UNIT I
DEFLECTIONS

Introduction

An initially straight beam deflects, when loaded, and its axis bends in a curve which is known as *elastic curve* or *deflection curve*. While designing a beam, the designer is not only concerned with stresses produced by the loads acting on the beam but also the deflection of the beam resulting from the loading. *Deflection of a point on the axis of the beam is the distance between its positions before and after the loading. Slope at any section in a deflected beam is defined as the angle in radians which the tangent at the section makes with the original axis of the beam.* From aesthetic and other considerations, deflection of a beam under the imposed loads is restricted to a certain ratio of the span. The ratio of maximum deflection of a beam to its span is called the *stiffness* of the beam.

7.1 RELATIONSHIP BETWEEN CURVATURE, DEFLECTION AND SLOPE

Due to imposed loads, let the beam *AB* bend to the curve* *A′PQB′* (Fig. 7.1). Take *XX′* and *YY′* as axes of reference.

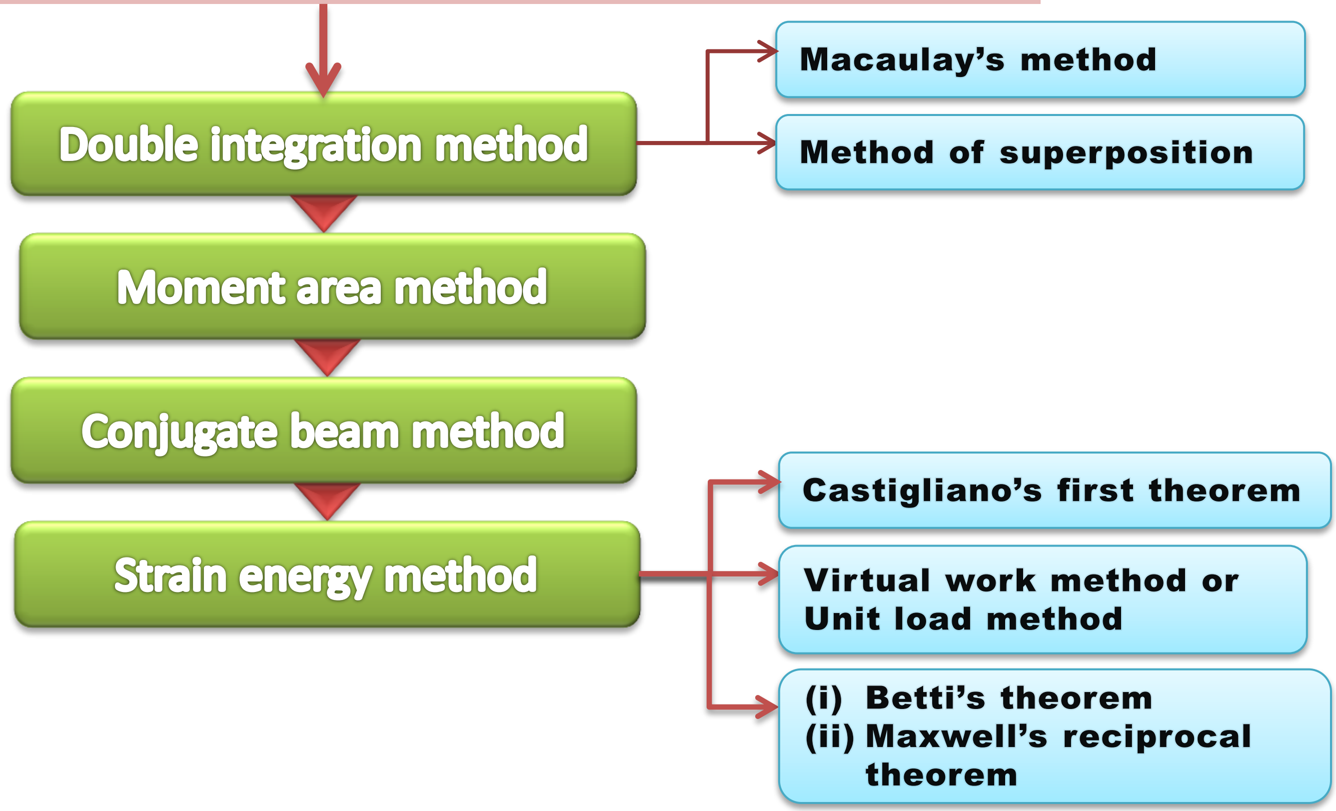
From the relation $$\frac{E}{R} = \frac{M}{I}$$

We have $$\frac{1}{R} = \frac{M}{EI} \qquad \text{...}(i)$$

* It was shown in Chapter 6 that a straight beam of uniform cross-section, when subjected to an end couple *M* applied about a perpendicular axis, bends into a circular arc of radius *R* given by the relation $\frac{M}{EI} = \frac{1}{R}$ where *EI* is the flexural rigidity of the beam. This equation holds good for elastic bending.

The axis of the beam, however, no longer bends into a circular arc when it bends due to combined effect of S.F. and B.M. We still assume that the above equation holds good at any point of the beam where the B.M. is *M* and with change of *M*, the radius of curvature changes from section to section.

Methods for calculating slope and deflection



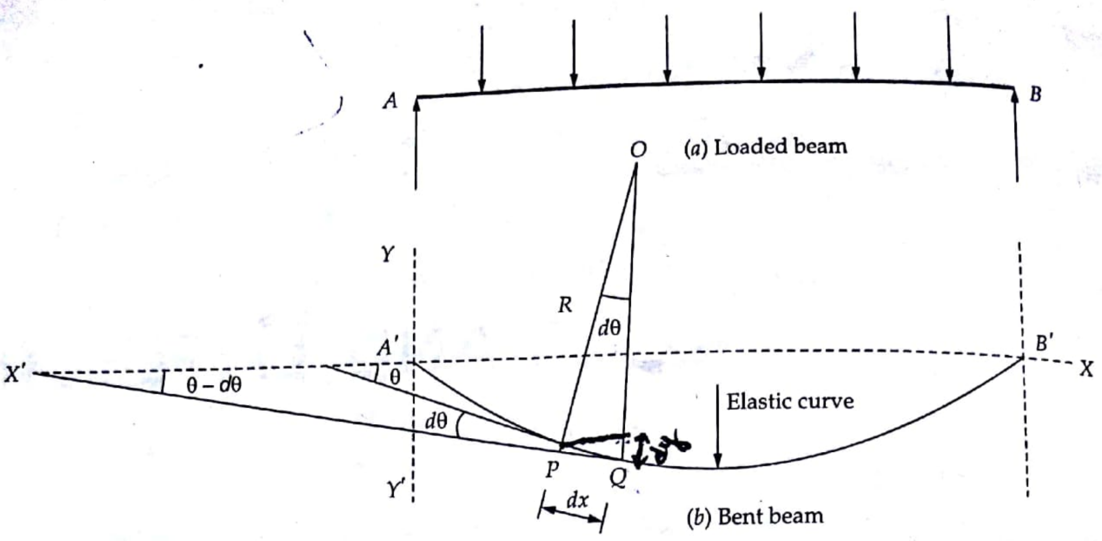


Fig. 7.1

The product EI is known as flexural rigidity; R is the radius of curvature and M the bending moment causing deflection of the beam. Consider a small element PQ on the elastic curve.

Let angle made by tangent at P with X-axis = θ

Angle between normals to the curve at P and $Q = d\theta$

The point of intersection O of the normals to the elastic curve at P and Q is the centre of curvature and the length OP of OQ is the radius of curvature R.

Now $$PQ = R\,d\theta \qquad \left(\because \theta = \frac{1}{r}\right)$$

or $$\frac{1}{R} = \frac{d\theta}{PQ} = \frac{d\theta}{dx} \quad (\because \textit{for very small deflection, arc } PQ = dx)$$

Slope of $P = \theta$ and at $Q = (\theta - d\theta)$, *i.e.*, the slope decreases with increase of dx therefore $\frac{d\theta}{dx}$ is $-ve$.

Hence $$\frac{1}{R} = -\frac{d\theta}{dx}$$

But $$\frac{dy}{dx} = \tan\theta \approx \theta \qquad (\textit{since } \theta \textit{ is very small})$$

∴ $$\frac{d^2y}{dx^2} = \frac{d\theta}{dx}$$

or $$= -\frac{1}{R} = -\frac{M}{EI}$$

∴ $$EI\frac{d^2y}{dx^2} = -M$$

or $$EI\frac{dy}{dx} = -\int M dx + C_1$$

and $$EI\,y = -\iint (M dx)\,dx + C_1x + C_2$$

where C_1 and C_2 are constants of integration.

Since $\frac{d\theta}{dx}$ is negative, so $\frac{d^2y}{dx^2}$ is also negative. The B.M. causing deflection is +ve.

In Fig. 7.2, however, the B.M. is negative and the slope at Q is more than at P. At P it is θ whereas at Q it is $(\theta + d\theta)$, and therefore here $\frac{d\theta}{dx}$ is positive and so is $\frac{d^2y}{dx^2}$.

In both cases, arc $PQ = Rd\theta$

Since deflection is small, the arc is very flat and therefore arc $PQ = dx$.

$$Rd\theta = dx$$

$\therefore$

$$\frac{d\theta}{dx} = \frac{1}{R}$$

or

$$\frac{1}{R} = \frac{d^2y}{dx^2} = \frac{-M}{EI}$$

or

(*from the equation* (*i*), *M is negative for a cantilever*)

$\therefore$
$$EI\frac{d^2y}{dx^2} = -M$$

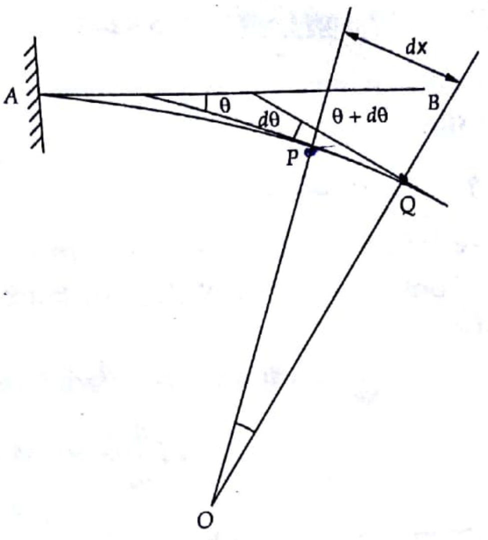


Fig. 7.2

When M is $+ve$ (as in the case of beams) $\frac{d^2y}{dx^2}$ is negative and when M is $-ve$ (as in case of cantilevers) $\frac{d^2y}{dx^2}$ is $+ve$. The general equation for deflection is

$\therefore$
$$EI\frac{d^2y}{dx^2} = -M \quad \text{... (7.1)}$$

By integrating it once, we get $\frac{dy}{dx}$, the slope, and by integrating it twice we get y, the *deflection*. The above equation is known as *differential equation of flexure*.

This method of determining slopes and deflections is called the *double integration method* or *Macaulay's method*.

7.2 SIGN CONVENTIONS

(1) When measured along the beam from left to right, x is taken as positive.

(2) Deflection y is positive downwards.

(3) Bending moment M is positive when sagging.

(4) Slope θ is positive if while going from left to right along the beam, the tangent to the elastic curve is inclined downwards.

Distance x; deflection y; bending moment M and slope θ_A are positive for the beam shown in Fig. 7.3, whereas θ_B is $-ve$.

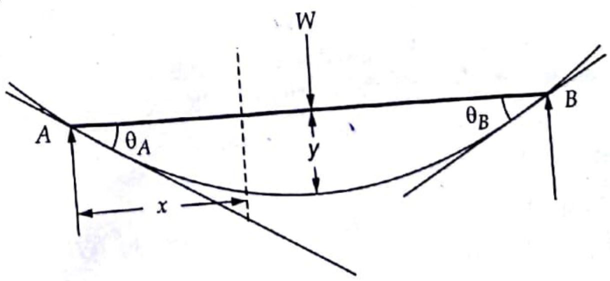


Fig. 7.3

7.3 STANDARD CASES

Deflection in case of few standard cases is determined below by using the differential equation of flexure.

7.3.1 Cantilevers

Case 1: *Concentrated load W at the free end:* (Fig. 7.4)

Consider a section XX' at a distance x from the fixed end A.

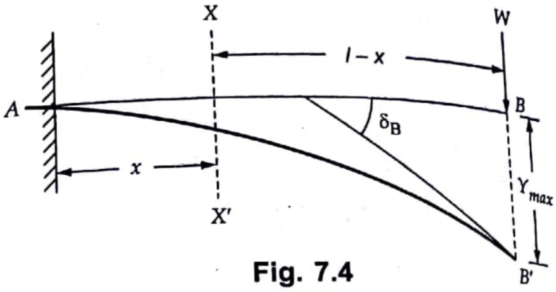


Fig. 7.4

$$M_x = -W(l - x)$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -M = W(l - x)$$

or $$= Wl - Wx$$

On integrating, we have

$$EI\frac{dy}{dx} = Wlx - \frac{Wx^2}{2} + C_1,$$ where C_1 is the constant of integration.

At A, when x is zero, the slope dy/dx is zero, therefore $C_1 = 0$

Hence $$EI\frac{dy}{dx} = Wlx - \frac{Wx^2}{2} \quad ...(i)$$

For slope at B, put $x = l$

$$\therefore \quad \theta_B = \frac{dy}{dx} = \frac{1}{EI}\left(Wl \times l - \frac{Wl^2}{2}\right) = \frac{Wl^2}{2EI} \quad ...(7.2)$$

For deflection, integrate the equation (*i*) above.

$$EI\,y = \frac{Wlx^2}{2} - \frac{Wx^3}{6} + C_2$$

where C_2 is the constant of integration.

Deflection at A is zero and thus $y = 0$ when $x = 0$

$$\therefore \quad C_2 = 0$$

Hence $$EI\,y = \frac{Wlx^2}{2} - \frac{Wx^3}{6} \quad ...(ii)$$

For deflection at B, put $x = l$

$$\therefore \quad y_B = \frac{1}{EI}\left(\frac{Wl \times l^2}{2} - \frac{Wl^3}{6}\right) = \frac{Wl^3}{3EI} \quad ...(7.3)$$

Equations (*i*) and (*ii*) give slope and deflection respectively at any section, whereas equations (7.1) and (7.3) give the maximum value of the slope and deflection at the free end.

Case 2: *Carrying U.D.L at the rate of w/unit length over the entire span:* (Fig. 7.5)

Consider a section XX' at a distance x from the fixed end A.

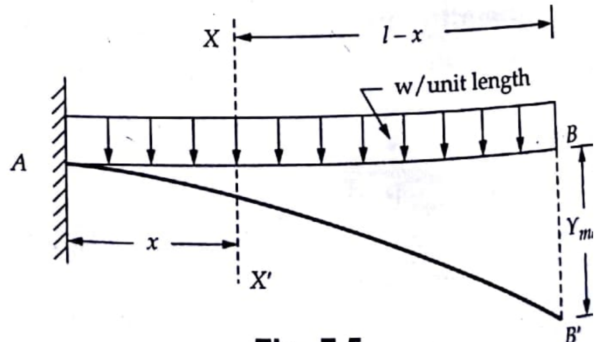


Fig. 7.5

$$M_x = \frac{-w(l-x)^2}{2}$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -M = \frac{w(l-x)^2}{2} = \frac{w}{2}(l^2 - 2lx + x^2)$$

On integrating both sides, we have

$$EI\frac{dy}{dx} = \frac{w}{2}\left[l^2x - \frac{2lx^2}{2} + \frac{x^3}{3}\right] + C_1$$

where C_1 is the constant of integration. At A, the slope is zero therefore on putting $\frac{dy}{dx} = 0$ when $x = 0$, we have $C_1 = 0$

$$EI\frac{dy}{dx} = \frac{w}{2}\left(l^2x - lx^2 + \frac{x^3}{3}\right) \quad ...(i)$$

For slope at B, put $x = l$

$$\therefore \quad \theta_B = \frac{dy}{dx} = \frac{w}{2EI}\left(l^2 \times l - l \times l^2 + \frac{l^3}{3}\right)$$

or $$= \frac{wl^3}{6EI} = \frac{Wl^2}{6EI} \quad (\text{where } W = wl) \quad ...(7.4)$$

Integrating the equation (*i*) above for deflection, we have

$$EI\,y = \frac{w}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) + C_2$$

where C_2 is the constant of integration. Deflection y at A is zero.

