CO
Unit-2

Part-1

Gate-Level Minimization

Why Logic Minimization ?

- Minimize the number of gates used
  - Reduce gate count = reduce cost
- Minimize total delay (critical path delay)
  - Reduce delay = improve performance
- Satisfy design constrains
  - Maximum fanins and fanouts, …
- Remove undesired circuit behavior
  - Hazard, race, …

THE MAP METHOD

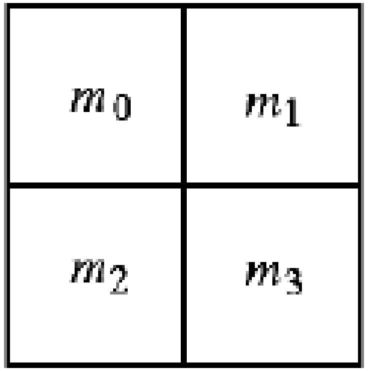
- The Boolean functions also can be simplified by map method as Karnaugh map or K-map.
- The map is made up of squares, with each square representing one minterm of the function.
- This produces a circuit diagram with a minimum number of gates and the minimum number of inputs to the gate.
- It is sometimes possible to find two or more expressions that satisfy the minimization criteria.

The Map Method

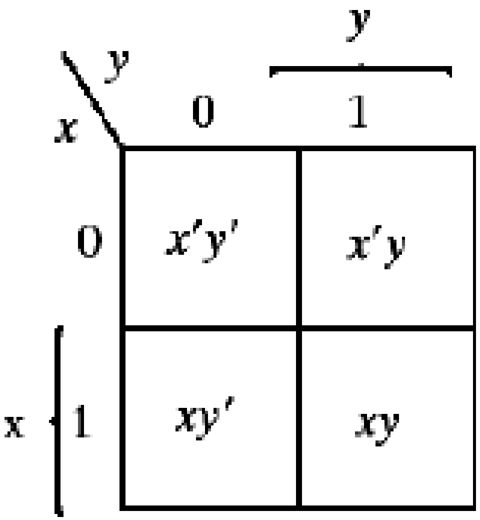
- The map method is also known as the **Karnaugh map** or **K-map**
- Provide a straightforward procedure for minimizing Boolean functions
- The simplified expressions are always in one of the two standard forms:
  - Sum of Products (SOP)
  - Product of Sums (POS)

Two-Variable Map (1/2)

- Two-variable function has **four** minterms
  - Four squares in the map for those minterms
- The corresponding minterm of each square is determined by the bit status shown outside



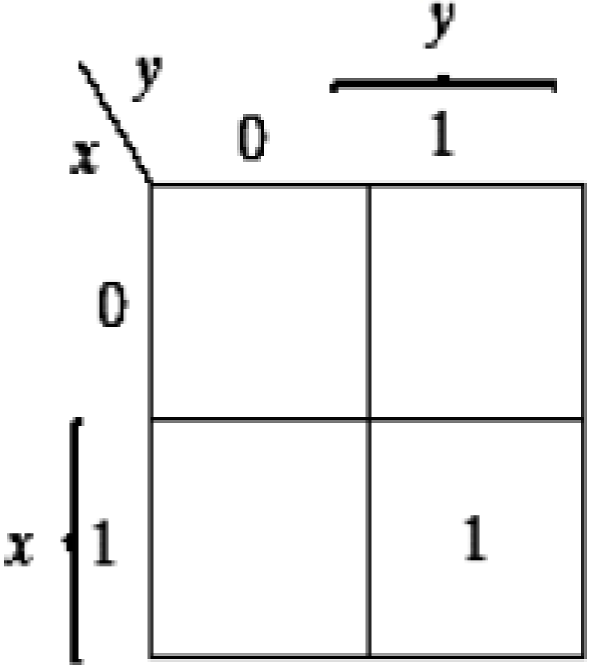
(a)



(b)

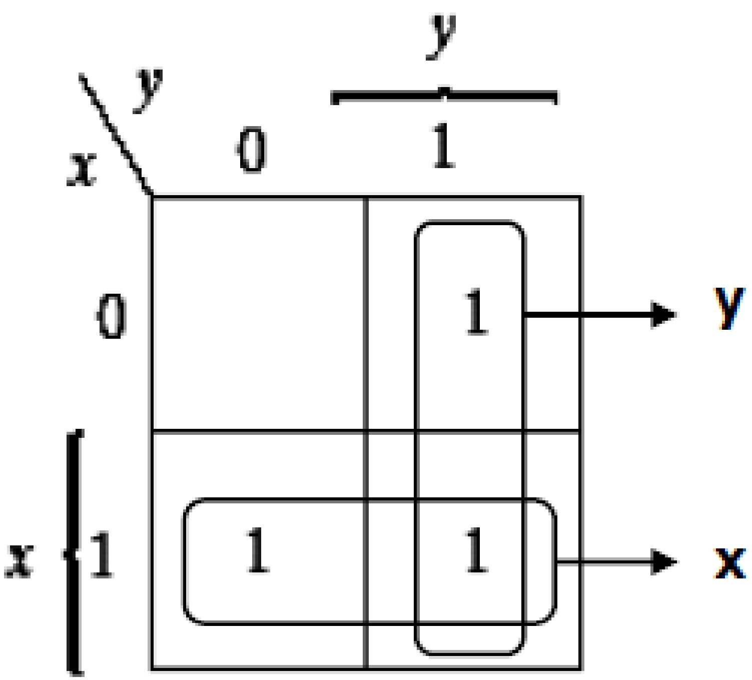
Two-Variable Map (2/2)

