UNIT-3 (statically indeterminate beams)

PROPPED CANTILEVERS & BEAMS

Propped cantilever beam: An additional support is provided at free end In a cantilever beam for avoiding excess deflections & reduce the value of Bending moment.

And the structure has changed into statically indeterminate.

static equ of equilibrium can't be analysed of this structures i.e statically indeterminate structure.

There two methods to analyse such beams

(i) compatability condition of deflection: (consistent deformation)

consider a cantilever AB carrying UDL at the rate w/unit length. The cantilever is propped at B. there are three unknowns R_A & R_B fixing moment M_A. Apart from the two equ of statics, we use the condition of deflection.

(i) Assume prop is removed ~~at B~~ & cantilever is allowed to deflect. deflection y_1 suffered by free end can be found for any type of loading.

(ii) Assume that the loads from the cantilever have been removed & the cantilever subject to prop reaction R_B. upward deflection y_2 caused by the reaction.

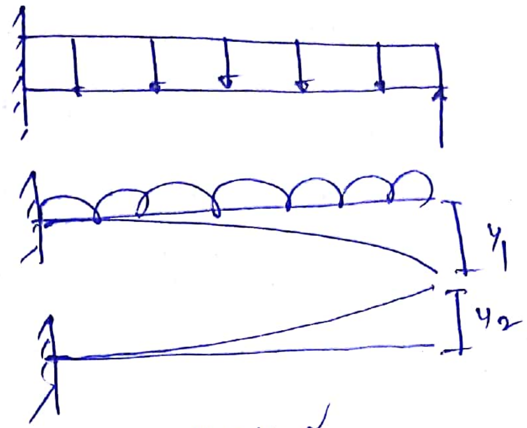
Now if the level of prop & the fixed end A be the same

$$y_1 - y_2 = 0$$

If, however, the prop sinks by an amount $\bar{y}$

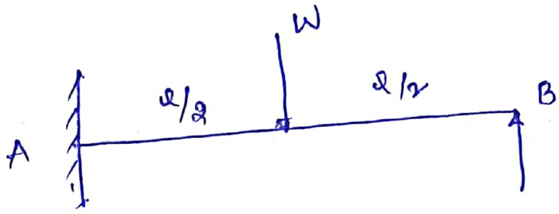
$$y_1 - y_2 = \bar{y}$$

From this relation, the value of the prop reaction R_B can be determined & then rest of unknowns can be determined.

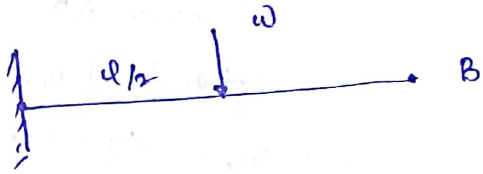


$y_1 = y_2$ ✓

propped Cantilever carrying a load w at mid span.



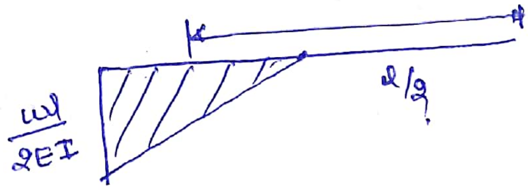
step(1): Assume prop at B is removed.



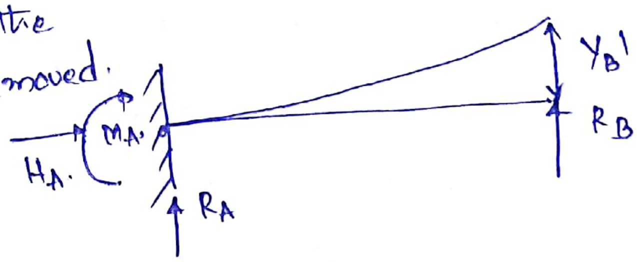
$$Y_B = \frac{1}{2}\cdot\frac{wl}{2EI}\cdot\frac{l}{2}\cdot\left(\frac{l}{2} + \frac{2}{3}\cdot\frac{l}{2}\right)$$

$$= \frac{wl^2}{8EI}\left(\frac{3l+2l}{6}\right)$$

$$Y_B = \frac{5wl^3}{48EI}$$



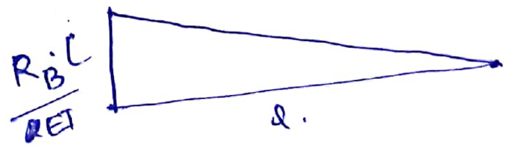
step(2): Assume loads on the propped cantilever is removed.



upward deflection

$$Y_B' = \frac{1}{\cancel{2}}\cdot\frac{R_B L}{\cancel{2}EI}\cdot l\cdot\frac{\cancel{2}l}{3}$$

