UNIT V

ENERGY THEOREMS

UNIT-V

ENERGY THEOREMS

When an external load acts on a structure, it undergoes deformation and hence the work is done.

→ To resist these external forces, the internal forces develop gradually from zero to their final value and the work is done.

→ This internal work done is stored as energy in the structure and it helps the structure to bring back to the original shape and size.

∴ This internal work, which is stored as energy is due to the straining of the material and hence it is called "**STRAIN ENERGY**"

As per, Law of conservation of energy,

The work done by the external forces = Strain energy stored.

Strain energy = workdone ⇒ Real work (or) Actual work

⇒ $U = W = \frac{1}{2} P\delta$ [Area under the Load deformation curve]

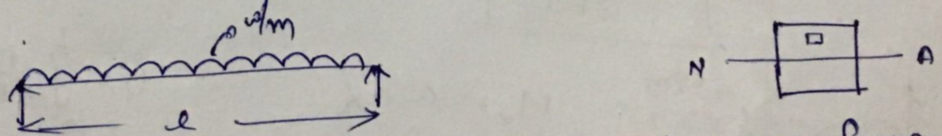
If there are 'n' no. of loads, $\boxed{W = \Sigma \frac{1}{2} \delta P}$ (Axial loads)

$U = \frac{1}{2} \sigma e V$

$= \frac{1}{2} \sigma \frac{\sigma}{E} V$ $\boxed{U = \frac{\sigma^2}{2E} V}$

(I) STRAIN ENERGY DUE TO BENDING

Consider the beam subjected to pure bending.



Let the area of the element δa & distance from N.A be 'y'.

We know that $\sigma_b = \frac{M}{I} y$

Strain $e = \frac{\sigma}{E} = \frac{M y}{EI}$.

Strain energy in the element $= \frac{1}{2} \times \text{Stress} \times \text{Strain} \times dv$

$$= \frac{1}{2} \times \left(\frac{M}{I}\right) y \times \left(\frac{M}{EI}\right) y \times dv$$

$$= \frac{1}{2} \frac{M^2 y^2 dv}{EI^2}$$

$\therefore$ Strain energy in the beam

$$= \int_0^L \int_0^A \frac{1}{2} \frac{M^2 y^2}{EI^2} \delta a \, dx.$$

$$\boxed{\therefore \int_0^A y^2 \delta a = I}$$

$$= \int_0^L \frac{M^2}{2EI^2} \left[\int_0^A y^2 \delta a \right] dx$$

$$= \int_0^L \frac{M^2}{2EI^2} . I \, dx$$

$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx.}$$

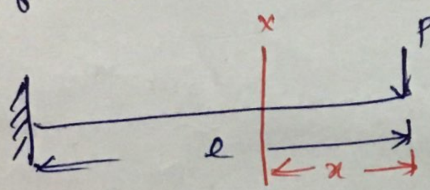
Note

Strain energy due to couple $= \frac{Ml^2}{2EI}$

P/o

1) Using strain energy method determine the deflection of the free end of cantilever of length l sub. to a conc. load P @ free end

Sol a)



$M_x = Px$

Strain energy $U = \int_0^L \frac{M^2}{2EI} dx$

$$= \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{2EI} \int_0^L x^2 dx$$

$$= \frac{P^2}{2EI} \left[\frac{x^3}{3}\right]_0^L = \frac{P^2 L^3}{6EI}$$

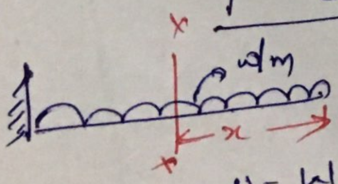
δ $U = W.$

$W = \frac{1}{2} P \times \delta$

$\frac{P^2 L^3}{6EI} = \frac{1}{2} P \times \delta.$

$$\boxed{\delta = \frac{PL^3}{3EI}}$$

b)



$M_x = wx^2/2$

$$U = \int_0^L \frac{M^2}{2EI} dx$$

$$= \int_0^L \frac{w^2 x^4}{8EI} dx$$

$$= \frac{w^2}{8EI} \left[\frac{x^5}{5}\right]_0^L = \frac{w^2 L^5}{40EI}$$

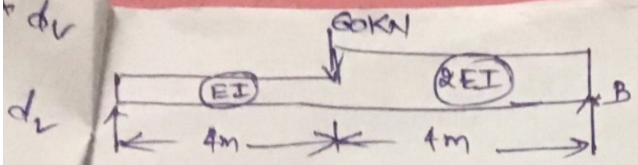
$U = W$

$\frac{w^2 L^5}{40EI} = \frac{1}{2} P \delta$

$$\boxed{\delta = \frac{w^2 L^5}{20PEI}}$$

Scanned with CamScanner

Deflection under load ?