Ex-1 $F(x,y)= \sum(3)$



(a) xy

Ex-2 $F(x,y) = \sum(1,2,3)$

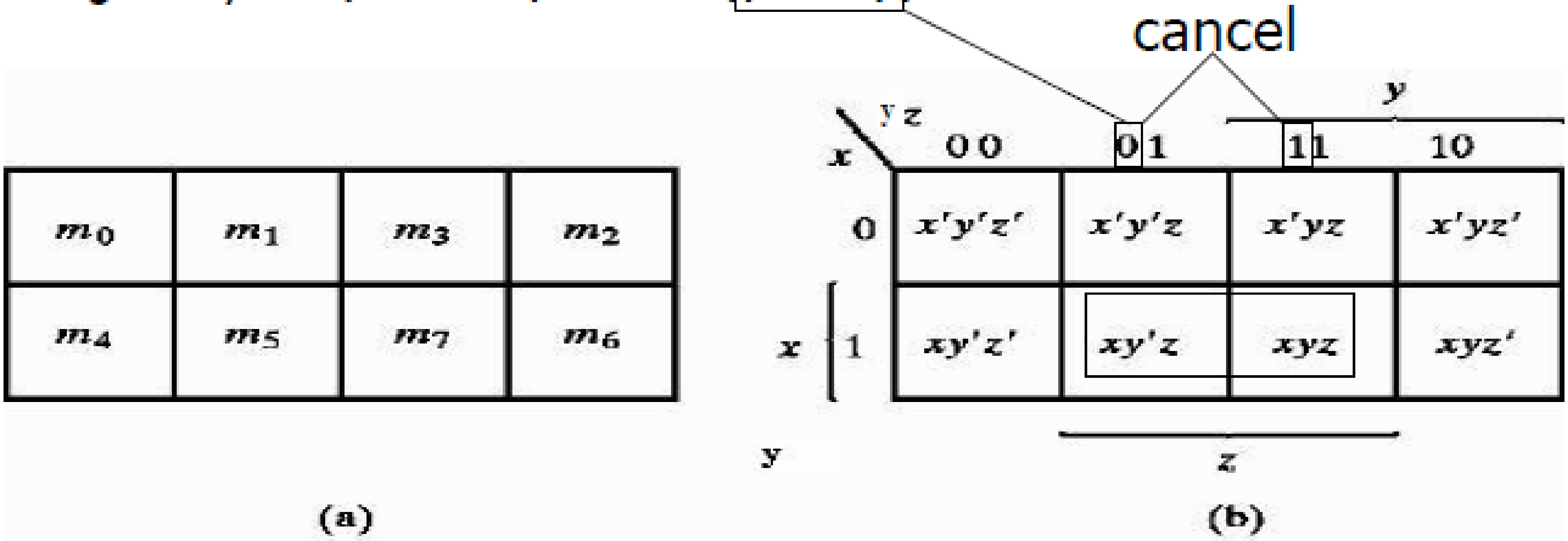


(b) $x + y$

$$
\begin{aligned}
x + y &= x(y+y') + y(x+x') \\
&= xy\ (m_3) + xy'\ (m_2) + x'y\ (m_1) \\
&= m_1 + m_2 + m_3
\end{aligned}
$$

Three-Variable map

- Note that the minterms are not arranged in a binary sequence, but similar to the Gray code.
- For simplifying Boolean functions, we must recognize the basic property possessed by adjacent squares.
- $m_5+m_7= xy'z + xyz = xz(\boxed{y' + y}) = xz$



cancel

m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6

(a)

x \ yz	00	01	11	10
0	$x'y'z'$	$x'y'z$	$x'yz$	$x'yz'$
1	$xy'z'$	$xy'z$	xyz	xyz'

(b)

Fig. 3-3 Three-variable Map

Simplification of the number of adjacent squares

- A larger number of adjacent squares are combined, we obtain a product term with fewer literals.

  1 square = 1 minterm = three literals.

  2 adjacent squares = 1 term = two literals.