$$Y_B' = \frac{R_B l^3}{3EI}$$



$$Y_B = Y_B'$$

$$\frac{5wl^3}{\quad} = \frac{R_B l^3}{3EI}$$

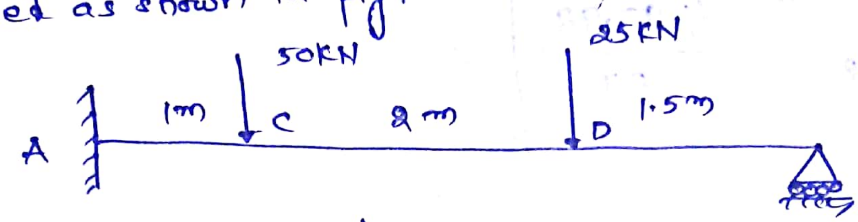
$$\boxed{R_B = \frac{5W}{16}}$$

$$R_A = W - \frac{5W}{16}$$

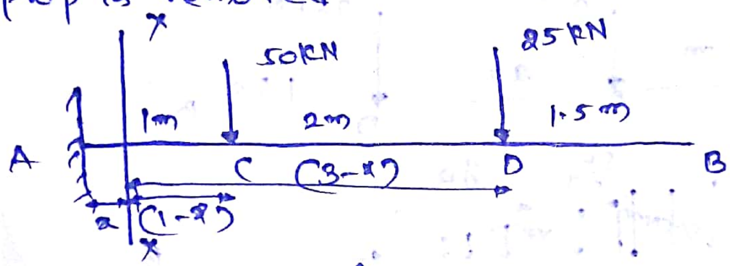
$$\boxed{R_A = \frac{11W}{16}}$$

Exercize problems:

1. Draw S.F & B.M Ds of the beam of span 4.5m supported & loaded as shown in fig.



step(i): Prop is removed



$$EI\frac{d^2y}{dx^2} = -M_x \Rightarrow EI\frac{d^2y}{dx^2} = +25(3-x)+50(1-x) \qquad M_x = -25(3-x)-50(1-x)$$

$M_A = -75-50$
$= -125\ kNm$
$M_C = -50\ kNm$
$M_D = 0$

put $$EI\frac{dy}{dx} = 25\frac{(3-x)^2}{2}\cdot\frac{1}{-1} + 50\frac{(1-x)^2}{2}\cdot\frac{1}{-1} + C_1$$

$$EIy = \frac{25}{2}\cdot\frac{(3-x)^3}{3} + \frac{50}{2}\cdot\frac{(1-x)^3}{3} + C_1x + C_2$$

put $x=0$; $\frac{dy}{dx} = 0$

$$0 = -\frac{25}{2}3^2 - \frac{50}{2}\cdot 1^2 + C_1$$

$$\Rightarrow C_1 = 112.5 + 25 = 137.5$$

$$\boxed{C_1 = 137.5}$$

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put $x = 0;\ y = 0$

$$0 = \frac{25}{6}3^3 + \frac{50}{6}1^3 + 0 + C_2$$

$$\Rightarrow \boxed{C_2 = -120.83}$$

Therefore, The deflection equ is

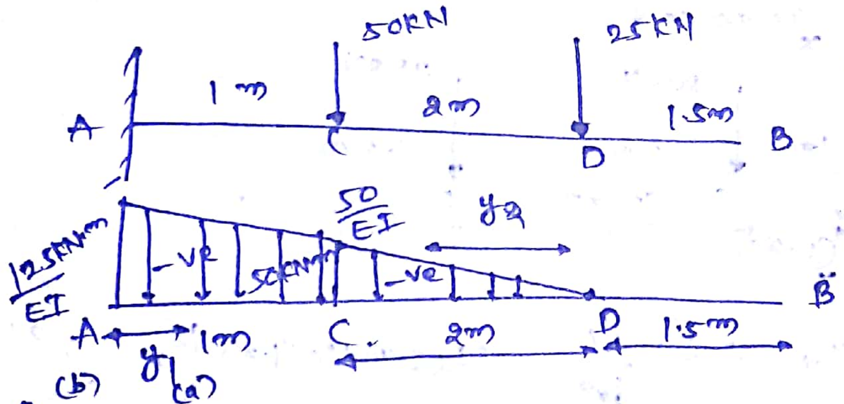
$$EIy = \frac{25}{6}(3-x)^3 + \frac{50}{6}(1-x)^3 + 137.5x - 120.83$$

put $x = 4.5\,m$ (-)ve terms have to be neglected

$$EIy_B = 137.5 \times 4.5 - 120.833$$

$$\boxed{y_B = \frac{497.92}{EI}}$$ ✓

Check:



$$A_1 = \left[\frac{\frac{125}{EI} + \frac{50}{EI}}{2}\right]1 \ ; \quad y_1 = \left[\frac{2a+b}{a+b}\right]\frac{b}{3} = \cancel{2\times\frac{125}{EI} + \frac{50}{E}}$$

$$= \frac{175}{2EI} \qquad = \left[\frac{2\times\frac{50}{EI} + \frac{125}{EI}}{\frac{50}{EI} + \frac{125}{EI}}\right]\cdot\frac{1}{3}$$

$$= \frac{87.5}{EI} \checkmark \qquad = \left(\frac{100+125}{50+125}\right)\frac{1}{3} = \frac{3}{7}$$

$$A_2 = \frac{1}{\cancel{2}}\cdot\frac{50}{EI}\times\cancel{2} \qquad y_2 = \frac{2}{3}\times 2 = \frac{4}{3}$$

Taking moments about B

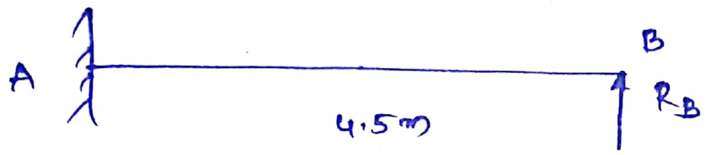
$$y_B = \frac{87.5}{EI}\left[4.5 - 0.43\right] + \frac{50}{EI}\left[1.5 + 1.33\right]$$