Support Reactions $R_A = R_B = 30KN.$

$M_x = R_A x = 30x$ KNm

$$U = \int_0^L \left(\frac{M^2}{2EI}\right) dx$$

$$= \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2}{2 \times 2EI} dx$$

$$= \int_0^4 \frac{900x^2}{2EI} dx + \int_0^4 \frac{900x^2}{4EI} dx$$

$$= \frac{900}{2EI}\left[\frac{x^3}{3}\right]_0^4 + \frac{900}{4EI}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{900}{2EI} \times \frac{4^3}{3} + \frac{900}{4EI} \times \frac{4^3}{3}$$

$$U = \frac{150 \times 64}{EI} + \frac{300 \times 16}{EI}$$

$$= \frac{9600 + 4800}{EI}$$

$$\boxed{U = \frac{14400}{EI}}$$

$W = \frac{1}{2} P\delta$

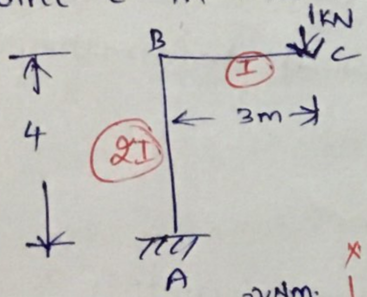
$W = \frac{1}{2} 60 \times \delta$

$U = W$

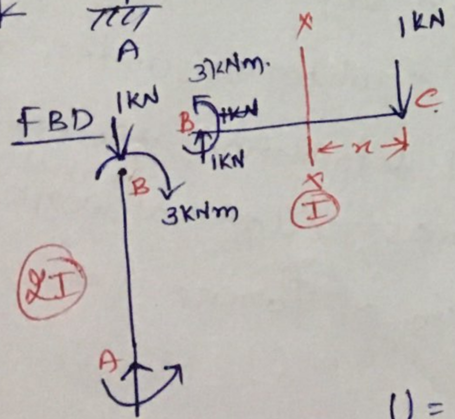
$\frac{14400}{EI} = \frac{1}{2} 60 \times \delta$

$$\boxed{\delta = \frac{480}{EI}}$$

3) Using strain energy determine the vertical deflection of Point 'c' in the frame. $E = 200 KN/mm^2$, $I = 30 \times 10^6 mm^4$.



Sol FBD



FBD for Various portions of Structure

Portion	origin	Limit	(M_x) Expression
BC	C	0 - 3	$1x = x$
AB	B	0 - 4	3

$$U = \int_0^3 \frac{x^2}{2EI} dx + \int_0^4 \frac{3^2}{(2EI)2} dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{9}{4EI}\left[x\right]_0^4$$

$$U = \frac{3^3}{6EI} + \frac{9}{4EI}(4) \qquad \therefore \boxed{U = \frac{13.5}{EI}}$$

$$\therefore W = \frac{1}{2} P\delta$$

$$\boxed{W = \frac{1}{2} 1 \cdot \delta} \qquad \rightarrow \frac{13.5}{EI} = \frac{\delta}{2} \quad \rightarrow \delta = \frac{27}{EI}$$

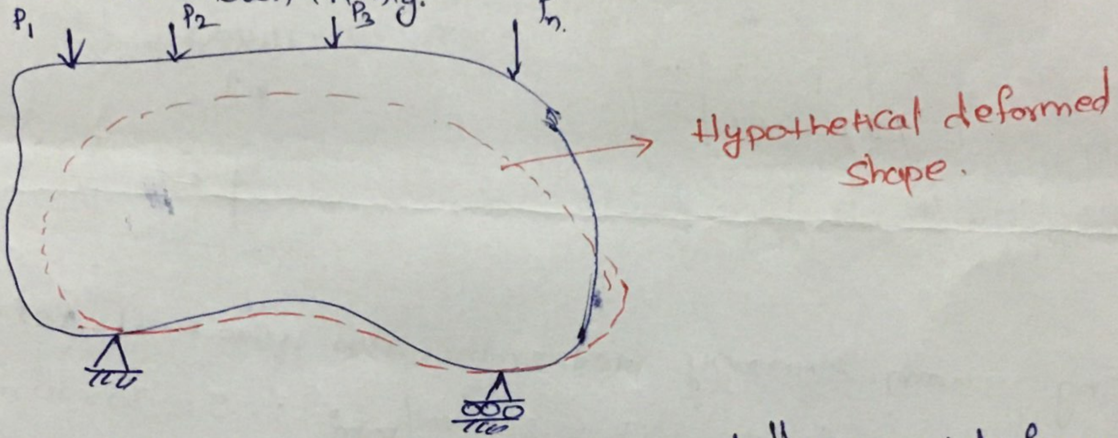
$$EI = 200 \times 30 \times 10^{6} \times 10^{-6} = 6000 \text{kN}$$

$$\delta = \frac{27}{60000} = 0.045 m$$

$$\boxed{\delta = 4.5 mm}$$

Virtual work

Consider the body subject to a set of real forces P_1, P_2, P_3 $P_4 \ldots P_n$. Let the body undergoes deformation due to some other forces as shown in fig.



This Hypothetical deformation is called "Virtual deformation" and the work done by real forces due to virtual deformation is called "Virtual work"

Principle of virtual work for Deformable Bodies:

If a deformable body in equilibrium under a system of forces is given virtual deformation, the virtual work done by the system of forces = Internal virtual work done by the stresses due to that system of forces