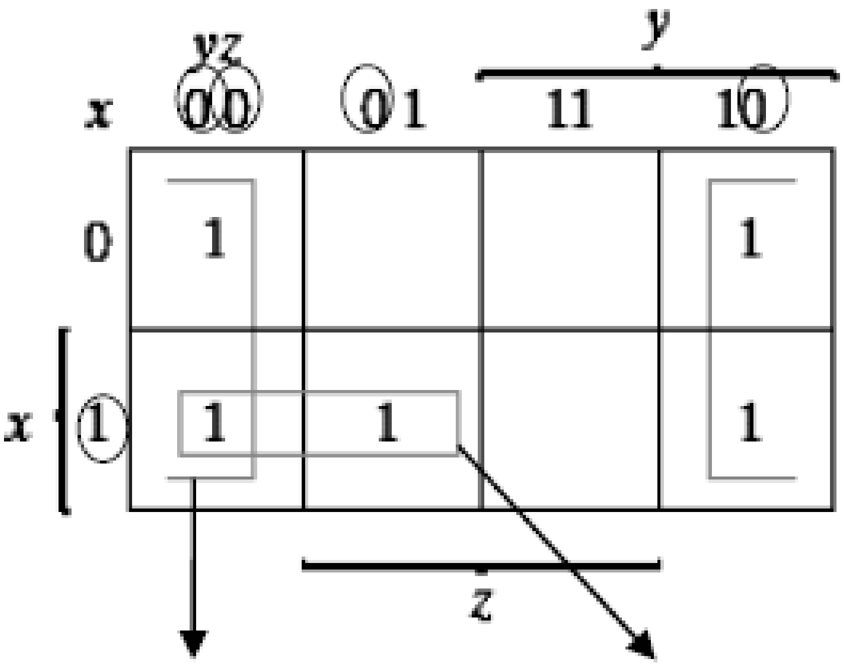
  4 adjacent squares = 1 term = one literal.

  8 adjacent squares encompass the entire map and produce a function that is always equal to 1.

- It is obviously to know the number of adjacent squares is combined in a power of two such as 1,2,4, and 8.

Example

Ex. 3-3 F(x, y, z) = ? (0, 2, 4, 5, 6)



z′
(x, y covered)

xy′
(all z covered)

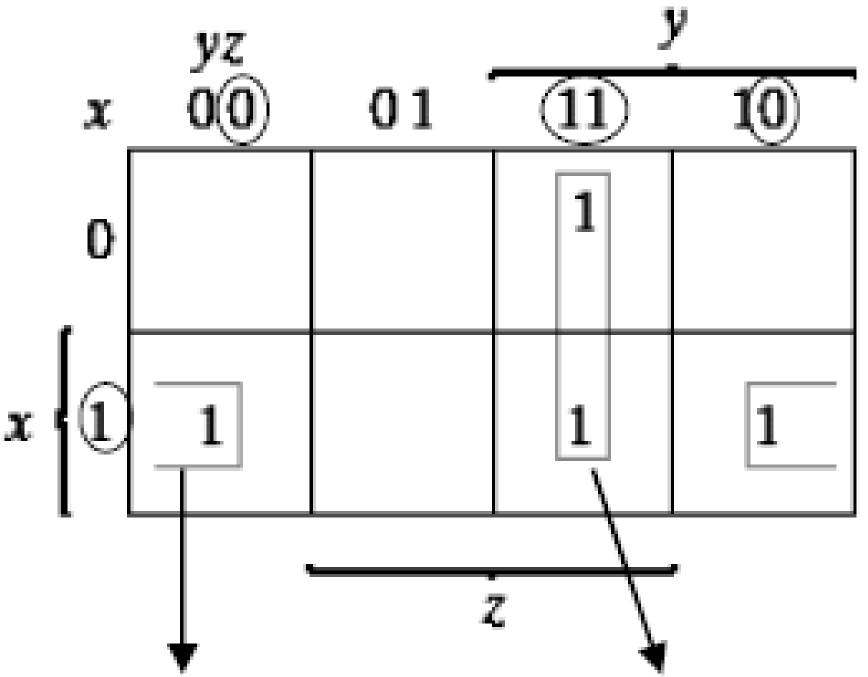
Fig. 3-6 Map for Example 3-3; $F(x, y, z) = \Sigma(0, 2, 4, 5, 6) = z' + xy'$

Example -2

$$F(x,y,z)=\sum(3,4,6,7)$$

Example-2

$$F(x,y,z)=\sum(3,4,6,7)=yz+xz'$$



xz′
(all y covered)

yz
(all x covered)