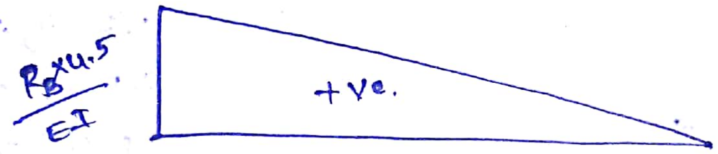
$$y_B = \frac{356.125}{EI} + \frac{141.5}{EI}$$

$$\boxed{y_B = \frac{497.68}{EI}}$$

$\therefore$ Hence proved

step(2):





$\frac{M}{EI}$ diagram

$$y_B' = A\bar{y}$$

$$= \frac{1}{\cancel{2}} \times \frac{4.5 R_B}{EI} \times 4.5 \times \frac{\cancel{2}}{3} \times 4.5$$

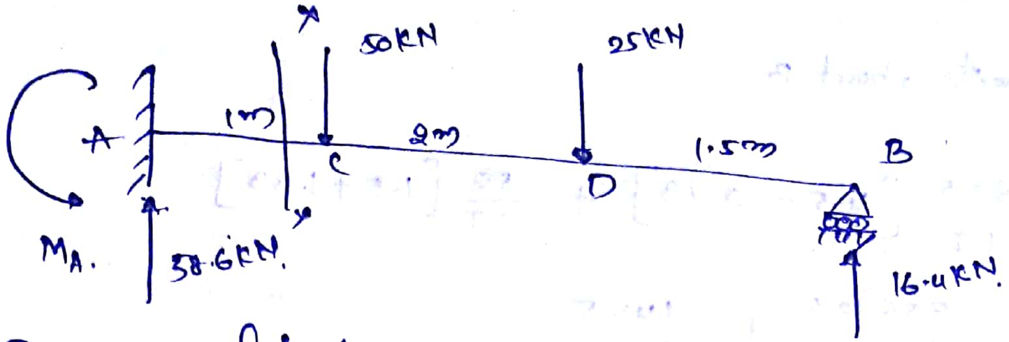
$$= \frac{R_B \times 4.5^3}{3EI}$$

step(3): Downward deflection = Upward deflection

$$\frac{497.98}{\cancel{EI}} = \frac{R_B (4.5)^3}{3\cancel{EI}}$$

$$\Rightarrow R_B = \underline{16.39} \quad kN.$$

$$\boxed{R_B = 16.4 \text{ kN}}$$



R_A = Total load − 16.4

= 75 − 16.4 = 58.6 KN

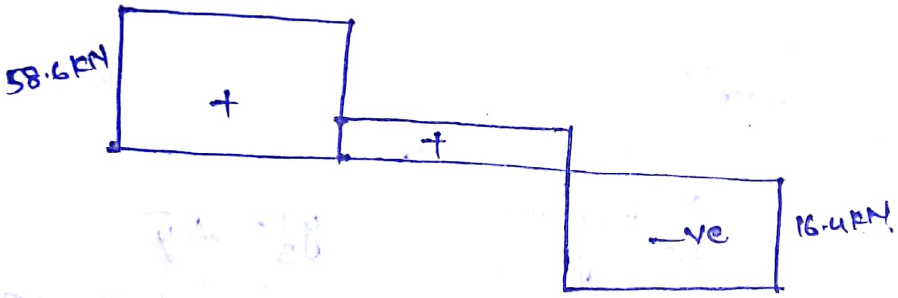
Taking moments about 'A'

$$-M_A + (50\times1) + (25\times3) - (16.4\times4.5) = 0$$

$$M_A = 50+75-73.8$$

$$\boxed{M_A = 51.2 \text{ KN.m}}$$

S.F.D:



S.F @ A+ = 58.6KN

S.F @ C− = 58.6KN

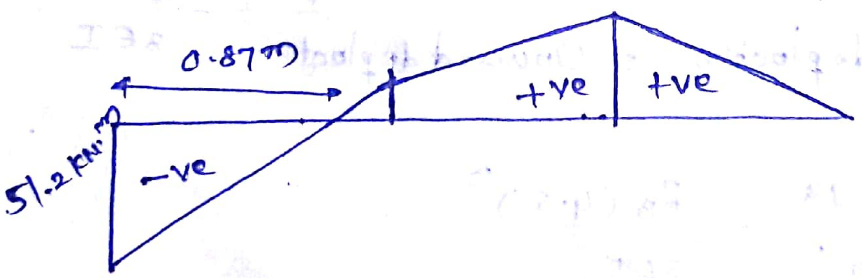
S.F @ C+ = 58.6 − 50 = 8.6KN

S.F @ D− = 8.6KN

S.F @ D+ = 8.6 − 25 = −16.4 KN

S.F @ B− = −16.4KN

B.M.D:



B.M @ A = −51.2KN.m

B.M @ C = −51.2 + (58.6 × 1) = +7.4 KN-m

B.M @ D = −51.2 + (58.6 × 3) − (50 × 2) = 24.6 KN-m

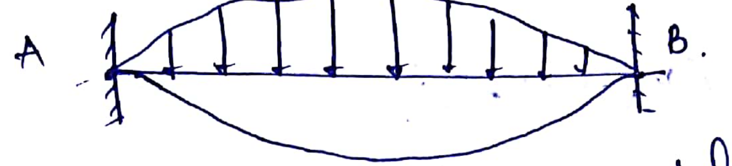
Point of contraflexure can be lie within AC portion. Therefore, B.M at 'x' within AC portion