Thus $y = 0$ when $x = 0$; therefore $C_2 = 0$

Hence $$EI\,y = \frac{w}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right) \quad ...(ii)$$

For deflection at B, put $x = l$

$$\therefore \quad y_B = \frac{w}{2EI}\left(\frac{l^2 \times l^2}{2} - \frac{l \times l^3}{3} + \frac{l^4}{12}\right)$$

or $$= \frac{wl^4}{8EI} = \frac{Wl^3}{8EI} \quad (\text{where } W = wl) \quad ...(7.5)$$

Thus, equations (*i*) and (*ii*) give slope and deflection at any section whereas equations (4) and (5) give slope and deflection at B which are the maximum.

Case 3: *A point load W, not at the free end:* (Fig. 7.6)

Consider a section XX' at a distance x from the fixed end A.

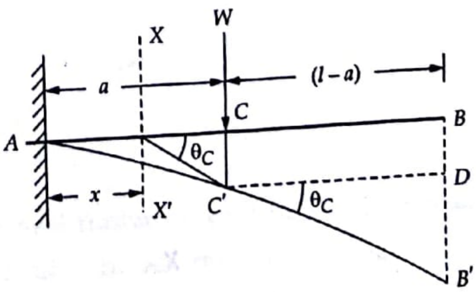


Fig. 7.6

$$M_x = -W(a - x)$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -M$$

or $$= W(a - x)$$

On integrating for slope, we have

$$EI\frac{dy}{dx} = Wax - \frac{Wx^2}{2} + C_1$$

where C_1 is a constant of integration.

But at A where $x = 0$; $\frac{dy}{dx}$ is zero, therefore $C_1 = 0$

Hence $$EI\frac{dy}{dx} = Wax - \frac{Wx^2}{2} \quad ...(i)$$

For slope at C, put $x = a_1$

$$\therefore \quad \theta_C = \frac{dy}{dx} = \frac{1}{EI}\left(Wa \times a - \frac{Wa^2}{2}\right) = \frac{Wa^2}{2EI} \quad ...(ii)$$

As there is no load on the portion BC, there will be no B.M. in that portion and the portion BC will not bend. It shall be straight.

$$\therefore \quad \theta_B = \theta_C = \frac{Wa^2}{2EI} \quad ...(7.6)$$

For deflection at C, integrate the equation (i) again.

$$EI\,y = \frac{Wax^2}{2} - \frac{Wx^3}{6} + C_2$$

where C_2 is the constant of integration.

At A where $x = 0$, y is zero; therefore $C_2 = 0$

Hence $$EIy = \frac{Wax^2}{2} - \frac{Wx^3}{6} \quad ...(iii)$$

For deflection at C, put $x = a$

$$\therefore \quad y_c = \frac{1}{EI}\left(Wa \times \frac{a^2}{2} - \frac{Wa^3}{6}\right) = \frac{Wa^3}{3EI} \quad ...(7.7)$$

But $$y_c = BD \quad \text{(Fig. 7.6)}$$

and $$B'D = DC' \tan\theta_C = BC \tan\theta_C = BC \times \theta_C$$

($\because$ θ_C is small $\therefore$ $\tan\theta_C = \theta_C$)

$$\therefore \quad B'D = (l - a) \times \frac{Wa^2}{2EI} \quad \left(\because \theta_C = \frac{Wa^2}{3EI}\right)$$

But $$y_B = BB' = BD + B'D$$

or $$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI} \times (l - a)$$

or $$= \frac{Wa^2}{6EI}(3l - a) \quad ...(7.8)$$

Case 4: *U.D.L. at the rate w/unit length on a part of span from the fixed end:*

Consider a section XX' at a distance x from the fixed end A [Fig. 7.7].

$$M_x = \frac{-w(a - x)^2}{2}$$

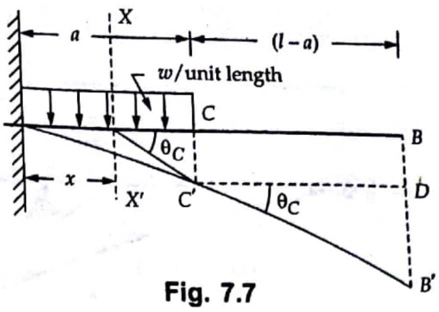


Fig. 7.7

But $$EI\frac{d^2y}{dx^2} = -M = \frac{w(a-x)^2}{2}$$

or $$= \frac{w}{2}(a^2 - 2ax + x^2)$$

On integrating for slope, we have

$$EI\frac{dy}{dx} = \frac{w}{2}\left(a^2x - \frac{2ax^2}{2} + \frac{x^3}{3}\right) + C_1$$

where C_1 is a constant of integration. At A, the slope is zero, *i.e.*,

$$\frac{dy}{dx} = 0 \text{ when } x = 0 \text{ and therefore } C_1 = 0$$

Hence $$EI\frac{dy}{dx} = \frac{w}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) \quad ...(i)$$

For slope at C, put $x = a$

$$\therefore \quad \theta_C = \frac{dy}{dx} = \frac{w}{2EI}\left(a^2 \times a - a \times a^2 + \frac{a^3}{3}\right) = \frac{wa^3}{6EI} \quad ...(ii)$$

Since the portion BC is not loaded, it does not bend and remains straight, therefore

$$\theta_B = \theta_C = \frac{wa^3}{6EI} = \frac{Wa^2}{6EI} \quad (\text{where } W = wa) \quad ...(7.9)$$

On integrating the equation (*i*) for deflection:

$$EIy = \frac{w}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_2$$

where C_2 is a constant of integration. At A, deflection is zero, *i.e.*, $y = 0$ when $x = 0$ and therefore $C_2 = 0$

Hence $$EIy = \frac{w}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right)$$

For deflection at C, put $x = a$

$$\therefore \quad y_c = \frac{wa^4}{8EI} = \frac{Wa^3}{8EI} \quad \text{where } W = wa$$

$$\therefore \quad CC' = BD = \frac{Wa^3}{8EI}$$

But $$B'D = C'D \tan\theta_C = BC \tan\theta_C = BC \times \theta_C \quad (\because \tan\theta \approx \theta \text{ when } \theta \text{ is small})$$

or $$= (l - a) \times \frac{Wa^2}{6EI} \quad \left(\because \theta_C = \frac{Wa^2}{6EI}\right)$$

$$y_B = BD + B'D = \frac{Wa^3}{8EI} + \frac{Wa^2}{6EI}(l - a) \quad ...(7.10)$$

or $$= \frac{Wa^2}{24EI}(4l - a)$$

Case 5: *U.D.L at the rate w/unit length on a part of span from the free end:* [Fig. 7.8(*a*)]

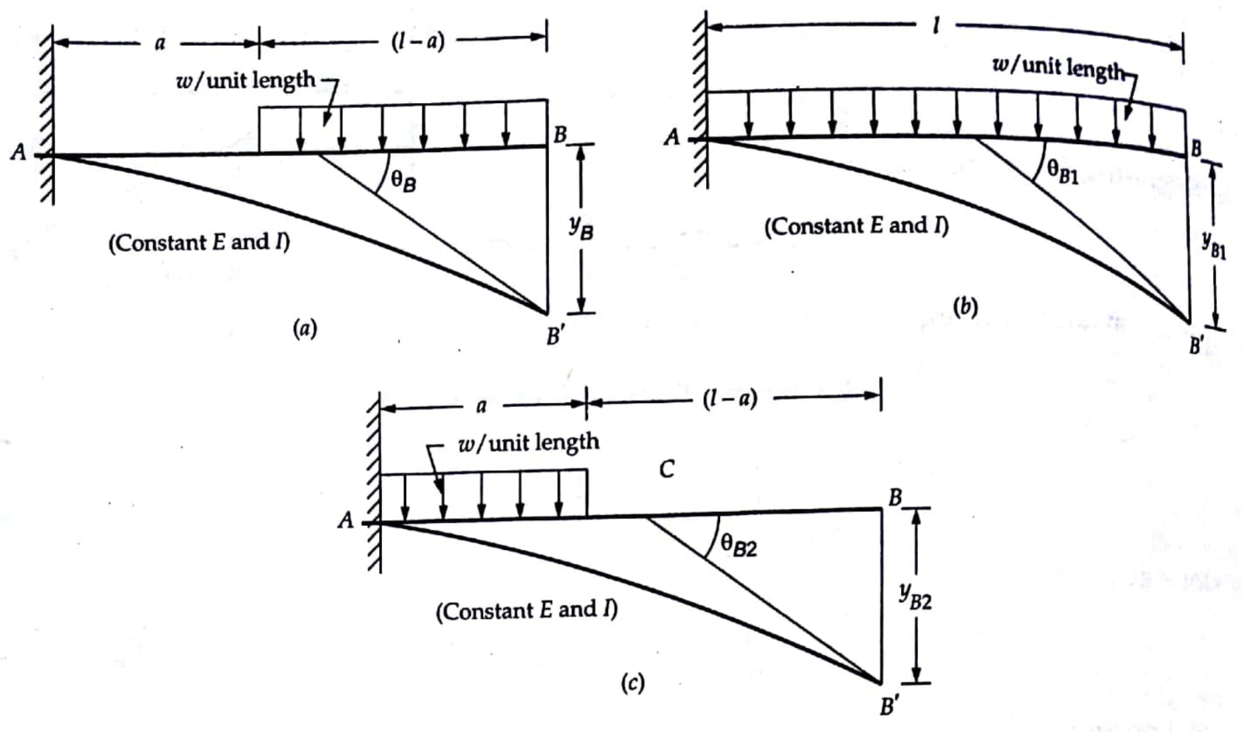


Fig. 7.8

From the figures it is obvious that to get the result in Case (*a*), take the differences of results in Case (*b*) and Case (*c*) thus:

$$\theta_B = \theta_{B_1} - \theta_{B_2}$$

and $$y_B = y_{B_1} - y_{B_2}$$

But from previous articles, we have

$$\theta_{B_1} = \frac{wl^3}{6EI}; \qquad y_{B_1} = \frac{wl^4}{8EI}$$

$$\theta_{B_2} = \frac{wa^3}{6EI}; \qquad y_{B_2} = \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l - a)$$

$\therefore$ $$\theta_B = \theta_{B_1} - \theta_{B_2} = \frac{wl^3}{6EI} - \frac{wa^3}{6EI} = \frac{w}{6EI}(l^3 - a^3) \qquad \text{...(7.}$$

$$y_B = y_{B_1} - y_{B_2}$$

or $$= \frac{wl^4}{8EI} - \left[\frac{wa^4}{8EI} + \frac{wa^3(l - a)}{6EI}\right]$$

or $$= \frac{w}{8EI}(l^4 - a^4) - \frac{wa^3}{6EI}(l - a)$$

or $$= \frac{w}{24EI}(3l^4 - 4la^3 + a^4) \qquad \text{...(7.}$$

Case 6: *A moment applied at the free end:* (Fig. 7.9)

Consider a section XX' at a distance x from the fixed end.

$$M_x = -M$$

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$$EI\frac{d^2y}{dx^2} = -M_x = M$$

Integrating for slope, we have

$$EI\frac{dy}{dx} = Mx + C_1$$

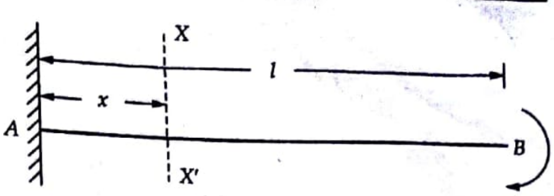


Fig. 7.9

where C_1 is the constant of integration. At A, the slope is zero, *i.e.*,

$$\frac{dy}{dx} = 0 \text{ when } x = 0, \text{ therefore } C_1 = 0$$

Hence $$EI\frac{dy}{dx} = Mx \qquad ...(i)$$

At $x = l$;

$$\theta_B = \frac{dy}{dx} = \frac{Ml}{EI} \qquad ...(7.13)$$

For deflection, on integrating equation (*i*), we have

$$EIy = \frac{Mx^2}{2} + C_2$$

At A, deflection is zero, *i.e.*, $y = 0$ when $x = 0$; therefore $C_2 = 0$

Hence $$EIy = \frac{Mx^2}{2} \qquad ...(ii)$$

Deflection at y when $x = l$ is

$$y_B = \frac{Ml^2}{2EI} \qquad ...(7.14)$$

7.3.2 Simply Supported Beams

Case 1: *Point load at mid-span:* (Fig. 7.10)

Consider a section XX' at a distance x from the support A but within the portion AC. By symmetry, the support reactions at A and B are each equal to $\frac{W}{2}$.

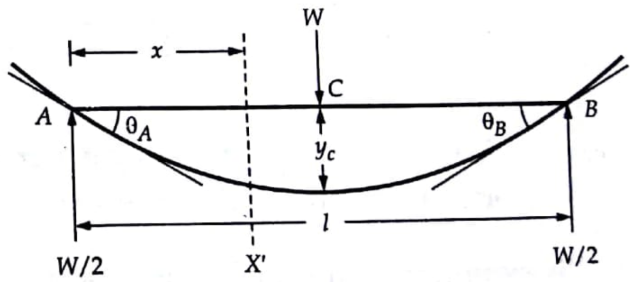


Fig. 7.10

$\therefore$ $$M_x = \frac{W}{2} \times x$$

But $$EI\frac{d^2y}{dx^2} = -M = -\frac{Wx}{2}$$

Integrating the above expression for slope, we have

$$EI\frac{dy}{dx} = -\frac{Wx^2}{4} + C_1$$

where C_1 is a constant of integration. At midspan C, the slope is zero

i.e., $$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2}$$

$$\therefore \quad \frac{-W\left(\frac{l}{2}\right)^2}{4} + C_1 = 0$$

or $$C_1 = \frac{Wl^2}{16}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{Wl^2}{16} \quad ...(i)$$

For slope at A, put $x = 0$

$$\therefore \quad \theta_A = \frac{dy}{dx} = \frac{Wl^2}{16EI} \quad ...(7.15)$$

By symmetry, $\theta_B = -\theta_A = \dfrac{-Wl^2}{16EI}$

For deflection, integrate expression (*i*) above,

$$EI\,y = \frac{-Wx^3}{12} + \frac{Wl^2x}{16} + C_2$$

where C_2 is a constant of integration. At A, the deflection is zero, *i.e.*,

$y = 0$ when $x = 0$; therefore $C_2 = 0$

$$\therefore \quad EI\,y = -\frac{Wx^3}{12} + \frac{Wl^2x}{16} \quad ...(ii)$$

For deflection at C, put $x = \dfrac{l}{2}$

$$\therefore \quad y_c = \frac{1}{EI}\left[\frac{-W\left(\frac{l}{2}\right)^3}{12} + \frac{Wl^2 \times \left(\frac{l}{2}\right)}{16}\right] = \frac{Wl^3}{48EI} \quad ...(7.16)$$

Case 2: *U.D.L. at the rate w/unit length over the whole span:* (Fig. 7.11)

Consider a section at a distance x from the support A and within the portion AC.