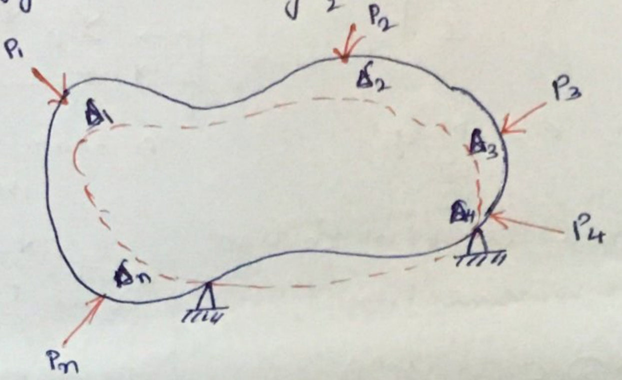
$$\boxed{W_e = W_i}$$

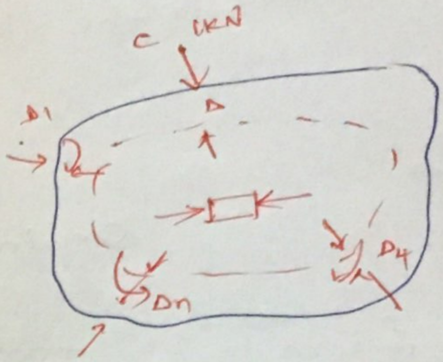
LOAD METHOD

Consider the body shown in fig. which is subjected to forces $P_1, P_2, P_3, P_4 \cdots P_n$ applied gradually. Let displacement under load @ points be $\Delta_1, \Delta_2, \Delta_3 \cdots \Delta_n$ and @ point c be δ.

$\therefore$ External workdone $= \frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \cdots + \frac{1}{2}\Delta_n P_n$

Strain energy stored $= \int \frac{1}{2}\sigma e \, dv$





Equating both

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \cdots + \frac{1}{2}\Delta_n P_n = \int \frac{1}{2}\sigma e \, dv \quad \text{--- (1)}$$

Now consider same body subjected to an unit load applied gradually @ c when it is free of system of P forces. Let the displacement @ 1, 2, 3 ..... n be $\delta_1, \delta_2 \ldots \delta_n$ respectively and the displacement @ c be δ.. Let the stress produced in the element be σ' and the strain be e'.

$\therefore$ External workdone $= \frac{1}{2} \times 1 \times \delta$

Internal " " $= \int \frac{1}{2}\sigma' e' \, dv$

Equating both $\frac{1}{2} \times 1 \times \delta = \int \frac{1}{2}\sigma' e' \, dv$ --- (2)

Equating (1) and (2) external & internal workdone

$$\frac{1}{2}\Delta_1 P_1 + \frac{1}{2}\Delta_2 P_2 + \cdots + \frac{1}{2}\Delta_n P_n + 1 \times \Delta = \int \frac{1}{2}\sigma e \, dv + \int \sigma' e \, dv \quad \text{--- (3)}$$

Substracting eq (1) & (3)

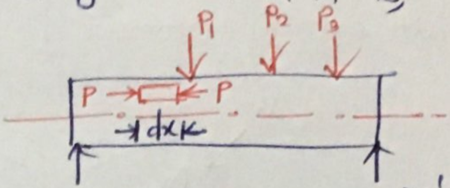
$$\boxed{\Delta = \int \sigma' e \, dv}$$

Application to beam deflections – Unit load method

Consider the beam shown in fig subjected ~~to~~ system of 'p' forces.

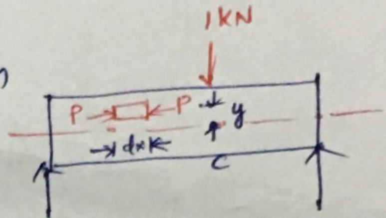
Stress in the element @ distance y from N.A is,

$$\sigma_b = \frac{M}{I} y'$$



A beam sub. to 'p' forces

Strain $e = \frac{M}{EI} y$



Beam sub. to unit load

Let 'm' be the moment @ the section where the element is considered, due to unit load acting @ C.

Then Stress $\sigma' = \frac{my}{I}$

We know that $\Delta = \int \sigma' e \, dv$

$$= \int \frac{m}{I} y \left(\frac{M}{EI} y\right) dv$$

$$= \int_0^L \frac{Mm}{EI^2} \left[\int_0^A y^2 dA\right] dx.$$

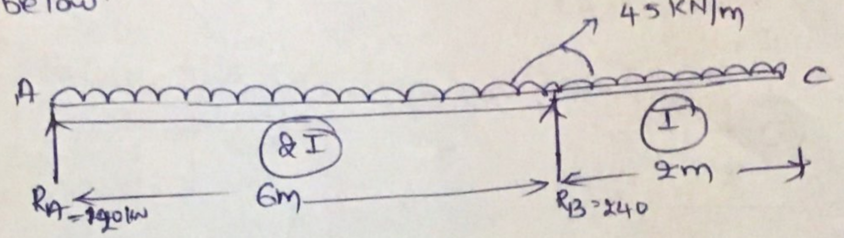
$$\int_0^A y^2 dA = I$$

$$= \int_0^L \frac{Mm}{EI^2} . I dx$$

$$\boxed{\Delta = \int_0^L \frac{Mm}{EI} dx}$$

Determine the deflection @ free end of the overhanging beam shown in fig. below.

(7)



Sol

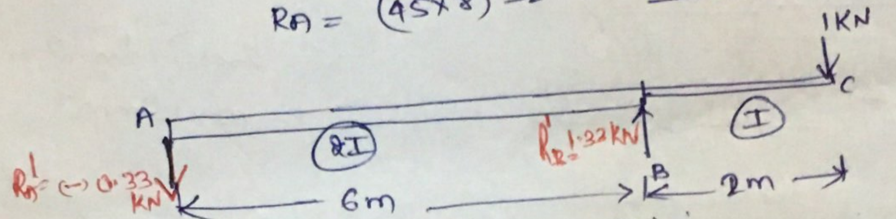
Support reactions

$\Sigma m_A = 0 \Rightarrow \quad R_B \times 6 - 45 \times 8 \times \frac{8}{2} = 0 \quad \Rightarrow 6R_B = 45 \times 8 \times 4$

$\boxed{R_B = 240KN}$

$R_A + R_B = (45 \times 8)$

$R_A = (45 \times 8) - 240 \quad \boxed{R_A = 120KN}$



When unit load is applied, (and acting) @ C,

$\Sigma m_A = 0 \Rightarrow \quad R'_B \times 6 - 1 \times 8 = 0 \quad \Rightarrow 6R'_B = 8 \Rightarrow \boxed{R'_B = 1.33KN}$.