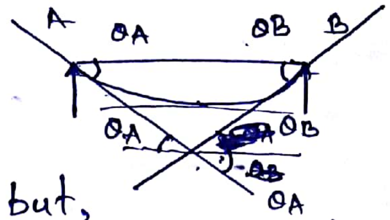
$$58.6x - 51.2 = 0$$

$$\Rightarrow x = \frac{51.2}{58.6} = 0.87\text{m}$$

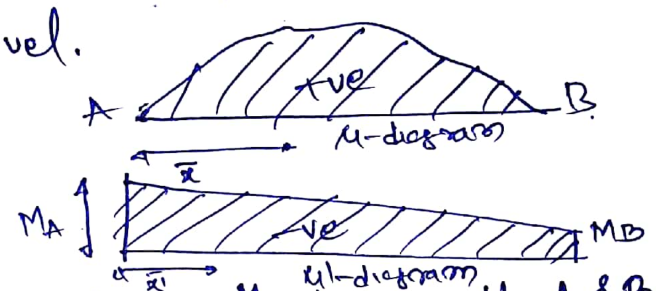
As per the moment area method from theorem-I, the angle b/n tangents to the elastic curve at two points on a straight beam under bending is equal to the Area of $\frac{M}{EI}$ diagram b/n those two points.

Theorem is applied to fixed beam





θ_A & θ_B makes angles w.r.t horizontal but, Tangents are drawn at the points which have horizontal lines & same level.



$\theta_A = \theta_B = 0$

Theorem-I:

$\theta_B - \theta_A$ = Area of $\frac{M}{EI}$ diagram b/n A & B

0 = Area of sagging moment b/n A&B − Area of hogging moment b/n A&B

$\Rightarrow \quad 0 = \mu - \mu'$

$\Rightarrow \quad \mu = \mu'$

$$\boxed{\mu = \frac{M_A + M_B}{2} \times l} \quad (1)$$

μ = Area of $\frac{M}{EI}$ diagram b/n A & B due to external loading.

Theorem-II: The vertical intercept made by the tangents to the elastic curve at A & B on a vertical line through A is equal to the moment of the $\frac{M}{EI}$ diagram about A.

$\therefore$ The intercepts of the two tangents on the vertical through A shall be zero.

$$\therefore \quad \mu\bar{x} - \mu'\bar{x}' = 0$$

$$\mu\bar{x} = \mu'\bar{x}' \qquad \bar{x}' = \frac{l}{3}\left(\frac{M_A + 2M_B}{M_A + M_B}\right)$$

$$\mu\bar{x} = \left(\frac{M_A + M_B}{2}\, l\right)\frac{l}{3}\left(\frac{M_A + 2M_B}{M_A + M_B}\right)$$

$$= (M_A + 2M_B)\frac{l^2}{6}$$

$$\boxed{M_A + 2M_B = \frac{6\mu\bar{x}}{l^2}} \quad \text{--- (2)}$$

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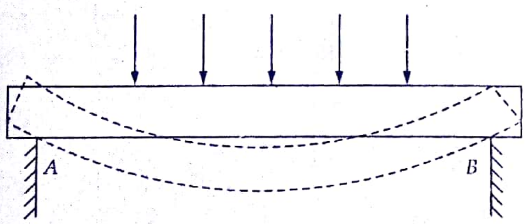
FIXED BEAMS

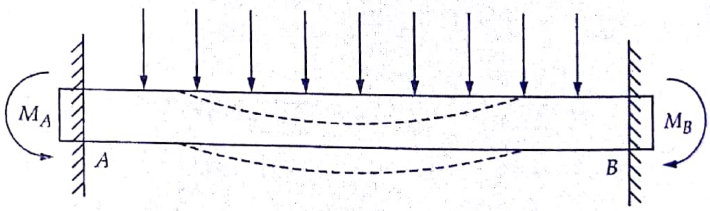
Introduction

Consider a simply supported beam AB carrying any pattern of loading (Fig. 9.1). The beam takes up the dotted position under the effect of the imposed loads and its ends lift up. If, however, the ends of the beam were firmly built in the supports or held firmly down in any other manner, then the beam shall deform in the manner shown by dotted lines in Fig. 9.2. Fixity at ends amounts to the application of moments M_A and M_B at ends A and B respectively in case of the beam considered in Fig. 9.1. The end moments are of such magnitude as to cause zero slope at the two ends. Beams with fixed ends (also called encastre beams or built-in beams) have reduced bending moments and deflections in the beam in contrast to the one of the same span and subjected to the same pattern of loading but simply supported at ends.



Simply supported beam

Fig. 9.1



Fixed beam

Fig. 9.2

End slopes of fixed beams are zero. End moments in case of fixed beams tend to bend the beam with convexity upwards whereas the normal downward loads tend to bend the beam with concavity upwards. The condition of maximum strength will be realized when the maximum hogging moments are equal in magnitude to the maximum sagging moments.

The analysis of the fixed beam may be divided in two stages, diagrammatically clarified by Figs 9.3(b) and 9.3(c).