By symmetry, support reactions at A and B are each equal to $\dfrac{wl}{2}$.

$$M_x = \frac{wlx}{2} - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = -M = -\left(\frac{wlx}{2} - \frac{wx^2}{2}\right)$$

or $$= \frac{-wlx}{2} + \frac{wx^2}{2}$$

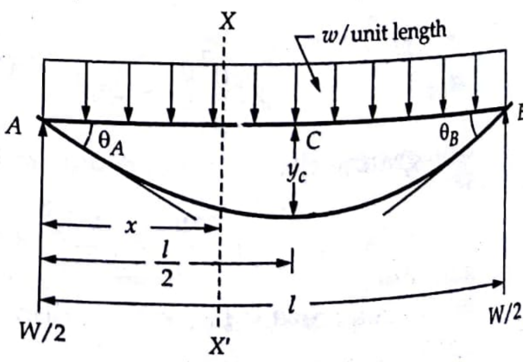


Fig. 7.11

On integrating the above for slope, we have

$$EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + C_1$$

where C_1 is a constant of integration. At midspan C, the slope is zero

That is, $\frac{dy}{dx} = 0$ when $x = \frac{l}{2}$

$$\therefore \quad -\frac{wl}{4} \times \left(\frac{l}{2}\right)^2 + \frac{w}{6} \times \left(\frac{l}{2}\right)^3 + C_1 = 0$$

$$\therefore \quad C_1 = +\frac{wl^3}{24}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{24} \qquad \text{... }(i)$$

For slope at A, put $x = 0$

$$\therefore \quad \theta_A = \frac{wl^3}{24EI} \qquad \text{... (7.17)}$$

By symmetry, $\theta_B = -\theta_A$

$$\text{or} \quad = \frac{-wl^3}{24EI} = \frac{-Wl^2}{24EI} \quad (\text{where } W = wl) \qquad \text{... }(ii)$$

For deflection, integrate expression (i) above

$$EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3x}{24} + C_2$$

where C_2 is the constant of integration. At A, the deflection is zero,

i.e., $y = 0$ at $x = 0$, therefore $C_2 = 0$

$$\text{Hence} \quad EI\,y = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3x}{24} \qquad \text{...}(iii)$$

For deflection at C, put $x = \frac{l}{2}$.

$$\therefore \quad y_C = \frac{1}{EI}\left[-\frac{wl}{12}\times\left(\frac{l}{2}\right)^3 + \frac{w}{24}\left(\frac{l}{2}\right)^4 + \frac{wl^3}{24}\left(\frac{l}{2}\right)\right]$$

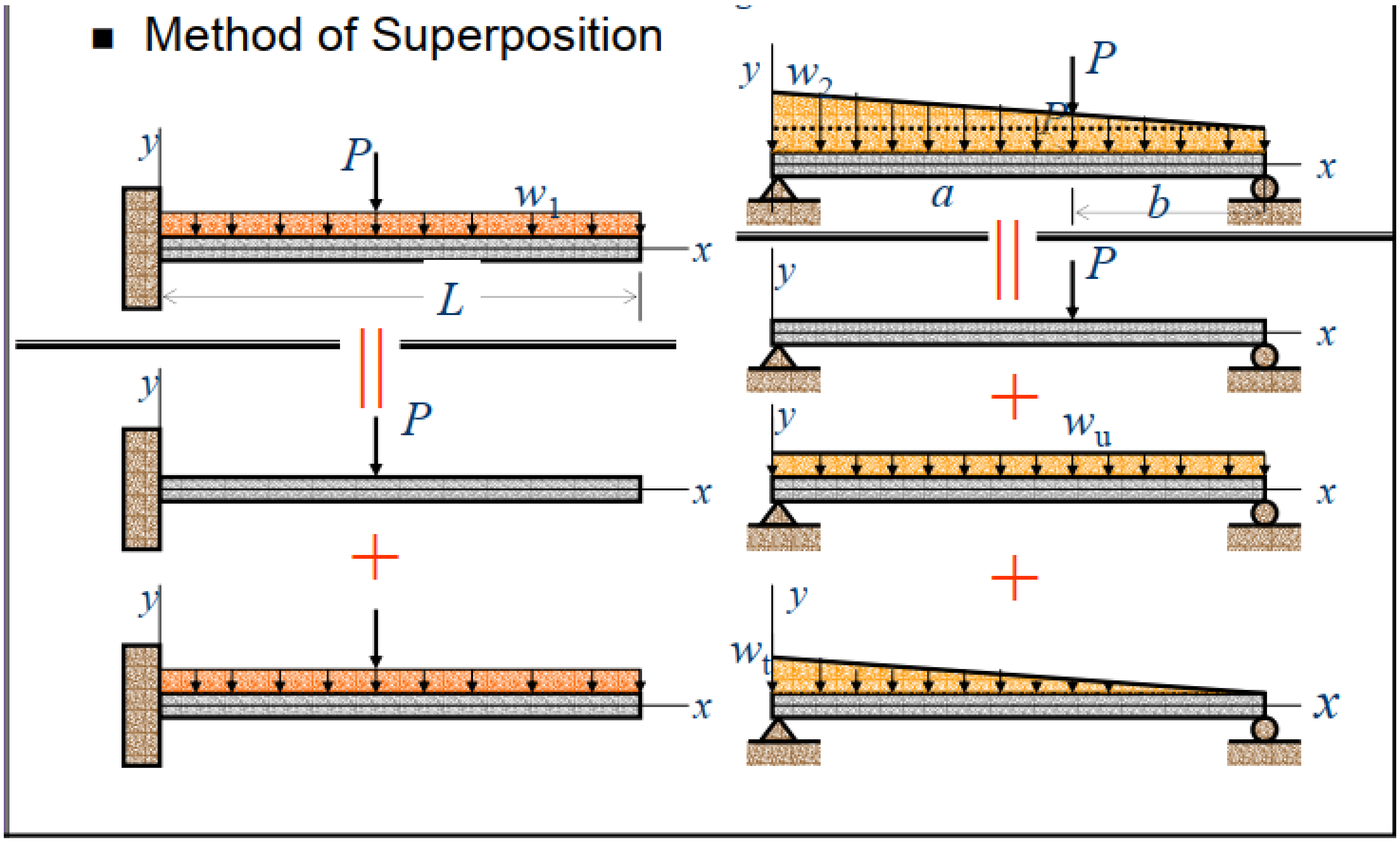
$$\text{or} \quad = \frac{5wl^4}{384EI} \qquad \text{...(7.18)}$$

$$\text{or} \quad = \frac{5Wl^3}{384EI} \quad (\text{where } W = wl)$$

Method of superposition:

- Deflections due to any complex loading on a beam or cantilever can be determined by treating the loading as a combination of simple loading is known as method of superposition.
- The resulting final deflection of a loaded beam is simply the sum of deflections caused by each of the individual loads.
- Slope or deflection at a point is determined as the resultant effect of each one of these loads at that point. However, a limitation on the application of this method in that the effect produced by each load must be independent of that produced by other loads, i.e., each independent load should not cause any appreciable change in original length or shape of the beam.
- This method is advantageous in solving problems wherein the loading can be broken up into components that can be treated as basic standard cases of loading.
- For partially distributed loads, this method is the best.

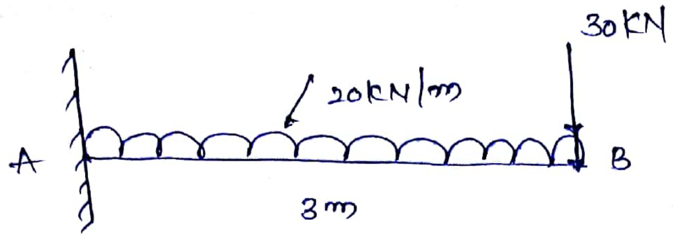
Examples:



Method of Superposition
y
P
w1
x
L
y
P
x
y
x
y
w2
P
x
a
b
y
P
x
y
wu
x
y
wt
x

Method of super position

Ex: 02



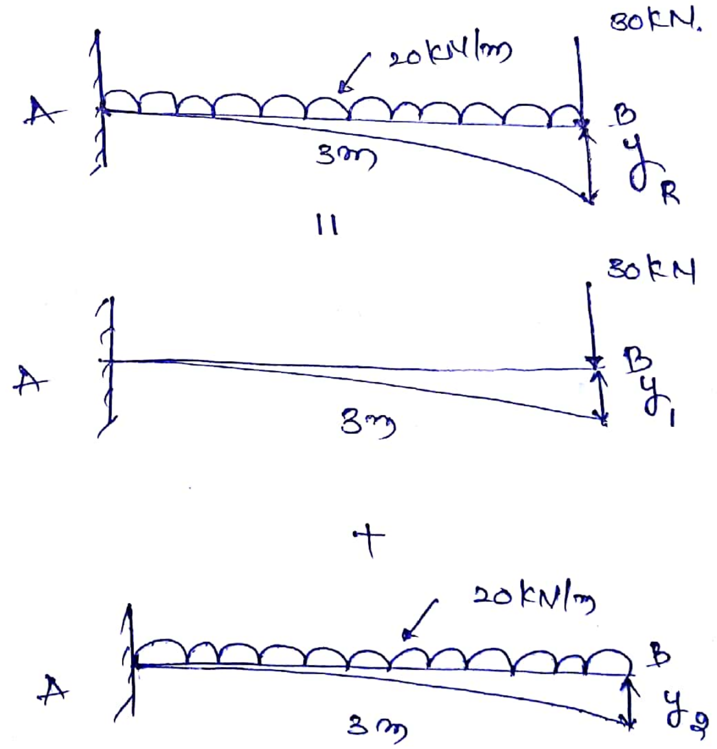
Calculate max, deflection,

Take $E = 210\ GN/m^2$; $I = 33750\ cm^4$

Sol:

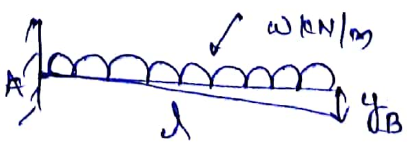
From method of super position, the resulting deflection of a loaded beam is equal to the sum of individual deflections ~~of~~ by individual loads of a same beam.

Therefore,



From standard cases

$$y_B = \frac{Wl^3}{3EI}$$



$$y_B = \frac{wl^4}{8EI}$$

$$\therefore \quad y_1 = \frac{Wl^3}{3EI}$$

$E = 210 \; GN/m^2$

$= 210 \times 10^9 \; N/m^2$

$= 210 \times 10^6 \; kN/m^2$

$I = 33750 \; cm^4$

$= 33750 (10^{-2})^4 m^4$

$= 33750 \times 10^{-8} \; m^4.$

$$y_1 = \frac{30 \times 3^3}{3 \times 210 \times 10^6 \times 33750 \times 10^{-8}}$$

$$y_1 = 3.809 \times 10^{-3} \; m$$

$$y_2 = \frac{wl^4}{8EI} = \frac{20 \times 3^4}{8 \times 210 \times 10^6 \times 33750 \times 10^{-8}}$$

$$y_2 = 2.86 \times 10^{-3} \; m$$

$$\therefore \quad y_R = (3.809 + 2.86) \times 10^{-3} \; m$$

$$= 6.669 \times 10^{-3} \; m = 6.67 \; mm$$

$$\boxed{y_R = 6.67 \; mm}$$

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Example 7.2

A 3 metre long cantilever of uniform rectangular cross-section, 15 cm wide and 30 cm deep, is loaded with a 30 kN load at its free end. In addition to this, it carries a U.D.L. of 20 kN per metre run over its entire length. Calculate (*a*) maximum slope and maximum deflection, and (*b*) the slope and deflection at 2 metres from the fixed end. Take $E = 210$ GN/m^2.

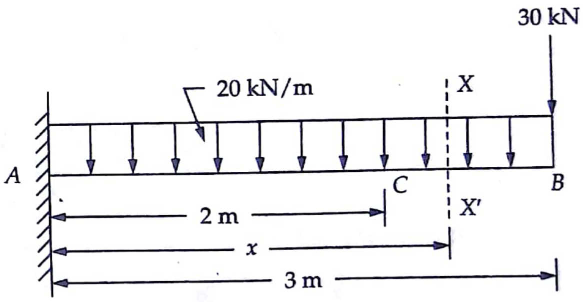


Fig. 7.15

Solution:

Consider a section XX' at distance x from the fixed end A (Fig. 7.15).

B.M. at the section is

$$M_x = -30 \times (3 - x) - 20 \times \frac{(3 - x)^2}{2} = 10 \times (-x^2 + 9x - 18)$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -M_x = 10\,(x^2 - 9x + 18) \text{ kNm} = 10^4\,(x^2 - 9x + 18) \text{ Nm}$$

Integrating the above successively, we have

$$EI\frac{dy}{dx} = 10^4\left[\frac{x^3}{3} - \frac{9x^2}{2} + 18x\right] + C_1$$

and $$EIy = 10^4\left[\frac{x^4}{12} - \frac{9x^3}{6} + \frac{18x^2}{2}\right] + C_1 x + C_2$$

where C_1 and C_2 are the constants of integration.

Applying conditions of zero slope and zero deflection at the end *A*, (*i.e*, when $x = 0$; $\frac{dy}{dx} = 0$ and $y = 0$), we have $C_1 = 0$ and $C_2 = 0$

$$\therefore \quad EI\frac{dy}{dx} = 10^4\left(\frac{x^3}{3} - \frac{9x^2}{2} + 18x\right) \quad ...(i)$$

and $$EIy = 10^4\left[\frac{x^4}{12} - \frac{3x^3}{2} + 9x^2\right] \quad ...(ii)$$

Now $$*\,I = \frac{bd^3}{12} = \frac{15\times 30^3}{12} = 33750 \text{ cm}^4$$

or $$= \frac{33750}{(100)^4}\text{ m}^4 = 33.75 \times 10^{-5} \text{ m}^4$$

and $$*\,E = 210 \times 10^9 \text{ N/m}^2$$

The maximum slope and deflection are obviously at the free end *B*, for which put $x = 3$ in the equations (*i*) and (*ii*).

$$\therefore \quad EI\theta_{max} = 10^4\left(\frac{3^3}{3} - \frac{9\times 3^2}{2} + 18\times 3\right) = 22.5 \times 10^4$$

$$\therefore \quad \theta_{max} = \frac{22.5\times 10^4}{EI} = \frac{22.5\times 10^4}{(210\times 10^9)\times(33.75\times 10^{-5})} = \mathbf{0.003175\ radian}$$

$$EI\,y_{max} = 10^4\left[\frac{3^4}{12} - \frac{3\times 3^3}{2} + 9\times 3^2\right] = 47.25 \times 10^4$$

$$\therefore \quad y_{max} = \frac{47.25\times 10^4}{EI} = \frac{47.25\times 10^4}{(210\times 10^9)\times(33.75\times 10^{-5})} = 0.00667 \text{ m} = \mathbf{0.667\ Cm}$$

... deflection at C (2 ... tions (i) and (ii)

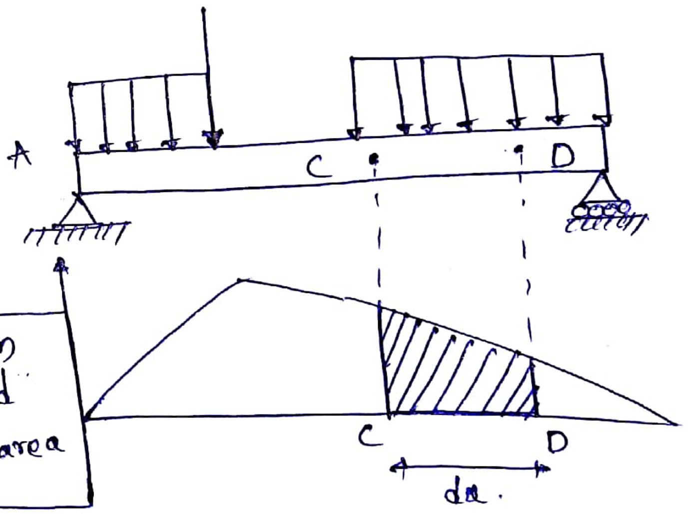
Moment Area Method:

It is considered to be the most effective of all methods, especially when deflection at a specific location is to be found out.