$R'_A + R'_B = 1 \Rightarrow \boxed{R'_A = -0.33KN} \; (\downarrow)$

Sagging moments +ve ; Hogging moments (-ve)

$$\boxed{\Delta = \int_0^L \frac{Mm}{EI} dx}$$

Portion	AB	BC
origin	A	C
Limit	0 – 6	0 – 2
M	$120x - \frac{45x^2}{2}$	$-\frac{45x^2}{2}$
m	$-0.33x$	$-x$
I	2I	I

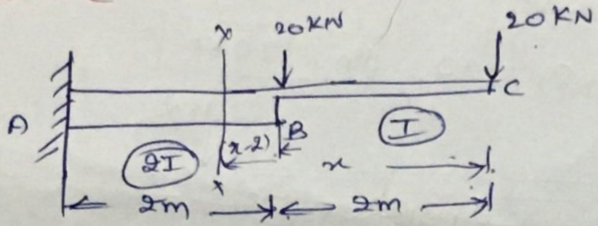
$$\Delta_c = \int_0^6 \frac{(120x - 22.5x^2)(-0.33x)}{E(2I)} dx + \int_0^2 \frac{(-22.5x^2)(-x)}{EI}$$

$$= \int_0^6 \frac{(-20x^2+3.7x^3)\,dx}{EI} + \frac{1}{EI}\int_0^2 22.5x^3\,dx$$

$$= \frac{1}{EI}\left[-\frac{20x^3}{3} + \frac{3.7x^4}{4}\right]_0^6 + \frac{1}{EI}\left[\frac{22.5x^4}{4}\right]_0^2$$

$$\boxed{\Delta_c = \frac{-135}{EI}}$$

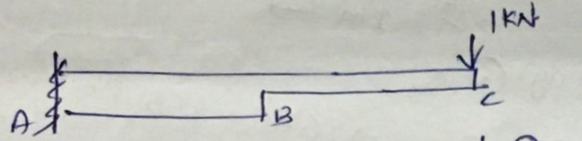
(2) Determine the deflection ~~and rotation~~



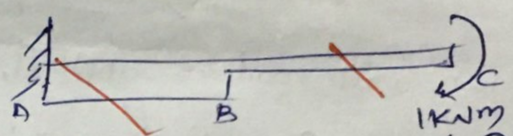
$E = 2\times10^5\ N/mm^2$

$I = 12\times10^6\ mm^4$

Sol



Beam with unit vertical load @C



Beam with unit moment @C

$M \rightarrow$ Given loads

$m_1 \rightarrow$ unit vertical load @ C

$m_2 \rightarrow$ Unit moment @C for various portions

Portion	CB	BA
origin	C	B
Limit	0–2	0–2
M	$-20x$	$-20x-20(x-2)$
m_1	$-x$	$-(x-2)$ $-[x+$
m_2	-1	-1
I	I	2I

$-20x - 20(x-2)$

$-20x - 20x + 40$

$\underline{40 - 40x}$.

$-[20x + 20(x-2)]$

$-[20x + 20x - 40]$

$\underline{40x - 40}$

$-[40x - 40]$

$\underline{-40x + 40}$

...al deflection of point D. The c/s area of ...ns are 1000mm²

$$\Delta = \int_0^L \frac{Mm_1}{EI}\,dx$$

$-[20x$

(9)

$$= \int_0^2 \frac{-20x\,(-x)}{EI}\,dx + \int_0^2 \frac{-20x - 20(x-2)(-x+2)}{2EI}\,dx$$

$$= \int_0^2 \frac{20x^2}{EI}\,dx + \int_2^2 (20x - 20x + 40)(-x-2)$$

$(40-40x)(-x+2)$

$= 40x + 80 + 40x^2$

$-80x$

$40x^2 - 120x$

$+80$

$$= \frac{20}{EI}\left[\frac{x^3}{3}\right]_0^2 + \int_0^2 \left(\frac{40 - 40x}{2EI}\right) dx$$

$$= \frac{20(2)^3}{3EI} + \frac{40}{2EI}\int_0^2 dx - 40\int_0^2 x\,dx$$

$$= \frac{20(2)^3}{3EI} + 40(x)_0^2 - 40\left(\frac{x^2}{2}\right)_0^2$$

$$= \frac{20(2)^3}{3EI} + \frac{40}{2EI}\left(2 - \frac{4}{2}\right)$$

$$\boxed{\Delta = \frac{160}{3EI}}$$

CASTIGLIANO'S THEOREMS

Theorem 1: (First Theorem)

In a linearly elastic structure, the partial deri[vative] of strain energy with respect to a load is equal to the deflection of the point where the load is acting. The deflection being measured in the direction of the load. $\frac{du}{dP_1} = \Delta_i$

Proof U: Total strain energy $P_j, M_j \rightarrow$ Loads; $\Delta i, \theta_j \rightarrow$ Deflections