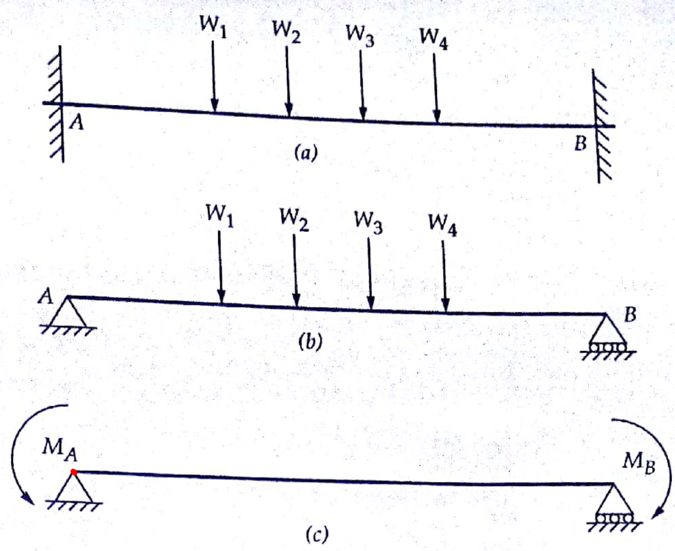


Fig. 9.3

In the first stage [Fig. 9.3(*b*)], the beam may be treated as a simply supported beam carrying the given system of loading. In the second stage [Fig. 9.3(c)], the simply supported beam may be considered under the action of *only* the end couples M_A and M_B without the given loading. Superimpose the effects of two stages of analysis incorporating the end conditions to determine the clamping moments.

9.1 MERITS AND DEMERITS OF FIXED BEAMS OVER SIMPLY SUPPORTED BEAMS (SPANS AND LOADS BEING SAME)

Merits

(*i*) Fixed beams have lesser values for maximum bending moments for same loadings.

(*ii*) Fixed beams have lesser values for maximum deflection for same loadings. In other words, built-in beams are stronger and stiffer, yet built-in beams are, at times, not used because of their following disadvantages.

Demerits

(*i*) A little sinking of one support sets in large stresses.

(*ii*) Extra care has to be taken in aligning supports accurately at the same level.

(*iii*) Even a small sinking of either support sets up large stresses.

(*iv*) Fluctuation of temperature sets up large stresses.

(*v*) Frequent fluctuations in loading, particularly in case of moving loads, renders the degree of fixity at the ends very uncertain. Many of the drawbacks of the fixed beams can, however, be avoided by having two cantilevers at the ends and bridging the gap by hinging a beam to their free end. The hinges are provided at points where points of contraflexure, in case of a fixed beam, would have been.

9.2 MOMENT AREA METHOD

Figure 9.4(*a*) shows a fixed beam carrying any pattern of load. Let M_A and M_B be the fixed end moments (FEM) at the ends *A* and *B* respectively. Figure 9.4(*b*) shows the B.M. diagram for the given beam if it had been simply supported. Let the area of this B.M. diagram be μ and the centroid of the B.M. diagram from A be x. Figure 9.4(c) shows the B.M. diagram for the beam taking into account only the fixed end moments M_A and M_B (ignoring the imposed load). Let the area of the FEM diagram [Fig. 9.4(c)] be μ' and the distance of its centroid from A be $\bar{x}'$.

Since the FEM M_A and M_B cause hogging of the beam, they are $-ve$ and the moments due to the load cause sagging of the beam, they are $+ve$.

By superposing Figs. 9.4(b) and (c), we get the B.M. diagram for the fixed beam [Fig. 9.4(d)].

According to the first moment area theorem, the change in slope between A and B is equal to the area of the M/EI diagram between A and B (or the area of B.M. diagram if EI is constant for the entire span of the beam). But due to end fixity, the slopes at both the ends remain zero, and therefore change in slope between A and B is zero. Hence, area or the B.M. diagram between A and B is zero, *i.e.*, the magnitude of the $+ve$ area (due to load) is equal to the magnitude of the $-ve$ area (due of FEM) or $\mu = \mu'$.

But $$\mu' = \frac{M_A + M_B}{2} \times l$$

$$\therefore \quad \mu = \frac{M_A + M_B}{2} \times l \qquad \text{...(9.1)}$$

Now according to the second 'moment area theorem', the intercept made by tangents to the elastic curve at A and B on a vertical line through A is equal to the moment of the $\frac{M}{EI}$ diagram about A (or the moment of the B.M. diagram about A if EI is constant).

Since the tangents to the elastic curve at A and B coincide (assuming the supports A and B to be at the same level), the intercept of the two tangents on the vertical through A shall be zero.

$$\therefore \quad \mu\bar{x} - \mu'\bar{x}' = 0$$

or $$^*\mu\bar{x} = \mu'\bar{x}'$$

and $$\bar{x}' = \frac{l}{3} \times \left(\frac{M_A + 2M_B}{M_A + M_B}\right)$$

or $$\mu\bar{x} = \left(\frac{M_A + M_B}{2} \times l\right) \times \frac{l}{3} \times \frac{M_A + 2M_B}{M_A + M_B}$$

or $$= (M_A + 2M_B) \times \frac{l^2}{6}$$

$$\therefore \quad M_A + 2M_B = \frac{6\mu\bar{x}}{l^2} \qquad \text{...(9.2)}$$

Knowing the values of μ and $\bar{x}$, we can by using equations 9.1 and 9.2, determine the fixed end moments M_A and M_B.