The following two theorems, help in determining the slopes & deflection in case of straight member under bending.

Theorem I: The angle in radians between the tangents to the elastic curve at two points on a straight member under bending is equal to the area of the $\frac{M}{EI}$ diagram between those two points.

Consider a beam subjected to arbitrary loading

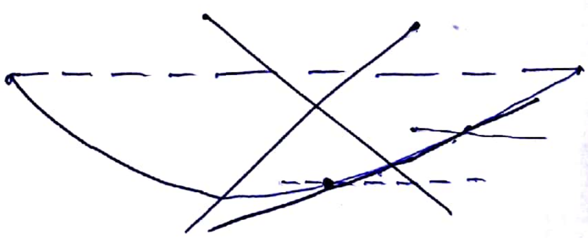


sign convention is ignored for using area

From, differential equation of flexure

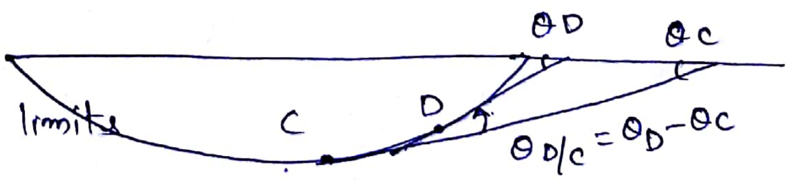
$$EI\frac{d^2y}{dx^2} = M$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$



but $\frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI}$

$$d\theta = \frac{M}{EI}.dx.$$

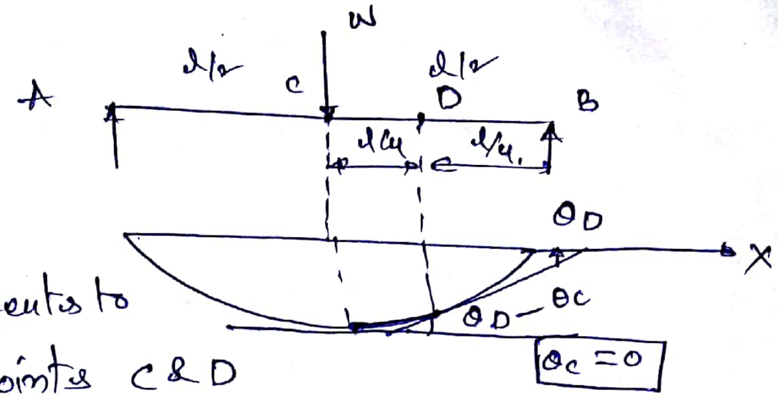


Integrating the above with limits

$$\theta_D - \theta_C = \int_C^D \frac{M}{EI}.dx.$$

For example, in a simply supported beam with symmetrical loading

At point D, we have to calculate slope.



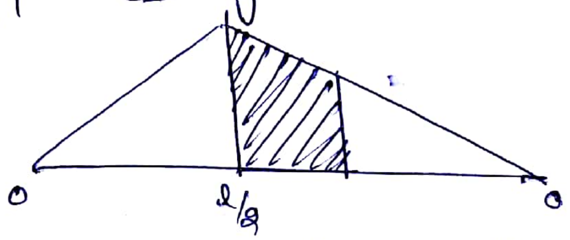
From, theorem I

angle in radians b/w the tangents to the elastic curve at two points C & D

Angle in radians b/w the tangent at D & tangent at C

$$= \theta_D - \theta_C = \theta_D - 0.$$

And it is equal to the area of $\frac{M}{EI}$ diagram b/w C & D



$$\therefore \theta_D - 0 = \int_C^D \frac{M}{EI} \cdot dx.$$

$$= \int_{l/4}^{l/2} \frac{\frac{Wx}{2}}{EI} dx$$

put $M_x = \frac{W}{2} x$.

$$= \frac{W}{2EI} \left[\frac{x^2}{2} \right]_{l/4}^{l/2} = \frac{W}{2EI} \cdot \frac{1}{2} \left[\frac{l^2}{4} - \frac{l^2}{16} \right]$$

$$= \frac{W}{4EI} \left[\frac{4l^2 - l^2}{16} \right]$$

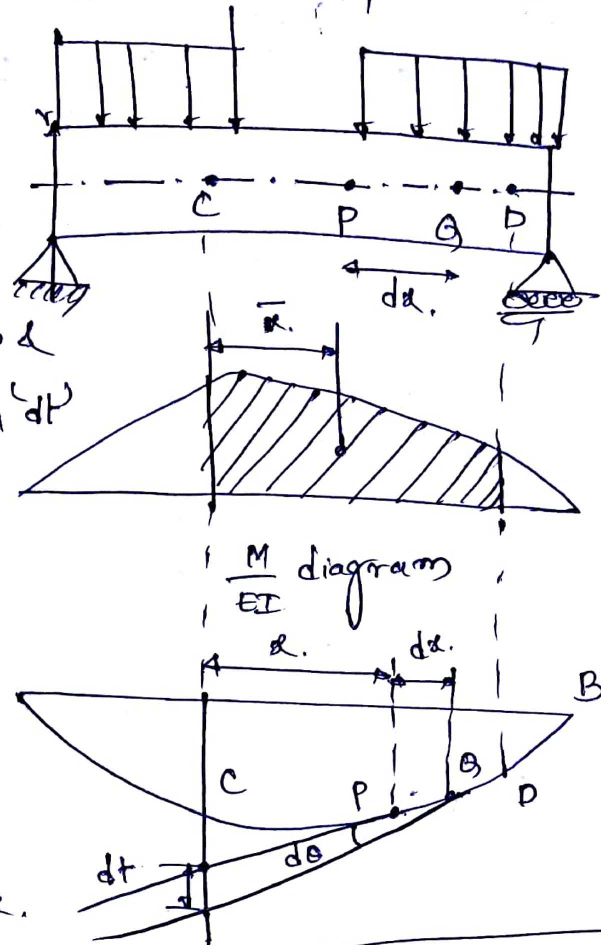
$$= \frac{W}{4EI} \frac{3l^2}{16} = \frac{3Wl^2}{64EI}$$ ✓

Hence, it is proved.

(using double integration method)

Theorem II: The deflection of a point on a straight member under bending in the direction perpendicular to the original straight axis of the member, measured from the tangent at another point on the member is equal to the moment of the $\frac{M}{EI}$ diagram between those two points, about the point where this deflection occurs.

Consider a beam subjected to arbitrary loading.



Tangents to the elastic curve at P & Q intercept a segment of length 'dt' on the vertical through C

$$dt = x\, d\theta.$$

$$dt = x \cdot \frac{M}{EI} \cdot dx$$

$$\therefore\ t_{C/D} = \int_{x_C}^{x_D} x \frac{M}{EI} \cdot dx = \frac{1}{EI}\int_{x_C}^{x_D} x\, M dx.$$

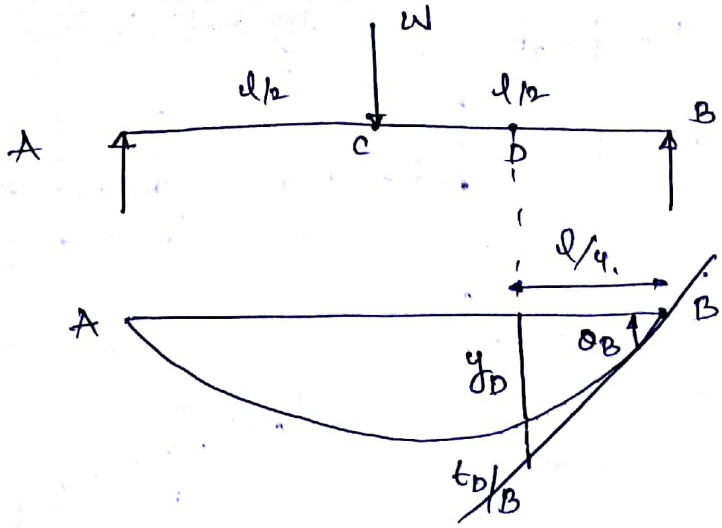
$$t_{C/D} = \frac{1}{EI} \cdot A\bar{x}$$

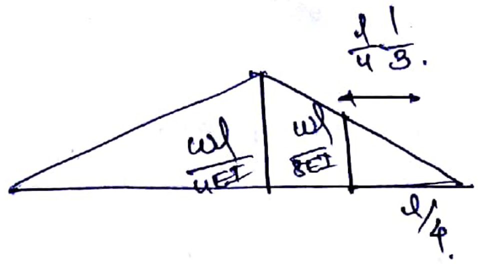
Theorem II

The tangential deviation of C w.r.t D is equal to the first moment w.r.t a vertical axis through C of area under the B.M b/n C & D

Here, A = Total area of B.M diagram b/n C & D

$\bar{x}$ = Distance of CG of B.M diagram from C





Using second theorem,

$$\theta_B \times \frac{l}{4} = y_D + t_{D/B}$$

$t_{D/B}$ = Vertical intercept / tangent deviation ~~of~~ at D w.r.t B

$t_{D/B}$ = moment of area of $\frac{M}{EI}$ diagram b/n D & B w.r.t to D (about D)

$$\theta_B = \frac{wl^2}{16EI}$$

$$\frac{wl^2}{16EI} \cdot \frac{l}{4} = y_D + t_{D/B}$$

$$t_{D/B} = \frac{1}{2} \times \frac{wl}{8EI} \times \frac{l}{4} \times \frac{l}{12} = \frac{wl^3}{768EI}.$$

$$\therefore y_D = t_{D/B} - \frac{wl^3}{64EI} = \frac{wl^3}{768EI} - \frac{wl^3}{64EI}$$

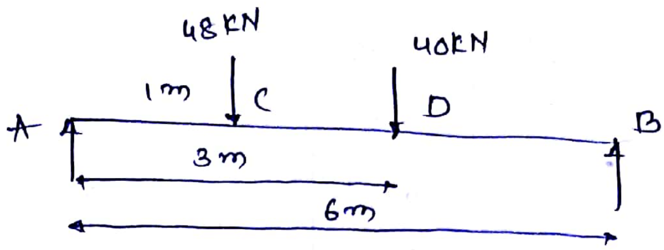
$$\boxed{y_D = \frac{11wl^3}{768EI}}$$

Hence it is proved.

Using moment area method:

Take $E = 2\times10^5\ N/mm^2$

$I = 8.5\times10^6\ mm^4$



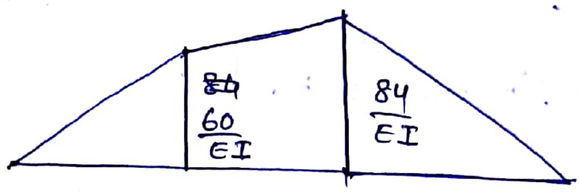
step(i):

Reactions:

$$(R_A\times6) - (48\times5) - (40\times3) = 0$$

$$R_A = 60\ kN$$

$$R_B = 28\ kN$$

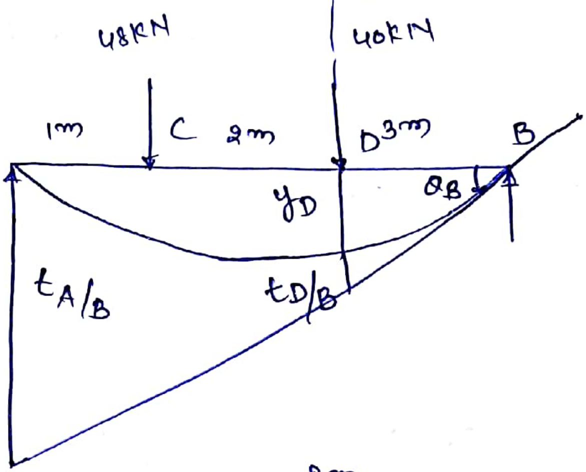


$\frac{M}{EI}$ diagram

step(ii): y_D is to be determined.

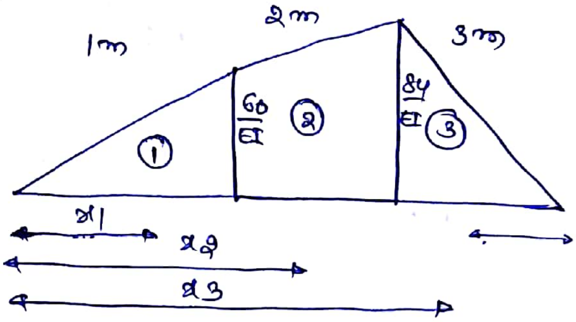
First, calculate slope at B or A

For convenience, θ_B is better,



$$\theta_B\times6 = t_{A/B}$$

$t_{A/B}$ = moment of Area of $\frac{M}{EI}$ diagram b/w B & A about A.



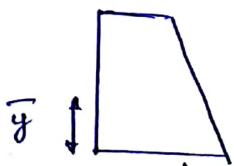
$$A_1 = \frac{1}{2}\times\frac{60}{EI}\times1 = \frac{30}{EI},\quad x_1 = \frac{2}{3}\times1$$

$$x_1 = 0.67\,m$$

$$A_2 = \frac{1}{2}\left[\frac{60}{EI}+\frac{84}{EI}\right]\times2 = \frac{144}{EI}$$

$$x_2 = 1+\left[2-\left(\frac{2\left(\frac{60}{EI}\right)+\frac{84}{EI}}{\frac{60}{EI}+\frac{84}{EI}}\right)\cdot\frac{2}{3}\right]$$

$$= 1+[2-0.94] \Rightarrow x_2 = 2.06\,m$$



$$\bar{y} = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$$

$$A_3 = \frac{1}{2}\times\frac{84}{EI}\times3 = \frac{126}{EI}$$

$$x_3 = 6-\frac{2}{3}\times3 = 4\,m$$

$$t_{A/B} = \left[\frac{30}{EI}\times0.67\right]+\left[\frac{144}{EI}\times2.06\right]+\left[\frac{126}{EI}\times4\right]$$

$$t_{A/B} = \frac{20.1+296.64+504}{EI}$$

$$t_{A/B} = \frac{820.74}{EI}$$

$$\theta_B \times 6 = \frac{820.74}{EI}$$

$$\boxed{\theta_B = \frac{136.79}{EI}}$$

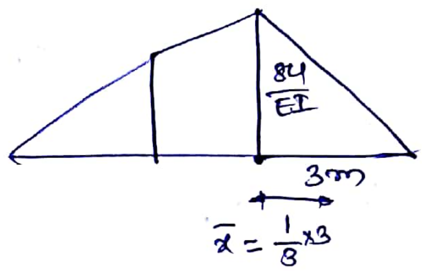
For deflection at point 'D'

$$\theta_B \times 3 = y_D + t_{D/B}$$

$t_{D/B}$ = moment of Area of $\frac{M}{EI}$ diagram b/n B & D about D

$$= \frac{1}{2} \times \frac{84}{EI} \times 3 \times \frac{1}{3} \times 3$$

$$= \frac{126}{EI}.$$



$$\Rightarrow \frac{136.79}{EI} \times 3 = y_D + \frac{126}{EI}.$$

$$EI = 2 \times 10^8 \times 85 \times 10^{-6}$$
$$= 17000 \text{ KN.m}^2$$

$$\Rightarrow y_D = \frac{410.37}{EI} - \frac{126}{EI}$$

$$y_D = \frac{284.37}{EI} = \frac{284.37}{17000} = 0.01672 \text{ m}$$

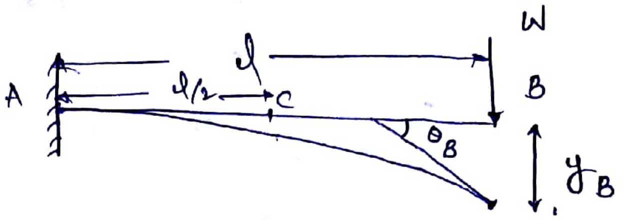
$$\boxed{\therefore y_D = 16.72 \text{ mm}}$$

Cantilever beams:

Case(i): Cantilever carrying point load at free end.

