I_2 L_2}$$




Multiplying throughout by $I_1$,

$$6M_A+12M_B+12M_B\times\frac{I_1}{I_2}+\frac{6M_C\times I_1}{I_2} = -\frac{6a_1'\,\overline{x}_1'}{L_1}-\frac{I_1}{I_2}\times\frac{6a_2'\,\overline{x}_2'}{L_2}+\frac{6EI_1\,\delta}{L_1}+\frac{6EI_2\,\delta}{L_2}\times\left(\frac{I_1}{I_2}\right)$$

But $I_1$ = 1.5I2, putting this value, we get

$$6M_A+12M_B+18M_B+9M_C$$
$$=-\frac{6}{6}\times 540-\frac{6}{6}\times 540\times 1.5+37.8+\frac{6}{6}\times 25,200\times(15)\times 0.001$$

$6M_A + 30M_B + 9M_C = -540 - 810 + 37.8 + 37.8$

$6M_A + 30M_B + 9M_C = -1274.4$ kNm

But $M_C = 0$