3-2. Four-variable map

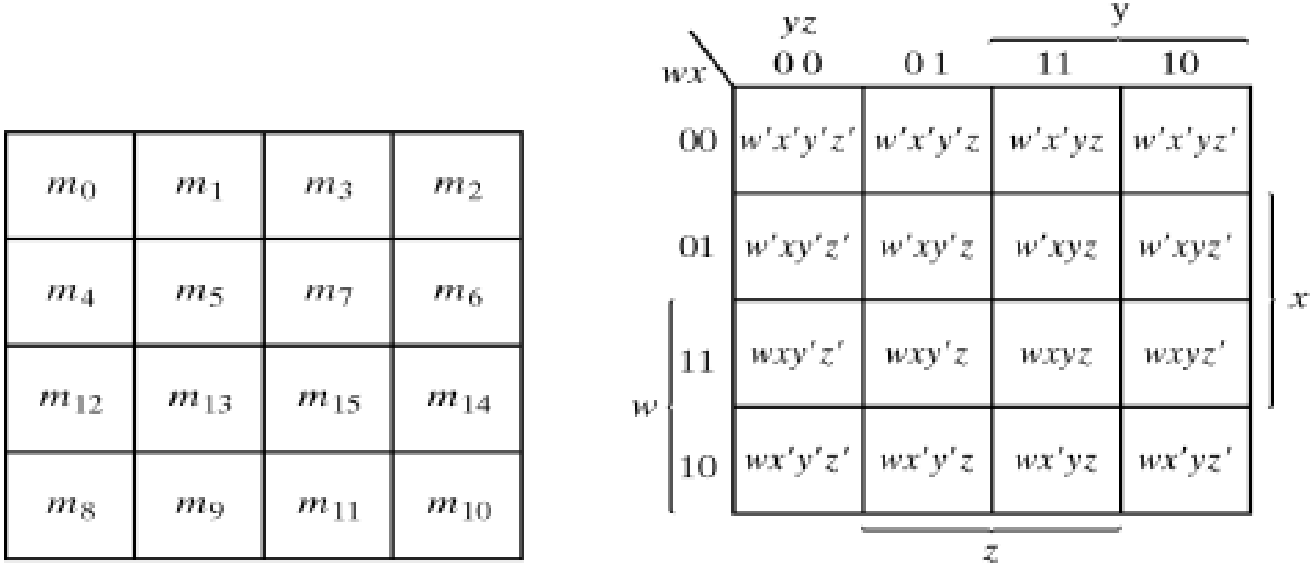
1 square = 1 minterm = 4 literals

2 adjacent squares = 1 term = 3 literals

4 adjacent squares = 1 term = 2 literals

8 adjacent squares = 1 term = 1 literal

16 adjacent squares = 1



m_0	m_1	m_3	m_2
m_4	m_5	m_7	m_6
m_{12}	m_{13}	m_{15}	m_{14}
m_8	m_9	m_{11}	m_{10}

(a)

wx \ yz	00	01	11	10
00	$w'x'y'z'$	$w'x'y'z$	$w'x'yz$	$w'x'yz'$
01	$w'xy'z'$	$w'xy'z$	$w'xyz$	$w'xyz'$
11	$wxy'z'$	$wxy'z$	$wxyz$	$wxyz'$
10	$wx'y'z'$	$wx'y'z$	$wx'yz$	$wx'yz'$

(b)