Unknowns; θ_B & y_B

And also, θ_C & y_C at mid span

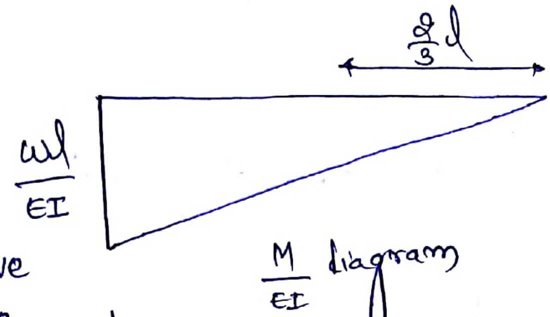


slope at B:

From Theorem I

slope or angle in radians b/n the two tangents on the elastic curve

$=$ Area of $\frac{M}{EI}$ diagram b/n ~~A & B~~ those points



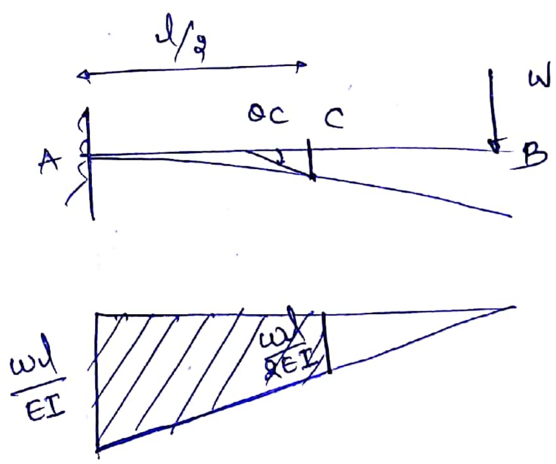
Hint: Any one of the tangent is flat or angle b/n tangent to curve is zero.

$$\theta_B - \theta_A = \frac{1}{2}\frac{wl}{EI} \times l$$

$\theta_A = 0$

$$\boxed{\theta_B = \frac{wl^2}{2EI}}$$

slope at C:



$$\theta_C - \theta_A = \frac{1}{2}\left(\frac{wl}{EI} + \frac{wl}{2EI}\right)\cdot\frac{l}{2}$$

$$\theta_C = \frac{l}{4}\left(\frac{2wl + wl}{2EI}\right)$$

$$\boxed{\theta_C = \frac{3wl^2}{8EI}}$$

Deflection at B:

From theorem II

Here, vertical intercept $t_{B/A} = y_B$

~~θ_B~~

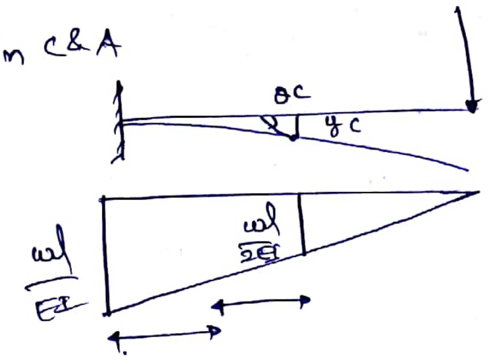
y_B = moment of Area of $\frac{M}{EI}$ diagram b/n A & B about B

$$= \frac{1}{2} \cdot \frac{wl}{EI} \times l \times \frac{2}{3} l$$

$$y_B = \frac{wl^3}{3EI}$$

Slope at C ; at $\frac{l}{2}$ distance

$\theta_C - \theta_A$ = Area of $\frac{M}{EI}$ diagram b/n C & A



Deflection at C ; at $\frac{l}{2}$ distance.

Vertical intercept at point C w.r.t tangent A = deflection at point C

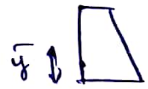
$t_{C/A} = y_C$ $\Rightarrow$ y_C = moment of Area of $\frac{M}{EI}$ diagram b/n C & A about C

$$y_C = \left[\frac{1}{2}\left(\frac{wl}{EI} + \frac{wl}{2EI}\right)\frac{l}{2}\right]\left[\frac{l}{2} - \frac{2l}{9}\right]$$

$$= \frac{l}{4}\left(\frac{3wl}{2EI}\right)\left[\frac{9l-4l}{18}\right]$$

$$= \frac{l}{4} \cdot \frac{3wl}{2EI} \cdot \frac{5l}{18}$$

$$y_C = \frac{15wl^3}{144EI}$$



$$\bar{y} = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$$

$$= \left(\frac{2\frac{wl}{2EI} + \frac{wl}{EI}}{\frac{wl}{2EI} + \frac{wl}{EI}}\right)\frac{l}{2} \cdot \frac{1}{3}$$

$$= \frac{\frac{2wl}{EI} \times \frac{l}{6}}{\frac{wl + 2wl}{2EI}}$$

$$= \frac{2wl}{6_3} \times \frac{2}{3wl} = \frac{2}{9} l.$$

7.8.3 A Cantilever Carrying a Point Load *W*, which is not at the Free End

Since E and I are constant, Fig. 7.43(b) shows the $\frac{M}{EI}$ diagram. End A of the cantilever is fixed, therefore, tangent to the elastic curve at A is horizontal, *i.e.*,

$$\theta_A = 0$$

$$\theta_C = \text{area of } \frac{M}{EI} \text{ diagram between } A \text{ and } C$$

or $$= \frac{Wa}{EI} \times \frac{a}{2} = \frac{Wa^2}{2EI}$$

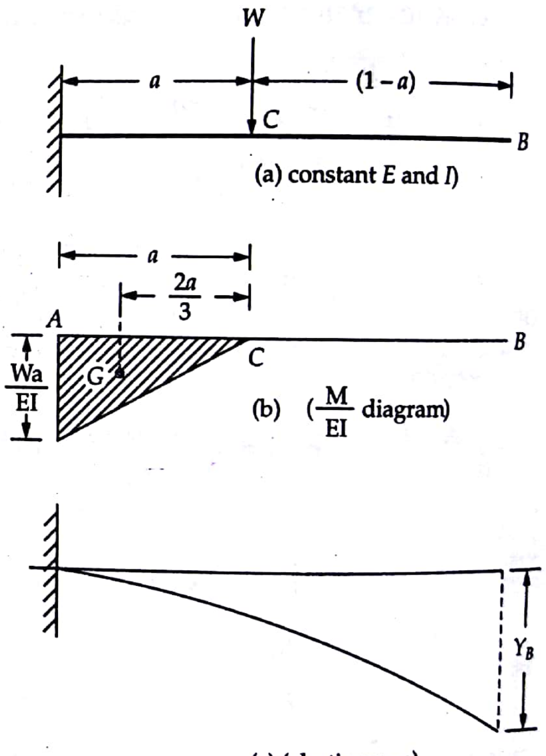


Fig. 7.43

Since there is no load on BC so slope at C is the same as the slope at B

$\therefore$ $$\theta_B = \theta_C = \frac{Wa^2}{2EI}$$

Now deflection at C is y_c = moment of area of $\frac{M}{EI}$ diagram about C

or $$= \left(\frac{1}{2} \times \frac{Wa}{EI} \times a\right) \frac{2a}{3}$$

or $$= \frac{Wa^3}{3EI}$$

Similarly, deflection at B is y_B = moment of area of $\frac{M}{EI}$ diagram about B

or $$= \frac{Wa^2}{2EI}\left(\frac{2a}{3} + l - a\right)$$

or $$= \frac{Wa^2}{6EI}(3l - a)$$

7.8.4 A Cantilever Carrying U.D.L. at the rate w/unit Length on a Part of the Span from the Fixed End

Since E and I are constants, Fig. 7.44(b) shows the $\frac{M}{EI}$ diagram. End A of the cantilever is so fixed that the tangent to the elastic curve at A is horizontal, *i.e.*,

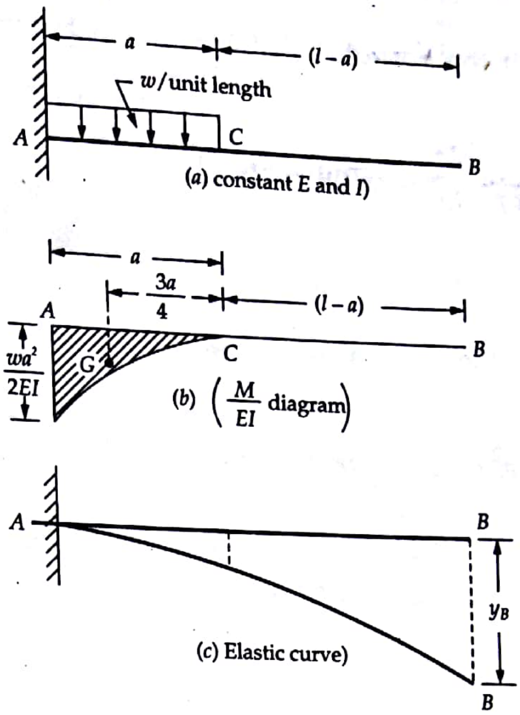


Fig. 7.44.

$$\theta_A = 0$$

$$\theta_C = \text{area of } \frac{M}{EI} \text{ diagram between } A \text{ and } C$$

or $$= \frac{1}{3} \times \frac{wa^2}{2EI} \times a = \frac{wa^3}{6EI} \qquad (\textit{where } W = wa_1)$$

As length BC carriers no load, so it shall be straight and $\theta_B = \theta_C = \frac{wa^3}{6EI}$

Deflection at C is

$$y_c = \text{moment of the area of } \frac{M}{EI} \text{ diagram between A and C about C.}$$

or $$= \left(\frac{1}{3} \times \frac{wa^2}{2EI} \times a\right) \times \frac{3a}{4} = \frac{wa^4}{8EI}$$

Similarly, $$y_B = \text{moment of area of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } B$$

or $$= \left(\frac{1}{3} \times \frac{wa^2}{2EI} \times a\right) \times \left(\frac{3a}{4} + l - a\right)$$

$$= \frac{wa^3 l}{8EI} - \frac{wa^4}{24EI}$$

7.8.6 Cantilever with a Moment Applied at the Free End

Figure 7.46(*b*) shows the $\frac{M}{EI}$ diagram. Since the end *A* of the cantilever is fixed, the tangent to the elastic curve at *A* shall be horizontal and thus:

$$\theta_A = 0$$

$$\theta_B = \text{area of } \frac{M}{EI} \text{ diagram}$$

$$= \frac{M}{EI} \times l = \frac{Ml}{EI}$$

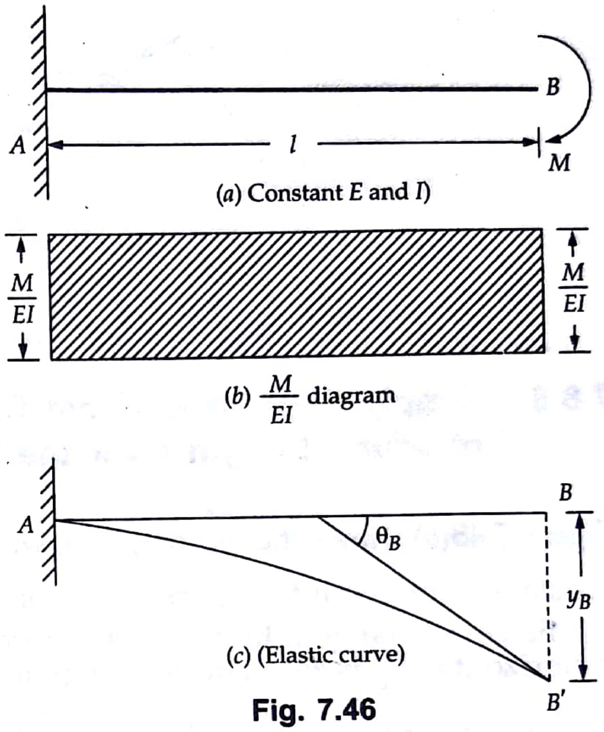


Fig. 7.46

Deflection y_B at *B* is the moment of area of $\frac{M}{EI}$ diagram about *B*.

$$y_B = \frac{Ml}{EI} \times \frac{l}{2} = \frac{Ml^2}{2EI}$$

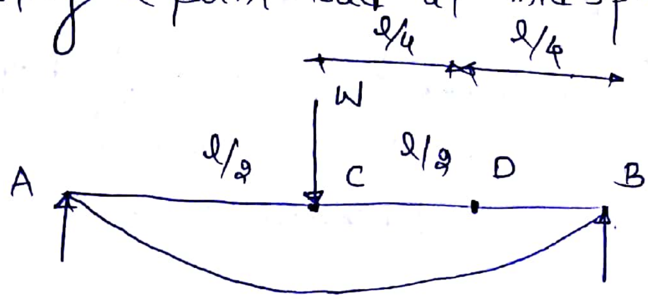
Moment Area method: It is the most effective method & deflection can be determined at any section on the beam.

Standard cases:

(i) simply supported beam carrying a point load at mid span

Sol: Unknowns

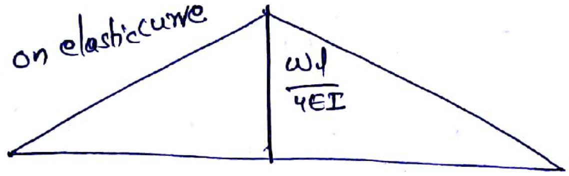
$\theta_A, \theta_B, y_C, \theta_D, \& \ y_D$



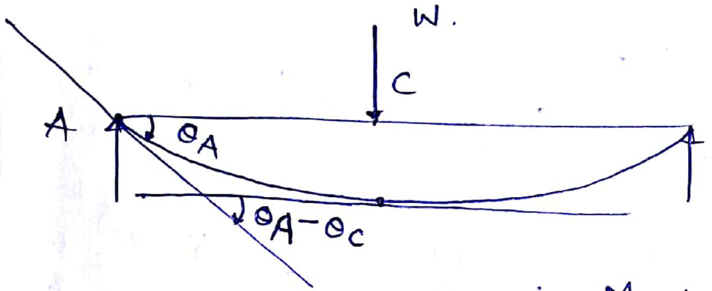
Slope θ_A:

Angle b/n the two tangents at two points on elastic curve

= Area of $\frac{M}{EI}$ diagram b/n these two points.



$\frac{M}{EI}$ diagram



why 'θ_c' is considered

As per theorem I, slope b/n two tangents
If two are not equal to zero,
required slope can't be determined.