Ily $\frac{du}{dM_j} = \theta_j$

Consider a SSB, on which loads P_1, P_2, P_3 are applied gradually. Let the deflections under the loads be $\Delta_1, \Delta_2, \Delta_3$ respectively

$$U = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \quad \text{--- (1)}$$

Let the additional load dP_1 be added after the loads P_1, P_2 and P_3

Let the " deflections be $d\Delta_1, d\Delta_2, d\Delta_3$. Then additional strain energy is

$$du = \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \quad \text{--- (2)}$$

Total strain energy of the system $U + du = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3$ (3)

$$U + du = \frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1) + \frac{1}{2}P_2(\Delta_2 + d\Delta_2) + \frac{1}{2}P_3(\Delta_3 + d\Delta_3) \quad \text{--- (4)}$$

Since final strain energy in both cases same, equating (3) & (4)

$$\frac{1}{2}P_1 d\Delta_1 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3 d\Delta_3 = \frac{1}{2}dP_1\Delta_1 \quad \text{--- (5)}$$

From eq (2) $\Rightarrow \frac{1}{2}(P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3) = \frac{1}{2}\left(du - \frac{1}{2}dP_1 d\Delta_1\right)$ --- (6)

From (5) & (6)

$$\frac{1}{2}\left(du - \frac{1}{2}dP_1 d\Delta_1\right) = \frac{1}{2}dP_1\Delta_1 \quad \text{--- (7)}$$

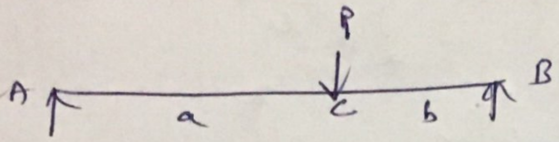
Neglecting small quantity of higher order from (7)

$$\frac{1}{2}du = \frac{1}{2}dP_1\Delta_1$$

$\boxed{\frac{du}{dP_1} = \Delta_1}$ Ily $\boxed{\frac{du}{dM} = \theta}$ $\boxed{\frac{\partial M}{\partial P} = \Delta}$

Using ... tral ... hal ...

Using castigliano's theorem obtain the deflection under a single concentrated load applied to a SSB shown



P/o

Take $P = 60\,kN$
$a = 3\,m$
$b = 1\,m$
$EI = 2.2\,MNm^2$

Ans: $\boxed{\Delta = 20.45\,mm}$

Method-1

Support Reactions

$\Sigma M_B = 0 \Rightarrow R_B \times l - Pa = 0 \Rightarrow \boxed{R_B = \frac{Pa}{l}}$ $\boxed{R_A = \frac{Pb}{l}}$

calculation table

Position	AC	CB
origin	A	B
Limit	$0-a$	$0-b$
M	$\frac{Pbx}{l}$	$\frac{Pax}{l}$
Flexural rigidity	EI	EI

Shear energy of the beam

$$U = \int_0^l \frac{M^2}{2EI}dx$$

$$= \int_0^a \left(\frac{Pb}{l}x\right)^2 \frac{1}{2EI}dx + \int_0^b \left(\frac{Pa}{l}x\right)^2 \frac{1}{2EI}dx$$

$$= \frac{P^2b^2}{2EIl^2}\int_0^a x^2 dx + \frac{P^2a^2}{2EIl^2}\int_0^b x^2 dx$$

$$= \frac{P^2b^2}{2EIl^2}\left[\frac{x^3}{3}\right]_0^a + \frac{P^2a^2}{2EIl^2}\left[\frac{x^3}{3}\right]_0^b$$

$$= \frac{P^2b^2a^3}{6EIl^2} + \frac{P^2a^2b^3}{6EIl^2}$$

$$= \frac{P^2a^2b^2}{6EIl^2}\left(\underbrace{a+b}_{l}\right)$$

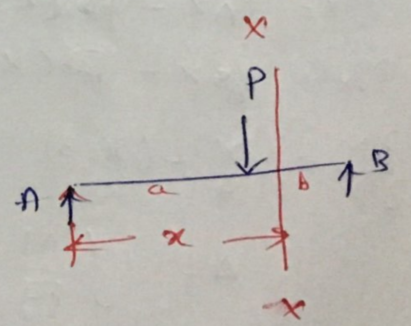
$$\therefore U = \frac{P^2a^2b^2}{6EIl^2}(l)$$

$$\boxed{U = \frac{P^2a^2b^2}{6EIl}}$$

$$\frac{\partial u}{\partial P} = \frac{2Pa^2b^2}{6EIl}$$

$$\Delta_c = \frac{\partial u}{\partial P} = \frac{Pa^2b^2}{3EIl} \qquad \therefore \boxed{\Delta_c = \frac{Pa^2b^2}{3EIl}}$$

Method 2



$$M_x = R_A x - P(x-a)$$

$$M_x = \frac{Pb}{l}x - P(x-a)$$

$$\frac{\partial M}{\partial P} = \frac{bx}{l} - (x-a)$$

$$\Delta = \int \frac{M}{EI}\cdot\frac{\partial M}{\partial P}\,dx.$$

$$\Delta = \int \frac{M}{EI}\frac{\partial M}{\partial P}dx$$

$$= \frac{1}{EI}\int_0^a \frac{Pbx}{l}\left(\frac{b}{l}x\right)dx + \frac{1}{EI}\int_a^l \left\{\frac{Pb}{l}x - P(x-a)\right\}\left\{\frac{bx}{l}\ldots\right.$$

$$= \frac{Pb^2}{EIl^2}\int_0^a x^2\,dx + \frac{P}{EI}\int_a^l \left[\left\{\frac{bx}{l} - (x-a)\right\}\right]^2 dx$$