Fig. 3-8 Four-variable Map

Example $\sum(0,1,2,6,8,9,10)$

Ex. 3-6 $F = \boxed{A'B'C'} + \boxed{B'CD'} + \boxed{A'BCD'} + \boxed{AB'C'}$

$= \boxed{B'D'} + \boxed{B'C'} + \boxed{A'CD'}$

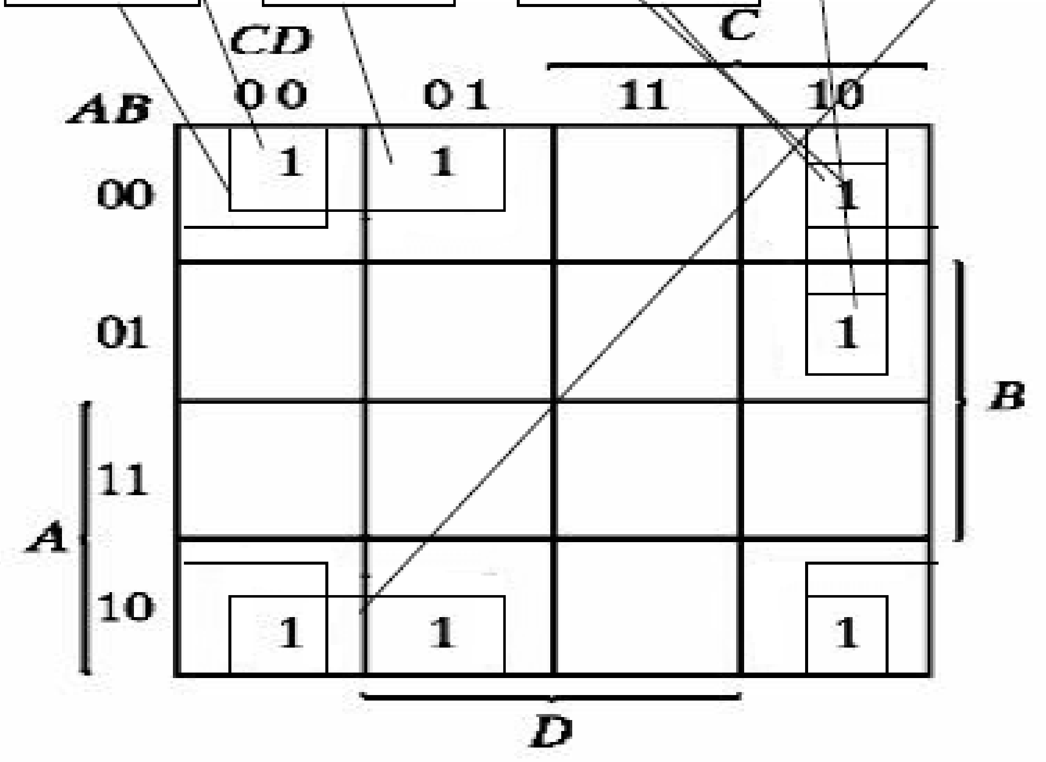
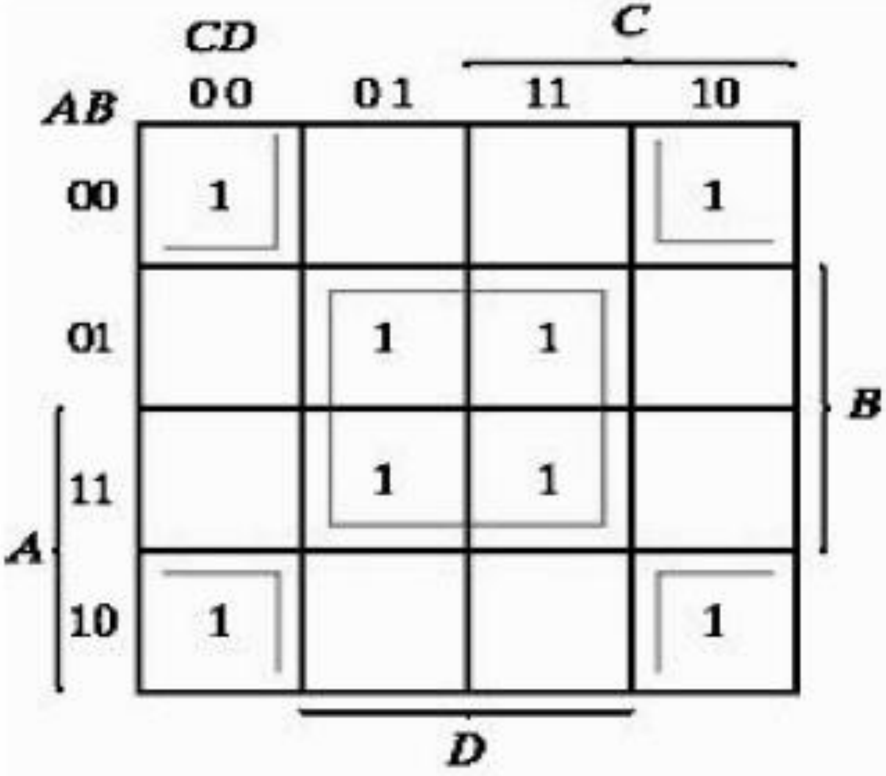


Fig.3-10 Map for Example 3-6; $A'B'C + B'CD' + A'BCD' + AB'C' = B'D' + B'C' + A'CD'$

Essential prime implicants

- If a minterm in a square is covered by only one prime implicant, that the prime implicant is said to be essential.



(a) Essential prime implicants BD and B′D′

Fig. 3-11 Simplification Using Prime Implicants

Prime implicant

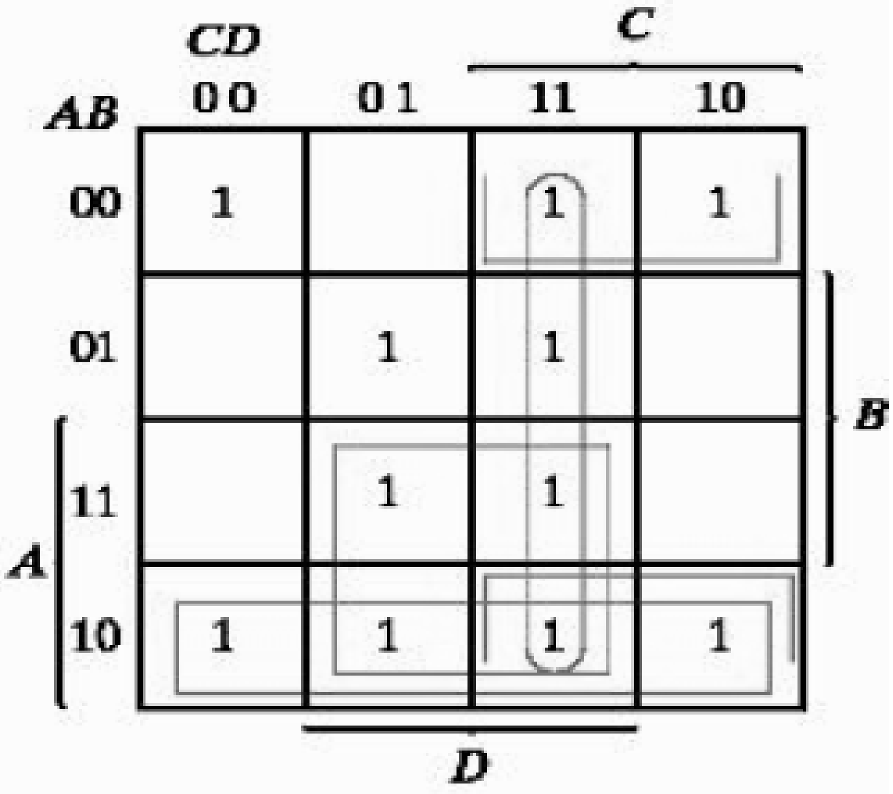
- A prime implicant is a product term obtained by combining the maximum possible number of adjacent squares in the map.
- This shows all possible ways that the three minterms(m_3,m_9,m_{11}) can be covered with prime implicants.