$\theta_A - \theta_C$ = Area of $\frac{M}{EI}$ diagram b/n A & C

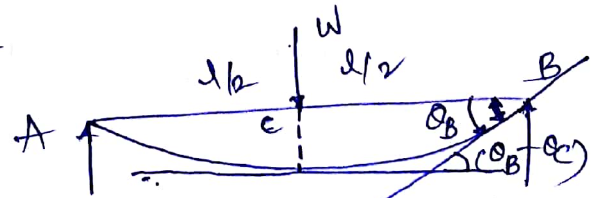
$$\theta_A - 0 = \frac{1}{2} \times \frac{wl}{4EI} \cdot \frac{l}{2} \Rightarrow \boxed{\theta_A = \frac{wl^2}{16EI}}$$

Slope θ_B:

Angle b/n the two tangents at θ_c & θ_B = Area of $\frac{M}{EI}$ diagram b/n B & C

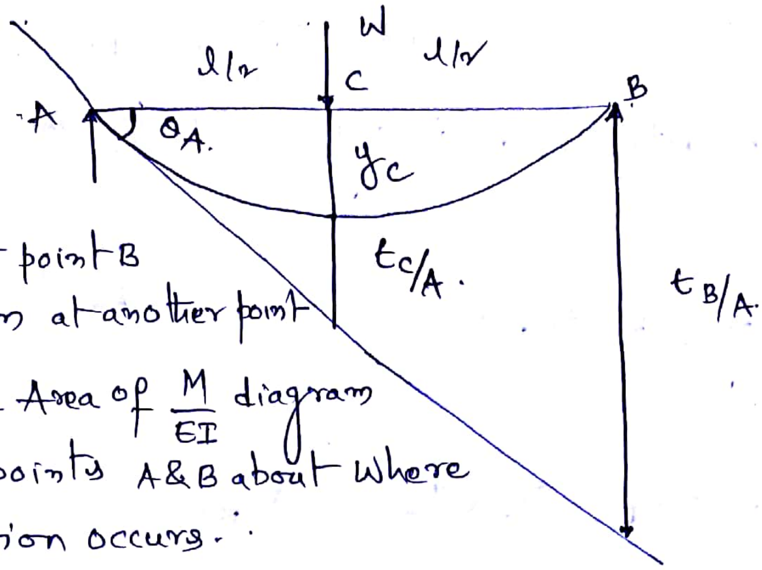
$$\theta_B - \theta_C = \frac{1}{2} \cdot \frac{wl}{4EI} \cdot \frac{l}{2}$$

$$\boxed{\theta_B = \frac{-wl^2}{16EI}}$$



rotation anticlockwise. −ve.

Deflection at point C:

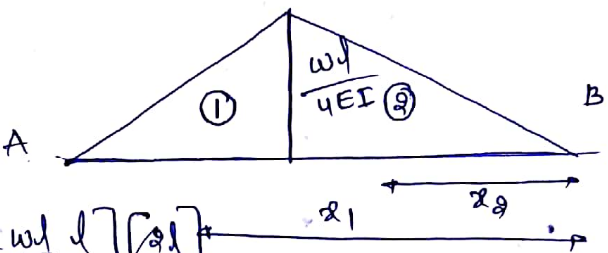


As per theorem II,

Vertical intercept at point B w.r.t tangent drawn at another point A
= Moment of Area of $\frac{M}{EI}$ diagram b/n the points A & B about where deflection occurs.

$$\theta_A l = t_{B/A}$$

$t_{B/A}$ = Moment of Area of $\frac{M}{EI}$ diagram b/n the points A & B about B



$$= A_1 x_1 + A_2 x_2$$

$$= \left[\frac{1}{2}\cdot\frac{wl}{4EI}\times\frac{l}{2}\right]\left[\frac{l}{2}+\frac{1}{3}\cdot\frac{l}{2}\right] + \left(\frac{1}{2}\cdot\frac{wl}{4EI}\cdot\frac{l}{2}\right)\left(\frac{2}{3}\cdot\frac{l}{2}\right)$$

$$= \frac{Wl^2}{16EI}\left[\frac{l}{2}+\frac{l}{6}+\frac{l}{3}\right] = \frac{Wl^2}{16EI}\left(\frac{3l+l+2l}{6}\right) = \frac{6wl^3}{96EI} = \frac{wl^3}{16EI}$$

But,

$$\therefore \theta_A l = \frac{wl^3}{16EI} \Rightarrow \boxed{\theta_A = \frac{wl^2}{16EI}}$$

But, we have to calculate 'y_c'

From fig. $\theta_A \times \frac{l}{2} = y_c + t_{C/A}$ — (1)

$t_{C/A}$ = moment of Area of $\frac{M}{EI}$ diagram b/n A & C about C.

$$t_{C/A} = \frac{1}{2} \times \frac{wl}{4EI} \times \frac{l}{2} \times \frac{l}{2} \cdot \frac{1}{3}$$

$$= \frac{wl^3}{96EI}$$

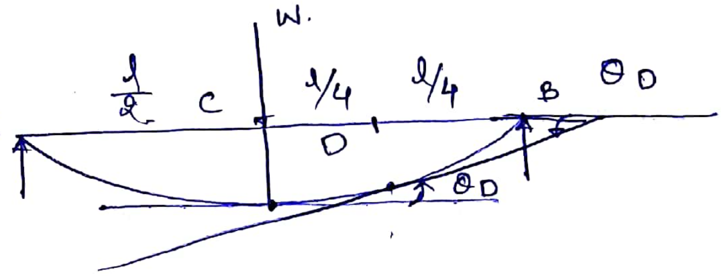
From eq (i);

$$\frac{wl^2}{16EI} \times \frac{l}{2} = y_c + \frac{wl^3}{96EI}$$

$$\Rightarrow \quad y_c = -\frac{wl^2}{96EI} + \frac{wl^3}{32EI} \quad \Rightarrow \quad \boxed{y_c = \frac{1}{48}\frac{wl^3}{EI}}$$

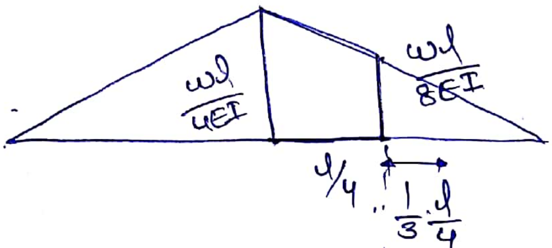
Slope at D:

$$\theta_D - \theta_C = \text{Area of } \frac{M}{EI} \text{ diagram A b/w C \& D}$$



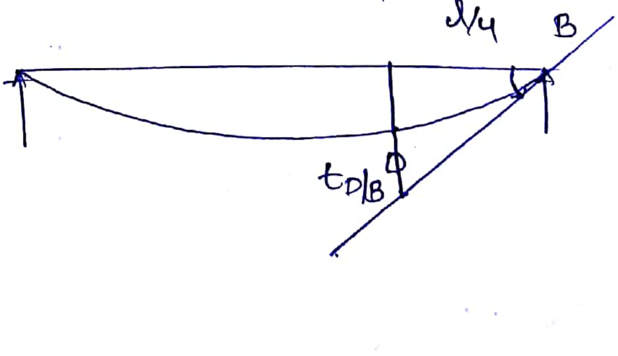
$$\boxed{\theta_C = 0}$$

$$\theta_D - 0 = \frac{1}{2}\left[\frac{wl}{4EI} + \frac{wl}{8EI}\right]\frac{l}{4}.$$



$$\theta_D = \frac{l}{8}\left[\frac{2wl + wl}{8EI}\right]$$

$$\boxed{\theta_D = \frac{3wl^3}{64EI}}$$



$$\theta_B = \frac{wl^2}{16EI};$$

From theorem II

$$t_{D/B} = \text{moment of Area of } \frac{M}{EI} \text{ diagram b/w B \& D about D}$$

$$t_{D/B} = \frac{1}{2} \times \frac{wl}{8EI} \times \frac{l}{4} \times \frac{1}{3} \cdot \frac{l}{4}. \qquad t_{D/B} = \frac{wl^3}{768EI}$$

From

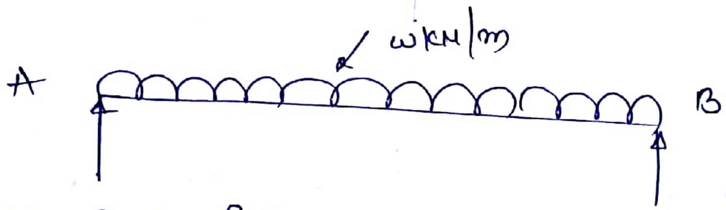
$$\theta_B \times \frac{l}{4} = y_D + t_{D/B}$$

$$\frac{wl^2}{16EI} \cdot \frac{l}{4} = y_D + \frac{wl^3}{768EI}$$

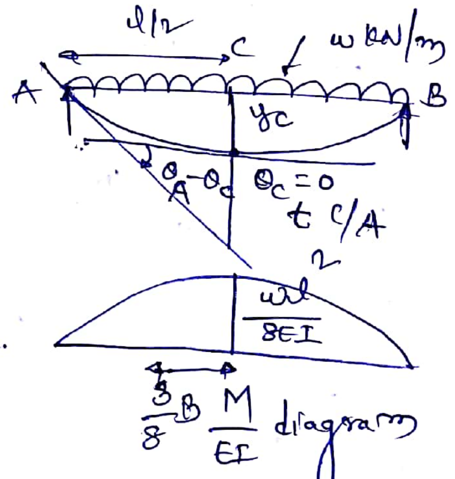
$$\Rightarrow \quad y_D = \frac{wl^3}{64EI} - \frac{wl^3}{768EI} = \frac{12wl^3 - wl^3}{768EI} = \frac{11wl^3}{768EI}$$

$$\boxed{y_D = \frac{11wl^3}{768EI}}$$

Case (ii):



Unknowns θ_A, θ_B & y_{max}.

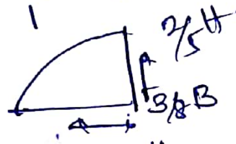


$$\theta_A - \theta_c = \frac{1}{EI} \cdot \frac{2}{3} \times \frac{wl^2}{8EI} \times \frac{l}{2}$$

$$\boxed{\theta_c = 0}$$

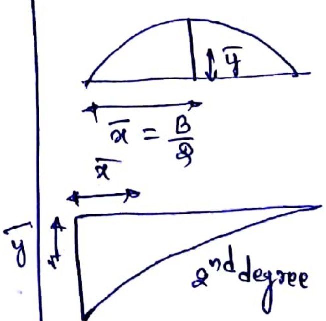
$$\boxed{\theta_A = \frac{wl^3}{24EI}}$$

$$\boxed{\theta_B = \frac{-wl^3}{24EI}}$$



$A = \frac{2}{3} BH$

$y_{max} = y_{midspan} = y_C = ?$



$A = \frac{2}{3} BH$

$\bar{y} = \frac{2}{5} H$

$A = \frac{1}{3} BH$

$\bar{x} = \frac{B}{4}; \bar{y} = \frac{3}{10} H$

$A = \frac{1}{4} BH$

$\bar{x} = \frac{B}{5}; \bar{y} = \frac{H}{5}$

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From eqn

$$\theta_A \frac{l}{2} = y_c + t_{c/A}$$

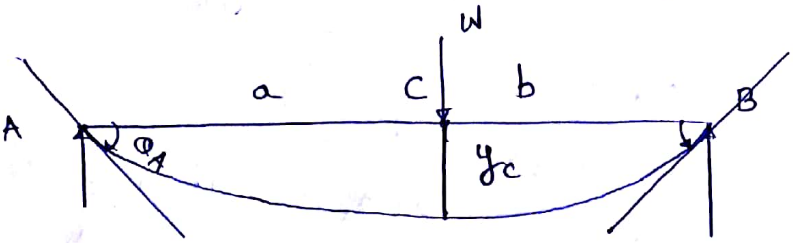
$$t_{c/A} = \left[\frac{1}{2} \cdot \frac{2}{3} \, l \times \frac{wl^2}{8EI}\right]\left[\frac{3}{8}\frac{l}{2}\right]$$

$$t_{c/A} = \frac{wl^4}{128EI}$$

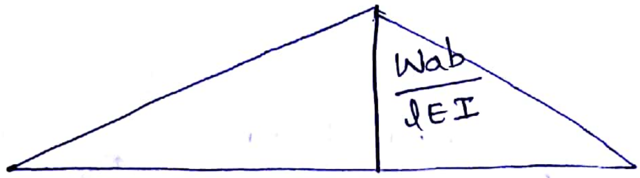
$$\frac{wl^3}{24EI} \cdot \frac{l}{2} = y_c + \frac{wl^4}{128EI}$$

$$y_c = \frac{wl^4}{48EI} - \frac{wl^4}{128EI} \Rightarrow \boxed{y_c = \frac{wl^4}{EI}\frac{5}{384}}$$

Case(iii): point load acts at distance of 'a' from left end $(a>b)$

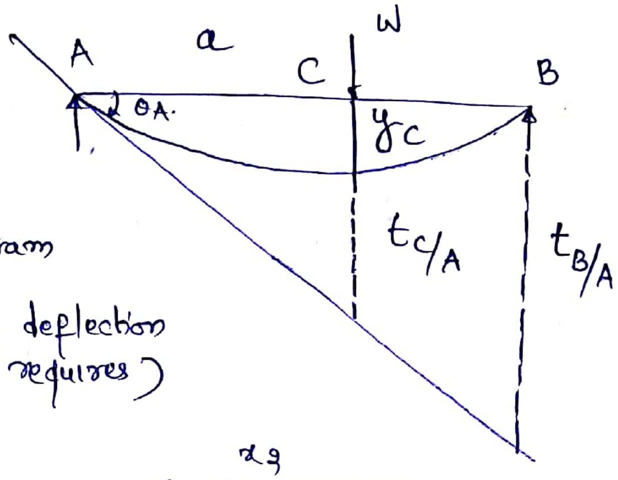


Sol: Unknowns
θ_A, θ_B & y_C



$\frac{M}{EI}$ diagram

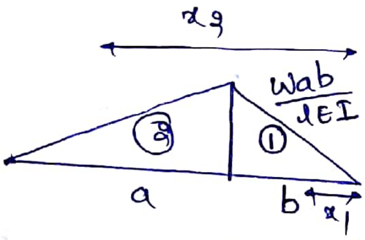
slope at A;



From theorem II ;

$t_{B/A}$ = moment of Area of $\frac{M}{EI}$ diagram b/n A & B about B (where deflection requires)

there, $\boxed{y_B = 0}$



$$A_1 x_1 = \frac{1}{2}\frac{Wab}{lEI} b \times \frac{2}{3} b$$

$$A_2 x_2 = \frac{1}{2}\frac{Wab}{lEI} a\left(\frac{a}{3}+b\right)$$

$$t_{B/A} = \frac{1}{2}\frac{Wab}{lEI} b \frac{2}{3} b + \frac{1}{2}\frac{Wab}{lEI} a\left(\frac{a+3b}{3}\right)$$

$$t_{B/A} = \frac{Wab^3}{3EIl} + \frac{Wa^2b(a+3b)}{6EIl}$$

$$t_{B/A} = \frac{2Wab^3 + Wa^2b(a+3b)}{6EIl} = \frac{Wab(2b^2+a(a+3b))}{6EIl}$$

$$t_{B/A} = \frac{Wab(2b^2+a^2+3ab)}{6EIl} = \frac{Wab(a^2+b^2+2ab+b^2+ab)}{6EIl}$$

$$t_{B/A} = \frac{Wab((a+b)^2+b(a+b))}{6EIl} = \frac{Wab(a+b)}{6EIl}(a+b+b)$$

$$t_{B/A} = \frac{Wab(a+2b)}{6EI}$$

$$\therefore \quad \theta_A \times l = y_B + t_{B/A} \qquad y_B = 0$$

$$\theta_A l = t_{B/A}$$

$$\theta_A l = \frac{Wab(a+2b)}{6EI} \Rightarrow \boxed{\theta_A = \frac{Wab(a+2b)}{6EIl}}$$

similarly for θ_B

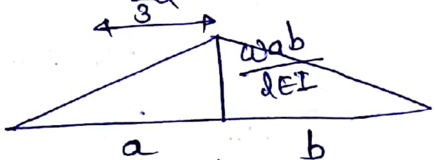
$$\theta_B = \frac{Wba(b+2a)}{6EIl} \qquad \boxed{\theta_B = \frac{Wab(2a+b)}{6EIl}}$$

For deflection at point C:

$$\theta_A \times a = y_C + t_{C/A}$$

$t_{C/A}$ = moment of Area of $\frac{M}{EI}$ diagram b/w C & A about C

$$t_{C/A} = \frac{1}{2} \cdot \frac{Wab}{lEI} \times a \times \frac{1}{3} a$$



$\frac{M}{EI}$ diagram

$$\therefore \quad \frac{Wab(a+2b)}{6EIl} \times a = y_C + \frac{Wa^3b}{6EIl}$$

$$\Rightarrow \quad y_C = \frac{Wa^2b(a+2b)}{6EIl} - \frac{Wa^3b}{6EIl} = \frac{Wa^2b[\cancel{a}+2b-\cancel{a}]}{6EIl}$$

$$y_C = \frac{Wa^2b(2b)}{6EIl} \Rightarrow \boxed{y_C = \frac{Wa^2b^2}{3EIl}}$$

7.9 CONJUGATE BEAM METHOD

It is a special case of the moment–area method and it can be stated in words as follows:

Conjugate beam Theorem I. *The angle between the tangent to the elastic curve of a beam AB at a point C and the chord AB is the same as the shear force at C in case of an imaginary simply supported beam AB loaded with* $\frac{M}{EI}$ *diagram.*

Conjugate beam Theorem II. *The deflection at any point C on the elastic curve of a beam AB measured from the chord AB is the same as the bending moment at C in case of an imaginary beam AB loaded with* $\frac{M}{EI}$ *diagram.*

In applying moment–area method for determining the slope and the deflection, we had known a specific point at which the tangent to the elastic curve was horizontal. The slope or deflection at any other point was determined with reference to that tangent. But in most of the cases where beams are not carrying symmetrical loads, the point where the tangent to the elastic curve is horizontal is not known. Application of moment–area method to such cases causes complications.