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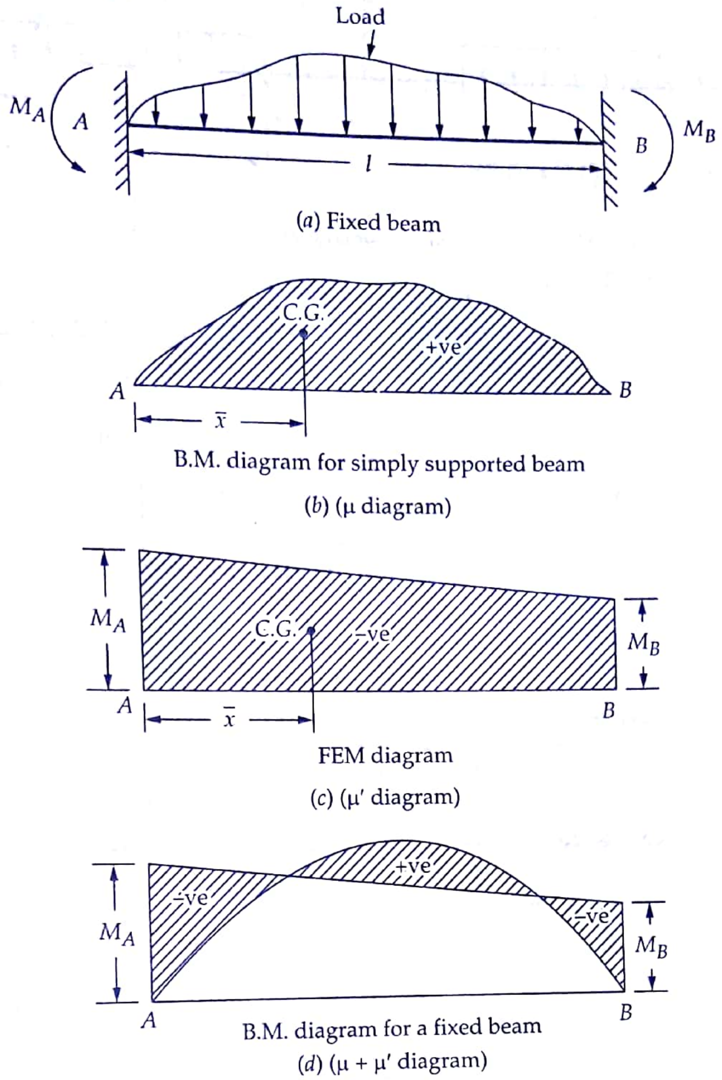
* We have already noticed that $\mu = \mu'$

Therefore, from the relation $\mu\bar{x} = \mu'\bar{x}'$, we conclude that for a fixed beam with end supports at the same level $\bar{x} = x'$, *i.e.*, the distance of the centroid of the B.M.D. from any end in simply supported end conditions, is the same as the distance of the centroid of the FEM diagram from the same end.



(*a*) Fixed beam

B.M. diagram for simply supported beam

(*b*) (μ diagram)

FEM diagram

(*c*) (μ' diagram)

B.M. diagram for a fixed beam

(*d*) ($\mu + \mu'$ diagram)

Fig. 9.4

Now apply conditions of zero slope and zero deflection at $x = l$ in both the slope and the deflection equations. From these two results, the two end moments can be determined. The application shall be abundantly clear from the following few examples.

9.4 POINT LOAD AT MIDSPAN

The B.M. diagrams for a given beam treated as simply supported [Fig. 9.5(*b*)] is positive and the FEM diagram [Fig. 9.5(*c*)] is negative.

The maximum B.M. at midspan [Fig. 9.5(*b*)] is $\frac{Wl}{4}$ and by symmetry, the two FEM's M_A and M_B are both equal.

$$\therefore \qquad M_A = M_B = M$$

Now
$$\mu = \frac{1}{2} \times \frac{Wl}{4} \times l = \frac{Wl^2}{8}$$

$$\mu' = Ml \qquad \text{But } \mu = \mu' \qquad \text{... (9.1)}$$

$$\therefore \qquad \frac{Wl^2}{8} = Ml$$

or
$$M = \frac{Wl}{8}$$

But
$$M_A = M_B = M$$

$$\therefore \qquad M_A = M_B = \frac{Wl}{8}$$

By symmetry, the end reactions R_A and R_B are equal.

$$\therefore \qquad R_A = R_B = W/2$$

To know the slope and the deflection at any point, determine the B.M. at any section in AC at a distance x from A

$$\therefore \qquad M_x = R_A x - M_A$$

or
$$= \frac{Wx}{2} - \frac{Wl}{8}$$

But
$$EI\frac{d^2y}{dx^2} = -M_x$$

or
$$= \frac{-Wx}{2} + \frac{Wl}{8}$$

On integrating successively the above equation twice, we get

$$\therefore \qquad EI\frac{dy}{dx} = \frac{-Wx^2}{4} + \frac{Wlx}{8} + C_1 \qquad \text{... (i)}$$