$F = BD+B'D'+CD+AD$

$= BD+B'D'+CD+AB'$

$= BD+B'D'+B'C+AD$

$= BD+B'D'+B'C+AB'$



AB \ CD	00	01	11	10
00	1		1	1
01		1	1	
11		1	1	
10	1	1	1	1

(b) Prime implicants CD, B'C AD, and AB'

Fig. 3-11 Simplification Using Prime Implicants

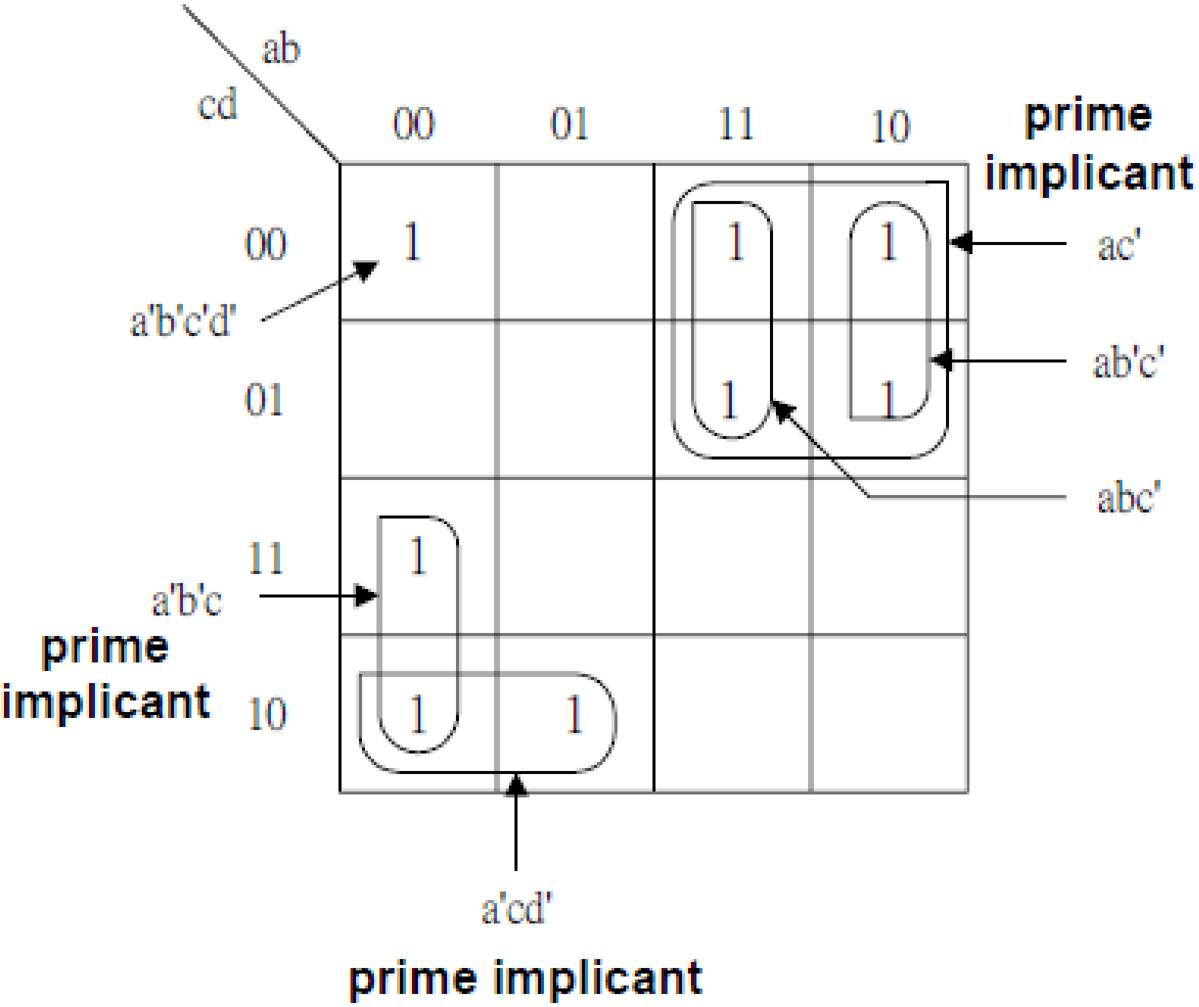
Prime Implicants

- Implicant (cube) :