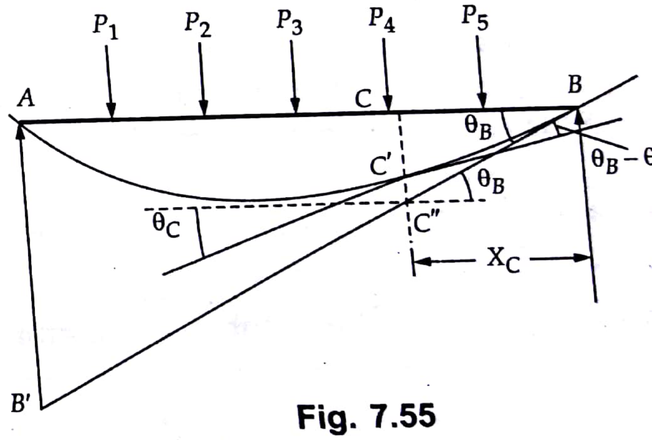


Fig. 7.55

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Consider the case of a simply supported beam loaded as shown in Fig. 7.55. Let the tangent to the elastic curve at C make an angle θ_C and the tangent to the elastic curve at B an angle θ_B with the horizontal. Now, by moment-area method, we have

$$\theta_B - \theta_C = \left(\text{Area of } \frac{M}{EI} \text{ diagram between } B \text{ and } C\right)$$

But $$\theta_B = \frac{AB}{AB'} = \frac{1}{l} \times \left(\text{Moment of area of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } A\right)$$

$\therefore$ $$\theta_C = \frac{\left(\text{Moment of area of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } A\right)}{l} - \left(\text{Area of } \frac{M}{EI} \text{ diagram between } B \text{ and } C\right) \quad \text{... (i)}$$

For deflection,

$$y_C = CC' = CC'' - C'C''$$

or $$= \theta_B \times x_C - \text{deflection of } C \text{ from tangent at } B$$

or $$= \theta_B \times x_C - \left(\text{Moment of } \frac{M}{EI} \text{ diagram between } B \text{ and } C \text{ about } C\right)$$

or $$= \frac{\left(\text{Moment of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } A\right)}{l} x_C - \left(\text{Moment of } \frac{M}{EI} \text{ diagram between } B \text{ and } C \text{ about } C\right) \quad \text{... (ii)}$$

Now assume an imaginary simply supported beam AB loaded with $\frac{M}{EI}$ diagram (Fig. 7.56). Let R_A and R_B be the support reactions in this case and F_C and M_C the shear force and the bending moment at C on this imaginary beam.

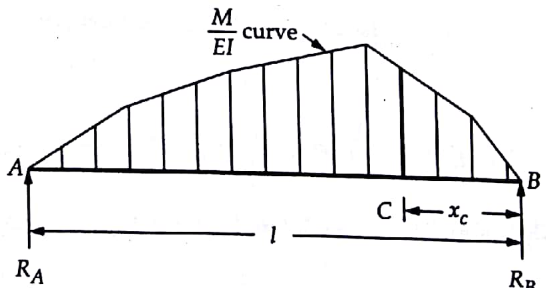


Fig. 7.56

Now $$R_B = \frac{\text{Moment of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } A}{l}$$

But $$F_C = R_B - \left(\text{Area of } \frac{M}{EI} \text{ diagram between } B \text{ and } C\right)$$

or $$= \frac{\text{Moment of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } A}{l} - \left(\text{Area of } \frac{M}{EI} \text{ diagram between } B \text{ and } C\right) = \theta_C \quad \text{.... [from Eq. (i)]}$$

Similarly, $M_C = R_B \times x_C - \left(\text{Moment of } \frac{M}{EI} \text{ diagram between } B \text{ and } C \text{ about } C\right)$

or $$= \left(\frac{\text{Moment of } \frac{M}{EI} \text{ diagram between } A \text{ and } B \text{ about } A}{l}\right) x_C$$

$$- \left(\text{Moment of } \frac{M}{EI} \text{ diagram between } B \text{ and } C \text{ about } C\right)$$

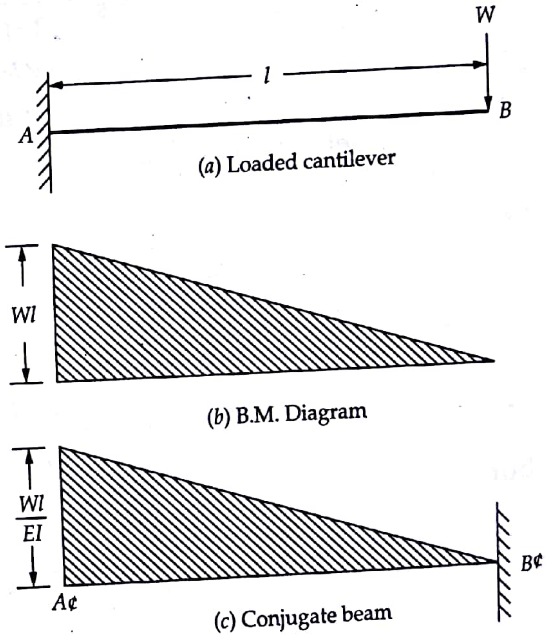
or $= y_c$ ... [from Eq. (*ii*)]

Thus, the above theorems can be applied to all cases where the chord AB is horizontal. The imaginary beam AB loaded with $\frac{M}{EI}$ diagram is also known as *auxiliary beam* or a *conjugate beam.*

7.10 CONJUGATE BEAM METHOD FOR CANTILEVERS

Conjugate beam corresponding to a simply supported beam as discussed in Sec. 7.6 is an imaginary simply supported beam loaded with the $\frac{M}{EI}$ diagram and has the same span.

Consider a cantilever AB of uniform cross-section loaded as shown in Fig. 7.57(a). Obviously, the deflection and the slope and deflection are both zero at the fixed end A. But we know that the slope and deflection at a point are respectively equal to the S.F. and the B.M. at that point in case of a conjugate beam. Thus, the conjugate beam $A'B'$ corresponding to the cantilever AB should be such that the S.F. and B.M. for the conjugate beam at A are both zero. It is possible only if the conjugate beam $A'B'$ is fixed at B' and free at A' and loaded with $\frac{M}{EI}$ diagram [Fig. 7.57(c)]. The conjugate beam $A'B'$ thus corresponds to the cantilever AB.



(*a*) Loaded cantilever
(*b*) B.M. Diagram
(*c*) Conjugate beam

Fig. 7.57

Example 7.29

A simply supported beam of span l carries a point load W (not at midspan). Using the conjugate beam method, determine the slopes at the ends of the beam and the deflection under the load.

Solution:

Support reactions at A and B are:

$$R_A = \frac{Wb}{l}; \quad R_B = \frac{Wa}{l}$$

B.M. at C is

$$M_C = \frac{Wab}{l}$$

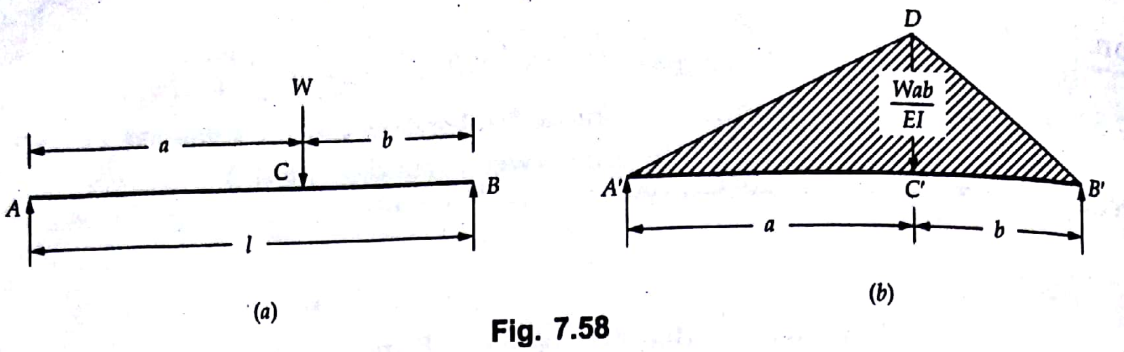


Fig. 7.58

Figure 7.58(b) shows the conjugate beam corresponding to the beam as shown in Fig. 7.58 (a).

Take moments about A' to find the support reaction R'_B for the conjugate beam.

$$R'_B \times l = \frac{l}{2} \times \frac{Wab}{EIl} \times a \times \frac{2}{3} a + \frac{1}{2} \times \frac{Wab}{EIl} \times b\left(a + \frac{b}{3}\right)$$

or
$$= \frac{Wa^3b}{3EIl} + \frac{Wab^2}{6EIl}(3a+b)$$

$$R'_B = \frac{Wa^3b}{3EIl^2} + \frac{Wab^2(3a+b)}{6EIl^2}$$

or
$$= \frac{Wab}{6EIl^2}(2a^2 + 3ab + b^2) \quad \text{... } (i)$$

Deflection at C is y_C = B.M. at C' for the conjugate beam

or
$$= R'_B \times b - \frac{1}{2} \times \frac{Wab}{EIl} \times b \times \frac{b}{3}$$

or
$$= \frac{Wab^2}{6EIl^2}(2a^2 + 3ab + b^2) - \frac{Wab^3}{6EIl}$$

or
$$= \frac{Wab^2}{6EIl^2}(2a^2 + 3ab + b^2 - bl)$$

But $b = l - a$

$\therefore$
$$y_C = \frac{Wa(l-a)^2}{6EIl^2}[2a^2 + 3a(l-a) + (l-a)^2 - l(l-a)]$$

or
$$= \frac{Wa^2(l-a)^2}{3EIl} = \frac{Wa^2b^2}{3EIl}$$

From the equation (i) $\theta_B = R'_B = \dfrac{Wab}{6EIl^2}(2a^2 + 3ab + b^2)$

or
$$= \frac{Wab}{6EIl^2}[2a^2 + 3a(l-a) + (l-a)^2] = \frac{Wab(l+a)}{6EIl}$$

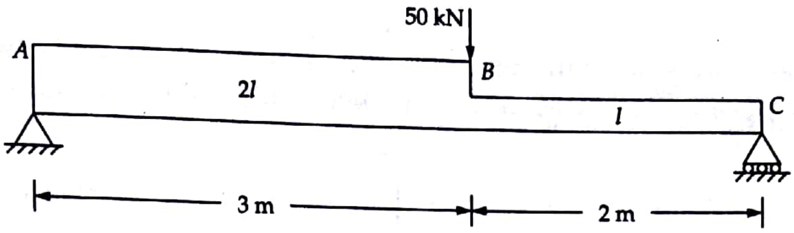
Similarly, $\theta_A = R'_A = \dfrac{Wab\,(l+b)}{6EIl}$

Example 7.28

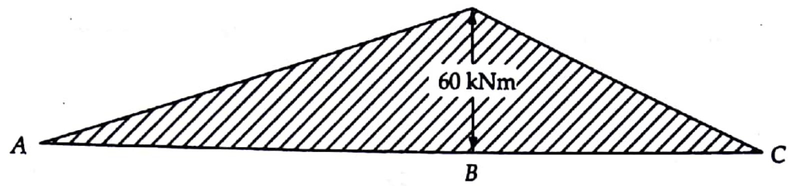
Work out slope at the support A and the vertical deflection at the point B in terms of EI for the beam AC shown in Fig. 7.59(a). Take $I = I_{BC}$ and $I_{AB} = 2I_{BC}$.

Solution:

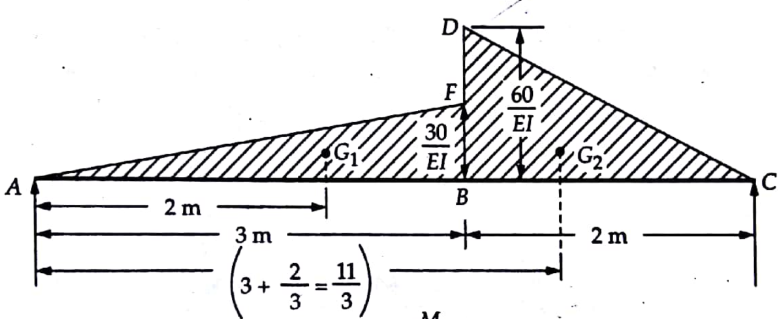
Figure 7.59(*b*) shows the B.M. diagram and Fig. 7.59(*c*) shows the $\frac{M}{EI}$ diagram. Assume a simply supported conjugate beam *ABC* loaded with the $\frac{M}{EI}$ diagram.



(*a*)



(*b*) B.M. diagram



(*c*) $\frac{M}{EI}$ diagram

Fig. 7.59

$$\text{Area of } \Delta BCD = \frac{1}{2} \times \frac{60}{EI} \times 2 = \frac{60}{EI}$$

$$\text{Area of } \Delta ABF = \frac{1}{2} \times \frac{30}{EI} \times 3 = \frac{45}{EI}$$

To find the support reactions, take moments about *A*.

$$R_C \times 5 = \frac{45}{EI} \times 2 + \frac{60}{EI} \times \frac{11}{3} = \frac{310}{EI}$$

$$\therefore \quad R_C = \frac{62}{EI}$$

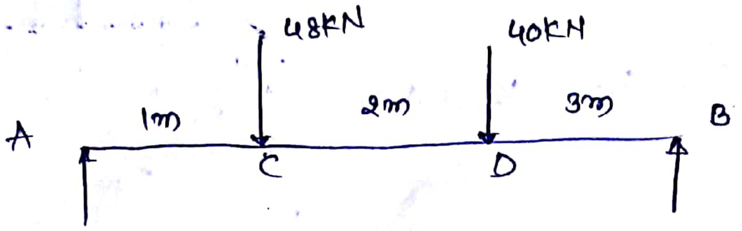
$$R_A = \left(\frac{60}{EI} + \frac{45}{EI}\right) - \frac{62}{EI} = \frac{43}{EI}$$

$$\text{S.F. at } A = R_A = \frac{43}{EI}$$

and $$\text{B.M. at } B = M_B = \frac{43}{EI} \times 3 - \frac{45}{EI} \times (3-2) = \frac{84}{EI}.$$

Using Conjugate beam method:

Take. $E = 2\times10^5 N/mm^2$; $I = 8.5\times10^6 mm^4$.