Note

$$\left\{\frac{Pb}{l}x - Px + Pa\right\}\left\{\frac{bx}{l} - x + a\right\}$$

$$\left\{\frac{Pbx - Pxl + Pal}{l}\right\}\left\{\frac{bx - xl + al}{l}\right\}$$

$l = a+b$ $\quad a^2+ab-ax$

$bx - x(a+b)$

$\cancel{bx} - ax - \cancel{bx} + a(a+b)$

a^2+ab

a^2+ab

$$= \frac{Pb^2}{EIl^2}\left[\frac{x^3}{3}\right]_0^a + \frac{P}{EI}\int_a^l \left(\frac{bx - xl + al}{l}\right)^2 dx$$

$$= \frac{Pb^2a^3}{3EIl^2} + \frac{P}{EIl^2}\int_a^l (a^2+ax+ab)^2\,dx$$

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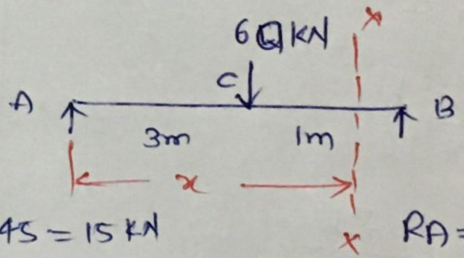
2) Using castigliano's theorem, obtain the deflection under a single concentrated load applied to a SSB, shown in the fig; $EI = 2.2\,MNm^2$

Sol Support reactions



$R_B \times 4 - 60 \times 3 = 0 \Rightarrow R_B = \frac{180}{4} = 45\,kN$

$R_A + R_B = 60$

$\Rightarrow R_A = 60 - 45 = 15\,kN$ $\qquad R_A = \frac{P}{4}$

$M_x = R_A x - P(x-3)$

$M_x = R_A x - P(x-3)$ $\qquad M_x = \frac{P}{4}x - P(x-3)$

$$\frac{\partial M}{\partial P} = \frac{x}{4} - (x-3)$$

$$\Delta = \int \frac{M}{EI}\frac{\partial M}{\partial P}dx$$

$$= \frac{1}{EI}\int_0^3 \frac{Px}{4}\left(\frac{x}{4}dx\right) + \frac{1}{EI}\int_3^4 \left\{\frac{Px}{4} - P(x-3)\right\}\times\left\{\frac{x}{4} - (x-3)\right\}dx$$

$$= \frac{P}{16EI}\int_0^3 x^2\,dx + \frac{P}{EI}\int_3^4 \left\{\left(\frac{x}{4} - (x-3)\right)\right\}^2 dx$$

$$\Delta = \frac{P}{16EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{P}{EI}\int_3^4 \left(x - 4x + 12\right)^2 dx$$

$$= \frac{3^3 P}{3(16EI)} + \frac{P}{EI}\int_3^4 (-3x+12)^2 dx$$

$$= \frac{9P}{16EI} + \frac{P}{EI}\int_3^4 (9x^2 + 144 - 72x)\, dx$$

$$= \frac{9P}{16EI} + \frac{9P}{16EI}\int_3^4 (x^2 - 8x + 16)\, dx$$

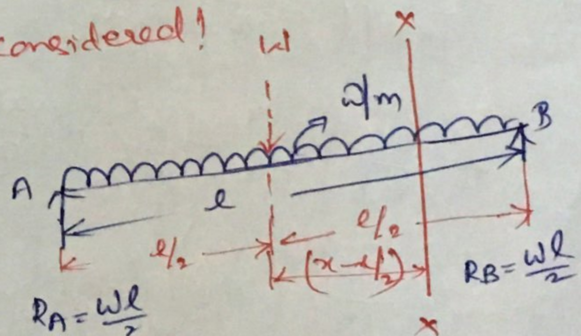
$$= \frac{9P}{16EI} + \frac{9P}{16EI}\left[\frac{x^3}{3} - 8\frac{x^2}{2} + 16x\right]_3^4$$

$$= \frac{9W}{16EI}(1+0.33) = \frac{0.75W}{EI}$$

$$\Delta = \frac{0.75(60\times10^3)}{2.2\times10^6} = 0.02\text{m (or) } 20.45\text{mm}$$

Q) Using castigliano's theorem, calculate the vertical deflection @ the middle of a SSB which carries a udl of intensity ω over the full span. EI of the beam is constant and only strain energy of bending is to be considered!

Sol Consider a dummy load 'W' acting @ the centre of the beam.



$$R_A = \frac{\omega l}{2} + \frac{W}{2}$$

$$M_x = R_A x - W\left(x - \frac{l}{2}\right) - \frac{\omega x^2}{2}$$

$$= \left(\frac{\omega l}{2} + \frac{W}{2}\right)x - \frac{\omega x^2}{2} - W\left(x - \frac{l}{2}\right)$$

$$M_x = \frac{\omega l x}{2} + \frac{Wx}{2} - \frac{\omega x^2}{2} - W\left(x - \frac{l}{2}\right)$$