  A group of minterms that form a cube

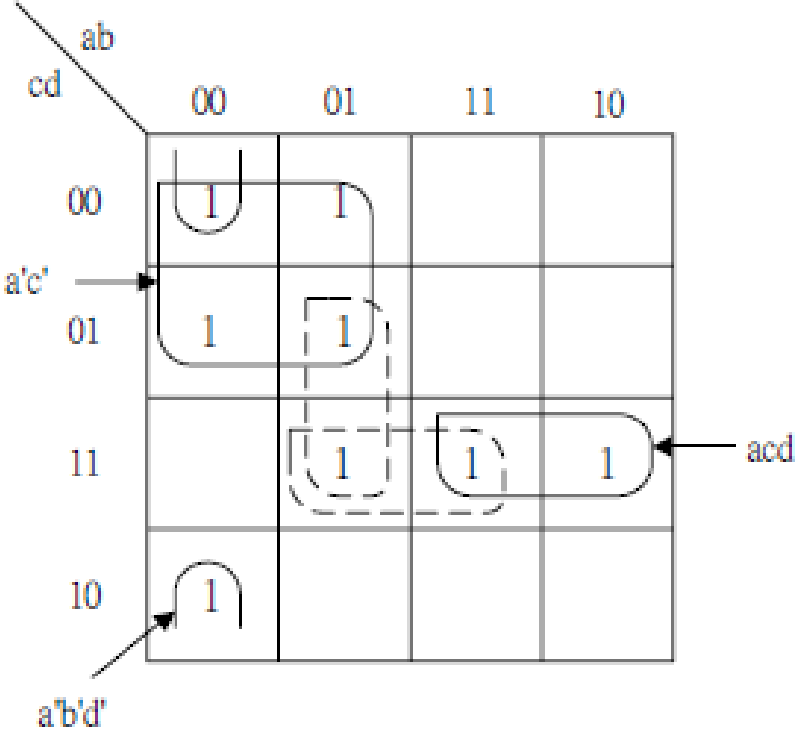
- Prime implicant :

  Combine maximum possible number of adjacent squares in the map



Essential Prime Implicants

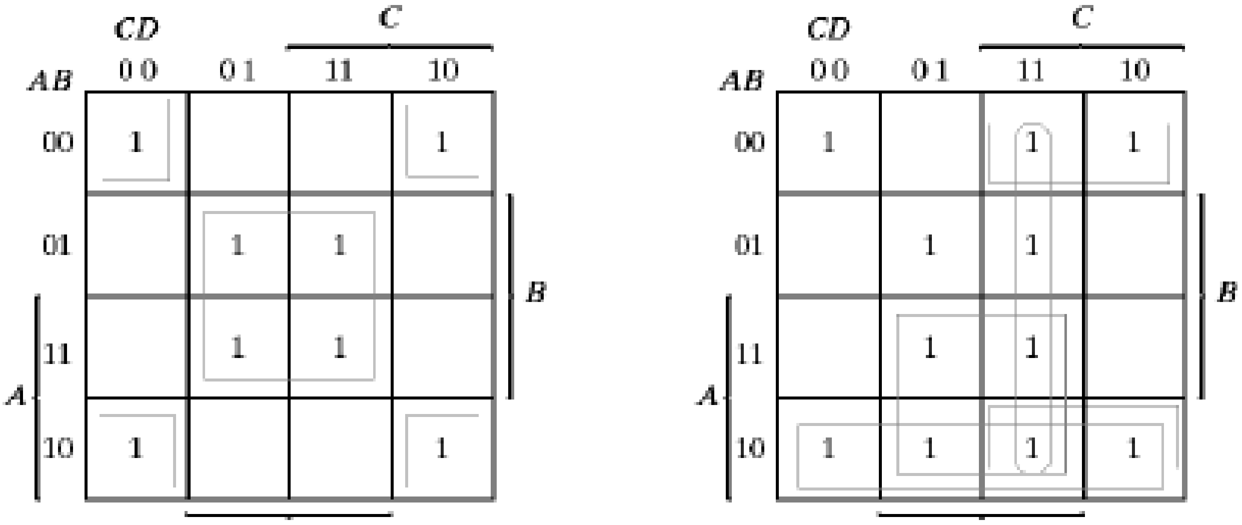
- If a minterm is covered by only one prime implicant, that prime implicant is **essential** and must be included



Note: 1's in red color are covered by only one prime implicant. All other 1's are covered by at least two prime implicants