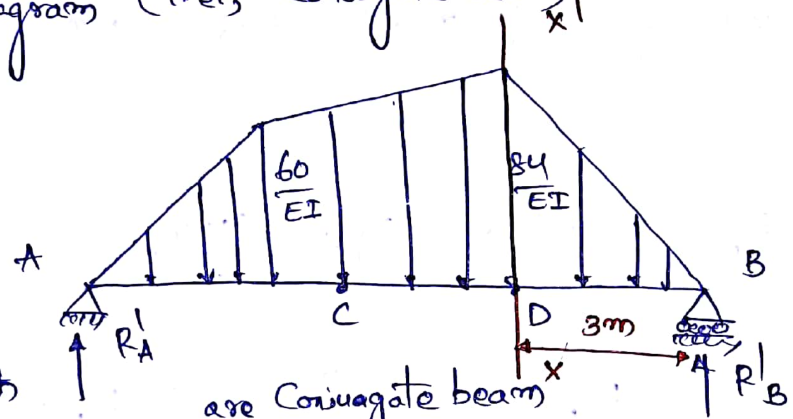


step(i): Reactions;

$(R_A \times 6) - (48\times5) - (40\times3) = 0 \Rightarrow R_A = 60KN$

$R_B = 28KN$

step(ii): $\frac{M}{EI}$ diagram is constructed & make an imaginary bead loaded with $\frac{M}{EI}$ diagram (i.e., Conjugate beam)



both are Conjugate beam

Note: If end supports & simply supports, no changes will be considered.

step(iii): y_D has to be determined.

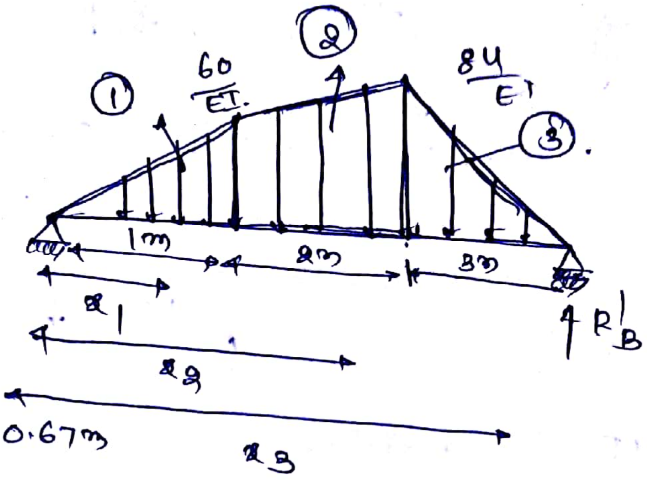
y_D = B.M of conjugate beam loaded with $\frac{M}{EI}$ diagram at point D.

$y_D = R'_B(3)$ - ~~Area o~~ Moment of Area of imaginary load b/n D&B about D.

~~$R'_B \times d$~~

From fig, $-R'_B \times 6 + A_1 x_1 + A_2 x_2 + A_3 x_3 = 0$

$(R'_B \times 6) = A_1 x_1 + A_2 x_2 + A_3 x_3$



$A_1 = \frac{1}{2} \frac{60}{EI} \times 1 = \frac{30}{EI}$; $x_1 = \frac{2}{3} \times 1 = 0.67m$

$A_2 = \frac{1}{2}\left[\frac{60}{EI} + \frac{84}{EI}\right] \times 2 = \frac{144}{EI}$; $x_2 = 1 + \left[2 - \left(\frac{2\left(\frac{60}{EI}\right) + \frac{84}{EI}}{\frac{60}{EI} + \frac{84}{EI}}\right)\frac{2}{3}\right]$

$= 1 + [2 - 0.94] \Rightarrow \boxed{x_2 = 2.06m}$

$A_3 = \frac{1}{2} \frac{84}{EI} \times 3 = \frac{126}{EI}$; $x_3 = 6 - \frac{2}{3} \times 3 = 4m$

$\therefore R'_B \times 6 = \left(\frac{30}{EI} \times 0.67\right) + \left(\frac{144}{EI} \times 2.06\right) + \left(\frac{126}{EI} \times 4\right)$

$R'_B = \frac{820.74}{6EI} \Rightarrow R'_B = \frac{136.79}{EI}$ ~~KN~~

$\therefore y_D = (R'_B \times 3) -$ Moment of Area of imaginary loading D&B about D

$= \left(\frac{136.79 \times 3}{EI}\right) - \left(\frac{1}{2} \times \frac{84}{EI} \times 3\right)\left(\frac{1}{3} \times 3\right)$

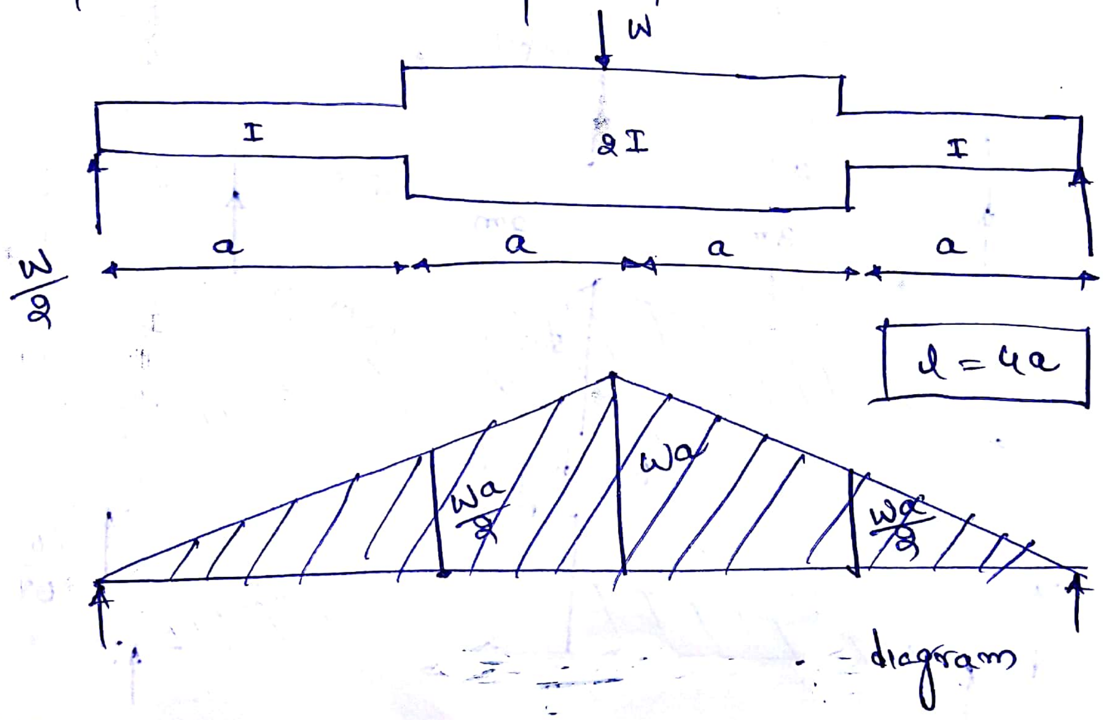
$= \frac{410.37}{EI} - \frac{126}{EI} = \frac{284.37}{EI}$

$EI = 17000 \ KN.m^2$

$y_D = \frac{284.37}{17000} = 0.01672m$

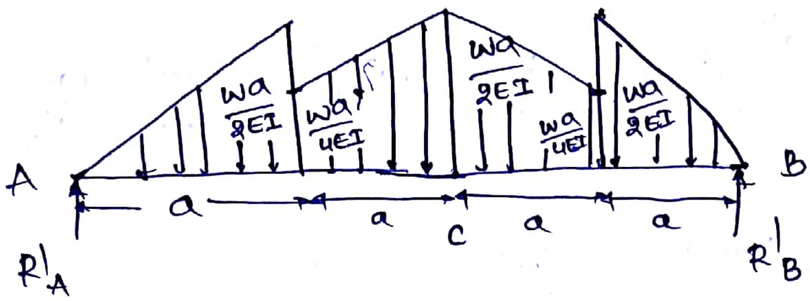
$\boxed{y_D = 16.72 \ mm}$

Ex2: The middle half of the beam shown in fig has a moment of inertia twice that of the rest of the beam & carries a point load W at midspan. Determine midspan deflection



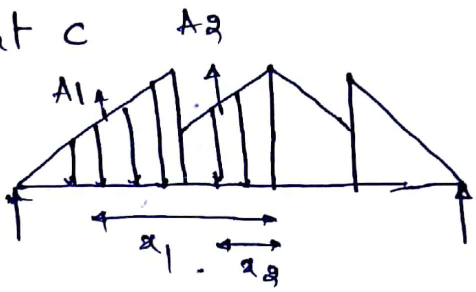
96EI

step(2): Conjugate beam loaded with $\frac{M}{EI}$ diagram



step(3): y at midspan has to be determined.

y_c = B.M of conjugate beam about C



$$y_c = (R'_A \times 2a) - (A_1 x_1 + A_2 x_2)$$

$$R'_A = \frac{\text{Total imaginary load}}{2}$$

$$\text{Total imaginary load} = 2\left[\frac{1}{2} \times \frac{wa}{2EI} \times a\right] + 2\left[\frac{1}{2}\left(\frac{wa}{4EI} + \frac{wa}{2EI}\right)a\right]$$

$$= \frac{wa^2}{2EI} + \left[\frac{wa + 2wa}{4EI}\right]a$$

$$= \frac{wa^2}{2EI} + \frac{3wa^2}{4EI} = \frac{2wa^2 + 3wa^2}{4EI} = \frac{5wa^2}{4EI}$$

$$\therefore R'_A = \frac{5wa^2}{4EI} \Big/ 2 \Rightarrow \boxed{R'_A = \frac{5wa^2}{8EI}}$$

$$A_1 = \frac{1}{2} \times \frac{wa}{2EI} \times a = \frac{wa^2}{4EI} \; ; \quad x_1 = a + \frac{1}{3}a = \frac{4a}{3}$$

$$A_2 = \frac{1}{2}\left[\frac{wa}{4EI} + \frac{wa}{2EI}\right]a = \frac{a}{2}\left(\frac{3wa}{4EI}\right) = \frac{3wa^2}{8EI}$$

$$x_2 = \frac{\left(2\left(\frac{wa}{4EI}\right) + \frac{wa}{2EI}\right)}{\left(\frac{wa}{4EI} + \frac{wa}{2EI}\right)} \times \frac{a}{3}$$

$$x_2 = \left[\frac{\frac{wa}{EI}}{\frac{3wa}{4EI}}\right]\frac{a}{3} \Rightarrow x_2 = \frac{wa}{EI} \times \frac{4EI}{3wa} \times \frac{a}{3} = \frac{4a}{9}$$

$$y_C = \frac{5Wa^2}{8EI} \times 2a - \left(\frac{Wa^2}{4EI} \times \frac{4a}{3}\right) - \frac{3Wa^2}{8EI} \times \frac{4a}{9}$$

$$y_C = \frac{5Wa^3}{4EI} - \frac{Wa^3}{3EI} - \frac{Wa^3}{6EI}$$

$$y_C = \frac{30Wa^3 - 8Wa^3 - 4Wa^3}{24EI} = \frac{18Wa^3}{24EI} = \frac{3}{4}\frac{Wa^3}{EI}$$

$$\boxed{y_C = \frac{3Wa^3}{4EI}}$$

Example 7.32

Using the conjugate-beam method, determine the deflection and the slope at the free end of a cantilever of span l carrying a point load W at a distance l_1 from the fixed end ($l_1 < l$).

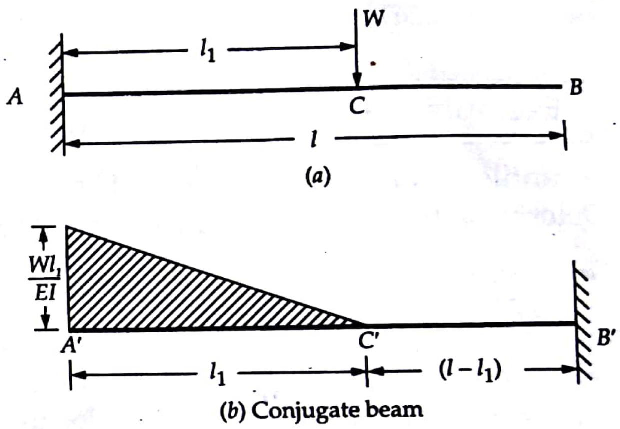


Fig. 7.61

Solution:

Figure 7.61(b) shows the conjugate beam $A'B'$ corresponding to the given cantilever AB [Fig. 7.61(a)]. Slope at the free end B of the cantilever AB = S.F. at fixed end B' of the conjugate beam.

$$\therefore \qquad \theta_B = \frac{1}{2} \times \frac{Wl_1}{EI} \times l_1 = \frac{Wl_1^2}{2EI}$$

Deflection at the free end B of the cantilever AB = B.M. at B' for the conjugate beam.

$$\therefore \qquad y_B = \left(\frac{1}{2} \times \frac{Wl_1}{EI} \times l_1\right) \times \left(l - \frac{l_1}{3}\right)$$

or

$$= \frac{Wl_1^2}{6EI}(3l - l_1)$$

Example 7.33

A 200 cm long cantilever carries a load of 3 kN at a distance of 100 cm from the fixed end and a load of 2 kN at the free end. Determine the deflection at the free end.

Take $E = 20 \times 10^6$ N/cm^2; $I = 1500$ cm^2

Solution:

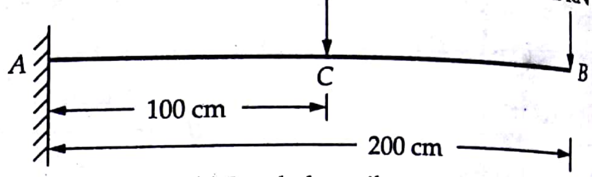
Deflection at B (Fig. 7.62) will be sum of the deflections caused individually by the two loads of 2.0 kN and 3.0 kN.

B.M. at A due to the 2.0 kN load = M_1 = $(2.0 \times 10^3) \times 200 = 4.0 \times 10^5$ Ncm

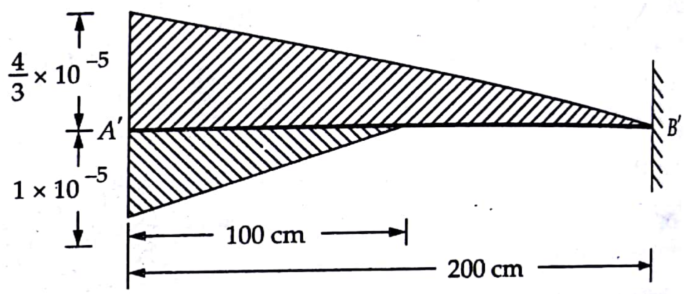
B.M. at A due to the 3.0 kN load = M_2 = $(3.0 \times 10^3) \times 100 = 3.0 \times 10^5$ Ncm

$$\frac{M_1}{EI} = \frac{4 \times 10^5}{20 \times 10^6 \times 1500} = \frac{4}{3} \times 10^{-5}$$

$$\frac{M_2}{EI} = \frac{3 \times 10^5}{20 \times 10^6 \times 1500} = 1 \times 10^{-5}$$



(a) Loaded cantilever



(b) Conjugate beam

Fig. 7.62

The corresponding conjugate beam is shown in Fig. 7.62(*b*). Deflection at the free end B is represented by the B.M. at B' in case of conjugate beam.

$$\therefore \quad y_B = \left[\left(\frac{1}{2} \times \frac{4}{3} \times 10^{-5} \times 200\right) \times \left(\frac{2}{3} \times 200\right)\right]$$

$$+ \left[\left(\frac{1}{2} \times 1 \times 10^{-5} \times 100\right) \times \left(100 + \frac{2}{3} \times 100\right)\right]$$

or $\quad$ **= 0.261 cm**