For, $x \le l/2$

$$M_x = \frac{\omega l x}{2} + \frac{Wx}{2} - \frac{\omega x^2}{2}$$

$$\frac{\partial M_x}{\partial W} = \frac{x}{2}.$$

Scanned with CamScanner

For $\frac{l}{2} \le x \le l$,

$$\frac{\partial M_x}{\partial W} = \frac{x}{2} - x + \frac{l}{2} = \frac{l}{2} - \frac{x}{2} = \left(\frac{l-x}{2}\right)$$

$$\Delta = \int \frac{M}{EI}\frac{\partial M}{\partial W}$$

$$= \frac{1}{EI}\int_0^{l/2}\left[\frac{wlx}{2} + \frac{Wx}{2} - \frac{wx^2}{2}\right]\frac{x}{2}\,dx + \frac{1}{EI}\int_{l/2}^{l}\left[\frac{wlx}{2} + \frac{Wx}{2} - \frac{wx^2}{2} - W\left(x - \frac{l}{2}\right)\right]\left[\frac{l-x}{2}\right]$$

Putting $W = 0$; we get

$$\Delta = \frac{1}{EI}\int_0^{l/2}\left[\frac{wlx^2}{4} - \frac{wx^3}{4}\right]dx + \frac{1}{EI}\int_{l/2}^{l}\left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(\frac{l-x}{2}\right)dx$$

$$= \frac{w}{4EI}\int_0^{l/2}(lx^2 - x^3)\,dx + \frac{w}{4EI}\int_{l/2}^{l}(l^2x - 2lx^2 + x^3)\,dx$$

$$= \frac{w}{4EI}\left[\frac{lx^3}{3} - \frac{x^4}{4}\right]_0^{l/2} + \frac{w}{4EI}\left[\frac{l^2x^2}{2} - 2l\frac{x^3}{3} + \frac{x^4}{4}\right]_{l/2}^{l}$$

$$= \frac{w}{4EI}\left[\left(\frac{l\left(\frac{l}{2}\right)^3}{3} - \frac{\left(\frac{l}{2}\right)^4}{4}\right) + \frac{l^2}{2}\left(l^2 - \frac{l^2}{4}\right) - \frac{2l}{3}\left\{l^3 - \left(\frac{l}{2}\right)^3\right\} + \frac{1}{4}\left\{l^4 - \left(\frac{l}{2}\right)^4\right\}\right]$$

$$= \frac{w}{4EI}\left[\frac{l^4}{24} - \frac{l^4}{64} + \frac{3l^4}{8} - \frac{7l^4}{12} + \frac{15l^4}{64}\right]$$

$$= \frac{wl^4}{4EI}\left(\frac{10}{192}\right)$$

$$\boxed{\Delta = \frac{5wl^4}{384EI}}$$

4) A beam simply supported over a span 3m carries a udl of 20 kN/m over the entire span. Taking $EI = 2.25\,MNm^2$ and using Castigliano's theorem determine the deflection @ the centre of the beam?

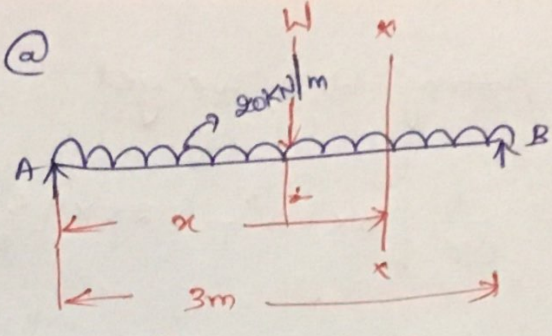
Sol: Given $l = 3m$

$w = 20\,kN/m$

$EI = 2.25\,MNm^2$

(17)

Let W(KN) be the dummy load @ mid span.



$$\Sigma M_B = 0 \Rightarrow R_A \times 3 - 20 \times 3 \times \frac{3}{2} - W(1.5)$$

$$R_A = (30 + 0.5W)\ kN$$

$$M_x = (30+0.5W)x - 20\frac{x^2}{2} - W(x-1.5)$$

$$\frac{\partial M_x}{\partial W} = 0.5x - (x-1.5)$$

$$\Delta = \int \frac{M}{EI}\frac{\partial M}{\partial W}dx$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(30+0.5W)x - 10x^2\right\}(0.5x)dx + \frac{1}{EI}\int_{1.5}^{3}\left\{(30+0.5W)x - 10x^2 - W(x-1.5)\right\}\left\{0.5x-(x-1.5)\right\}dx$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(15+0.25W)x^2 - 5x^3\right\}dx + \frac{1}{EI}\int_{1.5}^{3}\left\{(30x+0.5Wx-10x^2 - Wx+1.5W)(0.5x-x+1.5)dx\right\}$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(15+0.25W)x^2 - 5x^3\right\}dx + \frac{1}{EI}\int_{1.5}^{3}\left\{(30x-0.5W)x-10x^2+1.5W\right\}\left\{1.5-0.5x\right\}dx$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(15+0.25W)x^2 - 5x^3\right\}dx + \frac{1}{EI}\int_{1.5}^{3}(45-0.75W)x - 15x^2 + 2.25W - (15-0.25W)x^2 + 5x^3 - 0.75Wx\}dx$$

$$= \frac{1}{EI}\int_0^{1.5}\left\{(15+0.25W)x^2 - 5x^3\right\}dx + \frac{1}{EI}\int_{1.5}^{3}\left\{(45-1.5W)x + (0.25W-30)x^2 + 5x^3 + 2.25W\right\}dx$$

$$= \frac{1}{EI}\left[(15+0.25W)\frac{x^3}{3} - \frac{5x^4}{4}\right]_0^{1.5} + \frac{1}{EI}\left[(45-1.5W)\frac{x^2}{2} + (0.25W-30)\frac{x^3}{3} + \frac{5x^4}{4} + 2.25Wx\right]_{1.5}^{3}$$

Putting W=0, We get

$$A = \frac{1}{EI}\left[5\times1.5^3 - 1.25\times1.5^4\right] + \frac{1}{EI}\left[22.5\times6.75 - 10\times3.6 \ldots - 1.25\times7.594\right]$$

$$= \frac{21.09}{EI}$$

$$= \frac{21.09\times10^3}{2.25\times10^6}$$

$$= 9.37\times10^{-3}\,m$$

$$\boxed{\Delta = 9.37\,mm}$$

MAXWELL'S THEOREM (RECIPROCAL DEFLECTION)

→ Clerk Maxwell

1) **Statement**: Displacement at point A due to the load at point B is Same as displacement of point B due to the same load acting at Point A, the displacement is measured in the direction of loads.