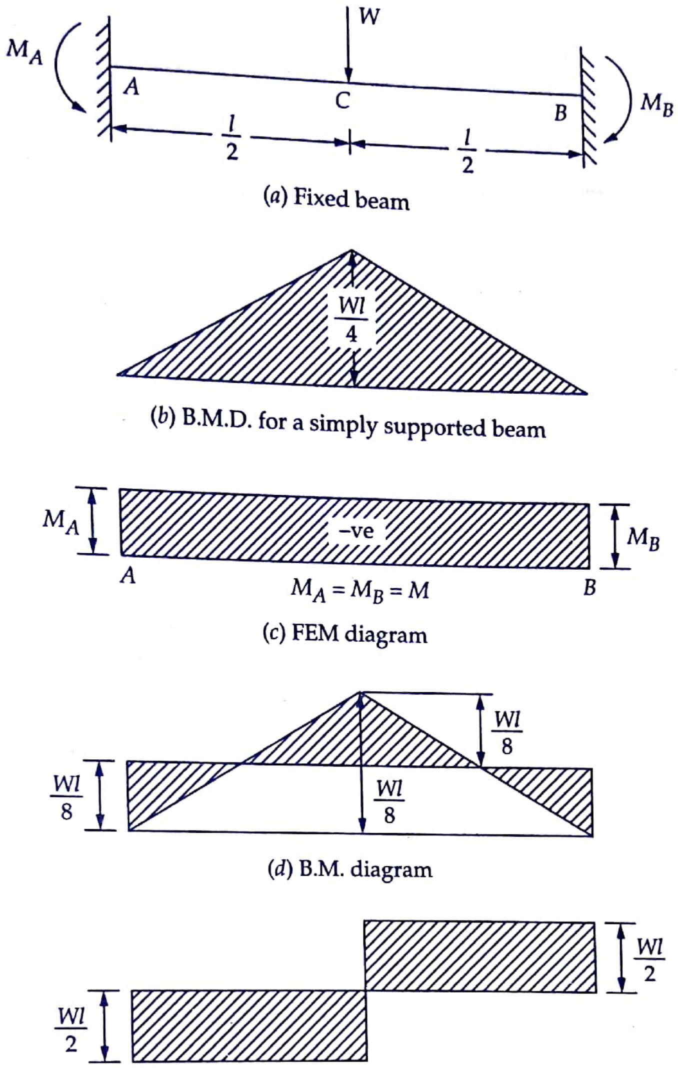
and
$$EIy = \frac{-Wx^3}{12} + \frac{Wlx^2}{16} + C_1 x + C_2 \qquad \text{... (ii)}$$

where C_1 and C_2 are the constants of integration.

Applying end conditions of zero slope and zero deflection at A, *i.e.*, when $x = 0$; $\frac{dy}{dx} = 0$ and $y = 0$ in the above equations, we have

$$C_1 = 0 \quad \text{and} \quad C_2 = 0$$

$$\therefore \qquad EI\frac{dy}{dx} = \frac{-Wx^2}{4} + \frac{Wlx}{8} \quad \text{and} \quad EIy = \frac{-Wx^3}{12} + \frac{Wlx^2}{16}$$



(a) Fixed beam

(b) B.M.D. for a simply supported beam

(c) FEM diagram

(d) B.M. diagram

(e) S.F. diagram

Fig. 9.5

Obviously the deflection is maximum at midspan, *i.e.*, when $x = \frac{l}{2}$

$$\therefore \quad y_{max} = y_C = \frac{1}{EI}\left(-\frac{Wl^3}{96} + \frac{Wl^3}{64}\right) = \frac{Wl^3}{192EI}$$

For the points of contraflexure, $M_x = 0$

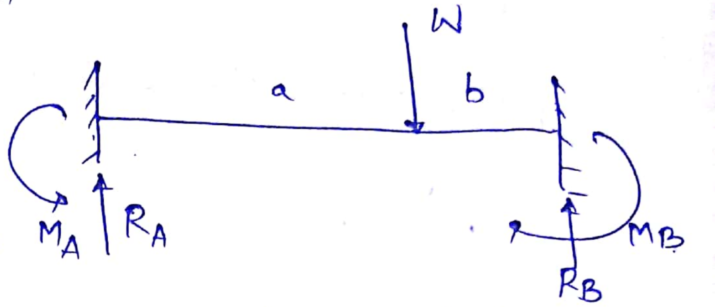
$$\therefore \quad \frac{Wx}{2} - \frac{Wl}{8} = 0$$

$$\therefore \quad x = \frac{l}{4}$$

By symmetry, the two points of contraflexure are at a distance of $\frac{l}{4}$ from either end.

Case(iii): Point load acts eccentrically

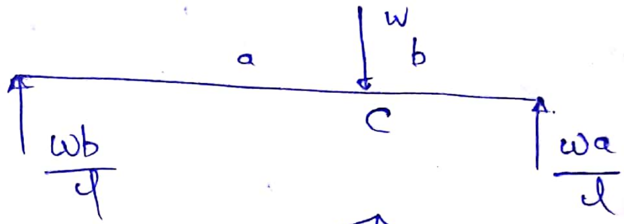
step(i): Fixity is removed & B.M.D is constructed.



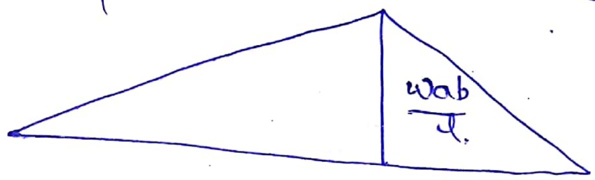
step(ii): Area of B.M.D is calculated

i.e $\mu = \frac{1}{2} \times l \times \frac{wab}{l}$

$$\boxed{\mu = \frac{wab}{2}}$$



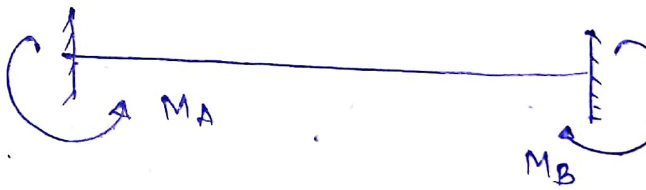
step(iii): Fig shows the FEM diagram which is a Trapezoidal shape since, M_A & M_B are unequal.