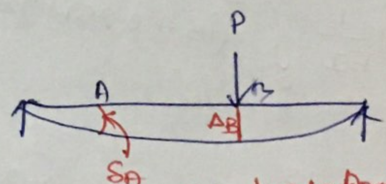
∴ It is valid for Linear as well as rotational displacement

δ_B = Displacement @ B due to load P @ A

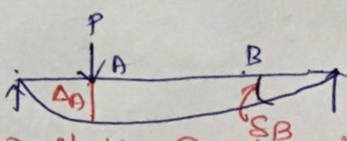
δ_A = " " A " P @ B

∴ According to this theorem,

$$\delta_A = \delta_B$$



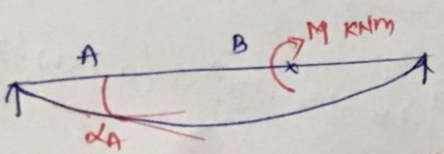
Deflection @ A due to load @ B



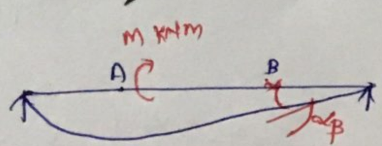
Deflection @ B due to load @ A

Let α_A = Rotation @ A due to moment M @ B

α_B = " " B " " " @ A



Rotation @ A due to moment @ B



Rotation @ B due to moment @ A

Pf:

When the load P is acting @ B, Let the displacement @ A be δ_A, & the displacement @ B be ΔB,

$$\text{Workdone} = \frac{1}{2} P \Delta_B \quad \text{(1)}$$

lly when the load 'P' acting @ A, Let the deflection @ A be Δ_A and deflection @ B be δ_B.

$$\text{Work done} = \frac{1}{2} P \Delta_A \quad \text{(2)}$$

Now Imagine,

If the load P is applied first at B and then at A, then

$$\text{External workdone} = \frac{1}{2} P \Delta B + P \delta_B + \frac{1}{2} P \Delta_A \quad \text{(3)}$$

If the load 'P' is applied first at A and then at B, then

$$\text{External work done} = \frac{1}{2} P \Delta_A + P \delta_A + \frac{1}{2} P \Delta_B \quad \text{(4)}$$

Eq. (3) & (4) workdone when 'P' acting @ both points A and (B)

$$\therefore \frac{1}{2} P \Delta_B + P \delta_B + \frac{1}{2} P \Delta_A = \frac{1}{2} P \Delta_A + P \delta_A + \frac{1}{2} P \Delta_B$$

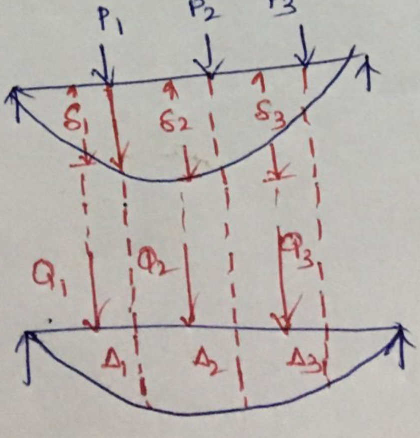
$$\therefore \boxed{\delta_B = \delta_A}$$

Betti's Theorem ~~In any material~~

Statement: The virtual work done by P system of loads is equal to the virtual work done by Q system of loads.

$$\therefore P_1 \Delta_1 + P_2 \Delta_2 = Q_1 \delta_1 + Q_2 \delta_2$$

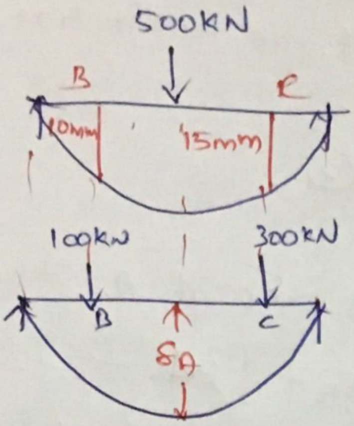
Proof: Consider a beam which is elastic and obey's Hook's law.



Virtual workdone by a system of forces $P_1, P_2, P_3 \ldots$ during the displacements caused by system of forces Q_1, Q_2, Q_3 is equal to workdone by system of forces $Q_1 Q_2 Q_3 \ldots$ during the displacements caused by forces $P_1, P_2, P_3 \ldots$

Plo 1) A load 500KN applied @ A, as shown in fig. produces a res deflection @ B and C of the beam as $\Delta b = 10mm$; $\Delta C = 15mm$. What deflection @ A when loads of 100KN and 300KN applied @ B&C?

Sol



$100(10) + 300(15) = \delta_A(500)$

$$\boxed{\delta_A = 11mm}$$

Deflection of Pin Jointed Trusses