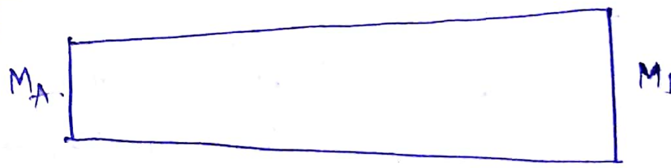
μ diagram

There being no change in slope b/n A & B, we have

$$\mu = \mu'$$



$$\mu = \frac{1}{2} \frac{wab}{l} \times l = \frac{wab}{2}$$



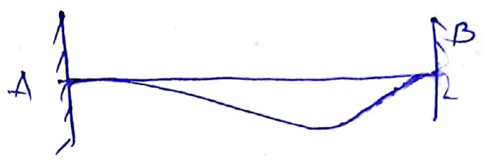
$$\mu' = \frac{M_A + M_B}{2} \times l$$

Therefore $\mu = \mu'$ (No change in slope b/n A & B)

$$\Rightarrow \frac{Wab}{\cancel{2}} = \left(\frac{M_A + M_B}{\cancel{2}}\right) l$$

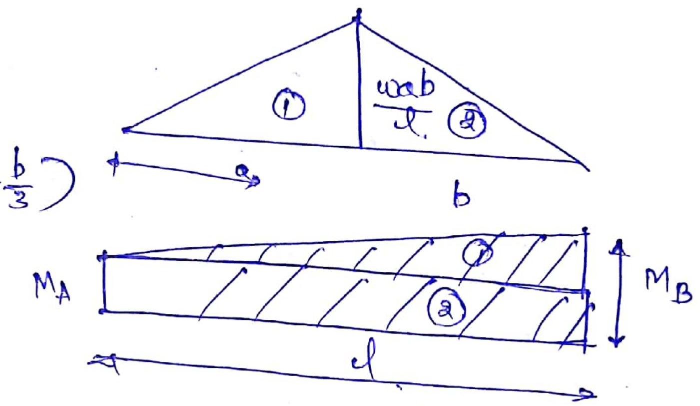
$$\Rightarrow \quad M_A + M_B = \frac{wab}{l} \quad \text{--- (1)}$$

The supports A & B are at the same level, the tangents to the elastic curve at A & B coincide & their intercept on the vertical line through A is zero.



Taking moments about A

$$\mu\bar{x} - \mu'\bar{x}' = 0$$



$$\mu\bar{x} = \frac{1}{\cancel{2}} \cdot \frac{wab}{l} \times a \times \frac{\cancel{2}}{3} a + \frac{1}{2} \cdot \frac{wab}{l} \times b \times \left(a + \frac{b}{3}\right)$$

$$= \frac{Wa^3b}{3l} + \frac{wab^2}{2l}\left(a + \frac{b}{3}\right)$$

$$= \frac{wa^3b}{3l} + \frac{Wa^2b^2}{2l} + \frac{Wab^3}{6l} = \frac{2wa^3b + 3wa^2b^2 + wab^3}{6l}$$

$$\mu'\bar{x}' = (M_A \times l) \times \frac{l}{2} + \frac{1}{\cancel{2}} \times (M_B - M_A) \times l \times \frac{\cancel{2}}{3} l$$

$$= \frac{M_A l^2}{2} + \frac{M_B l^2}{3} - \frac{M_A l^2}{3}$$

$$= \frac{M_A l^2}{6} + \frac{M_B l^2}{3} = \frac{M_A l^2 + 2M_B l^2}{6}$$

$$\mu\bar{x} - \mu'\bar{x}' = 0 \Rightarrow \frac{wab}{6l}\left[2a^2 + 3ab + b^2\right] = \frac{(M_A + 2M_B) l^2}{6}$$

$$\Rightarrow \frac{Wab}{\cancel{6}l}\left(2a^2+2ab+ab+b^2\right) = \frac{(M_A+2M_B)l^2}{\cancel{6}}$$

$$\Rightarrow \frac{Wab}{l^3}\left[2a(a+b)+b(a+b)\right] = (M_A+2M_B)$$

$$\Rightarrow \frac{Wab}{\cancel{l^3}_{\,l^2}}\left[(2a+b)\cancel{l}\right] = M_A+2M_B$$

$$\Rightarrow \frac{Wab\,(l+a)}{l^2} = M_A+2M_B \quad (2)$$

eq (2) − eq (1) ⇒

$$M_B = \frac{Wab\,(l+a)}{l^2} - \frac{Wab}{l}$$

$$= \frac{Wab(l+a) - Wabl}{l^2}$$

$$= \frac{Wab}{l^2}\left[\cancel{l}+a-\cancel{l}\right]$$

$$\boxed{M_B = \frac{Wa^2b}{l^2}}$$

From eq (1)

$$M_A = \frac{Wab}{l} - \frac{Wa^2b}{l^2}$$

$$= \frac{Wabl - Wa^2b}{l^2}$$

$$= \frac{Wab\,(l-a)}{l^2}$$

$$\boxed{M_A = \frac{Wab^2}{l^2}}$$

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