Systematic Simplification

- Identify all prime implicants on the k-map
- Select all essential prime implicants
- Select a minimum subset of the remaining prime implicants that cover all 1's
- Ex: $F(A,B,C,D)=\sum(0,2,3,5,7,8,9,10,11,13,15)$

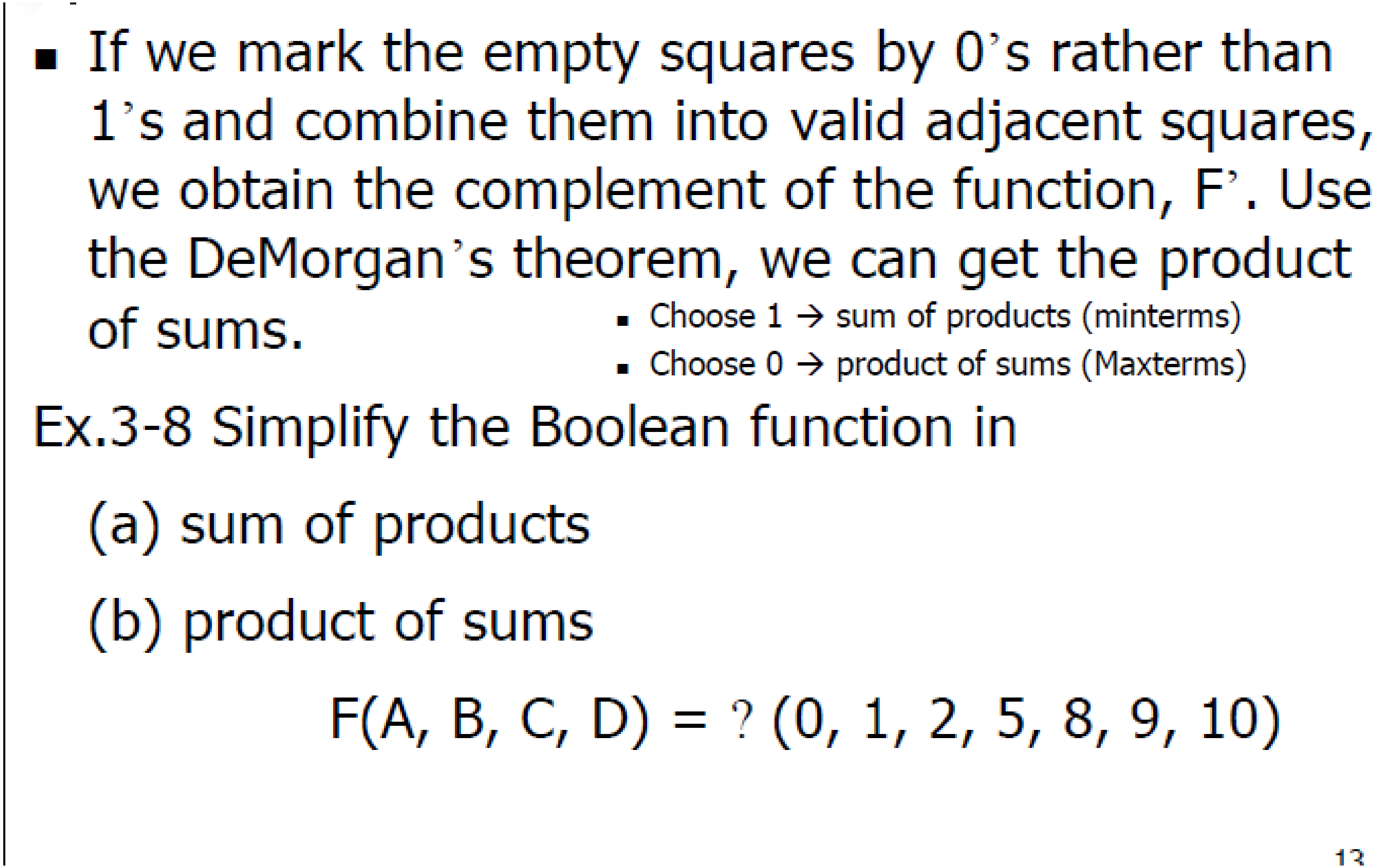


(a) Essential prime implicants BD and B′D′

(b) Prime implicants CD, B′C AD, and AB′

$$F = BD + B'D' + \begin{cases} CD + AD \\ CD + AB' \\ B'C + AD \\ B'C + AB' \end{cases}$$

Product of sums simplification



- If we mark the empty squares by 0's rather than 1's and combine them into valid adjacent squares, we obtain the complement of the function, F'. Use the DeMorgan's theorem, we can get the product of sums.
  - Choose 1 → sum of products (minterms)
  - Choose 0 → product of sums (Maxterms)

Ex.3-8 Simplify the Boolean function in

(a) sum of products

(b) product of sums

F(A, B, C, D) = ? (0, 1, 2, 5, 8, 9, 10)

Example

(a) SOPs

F= B’D’ + B’C’ + A’C’D

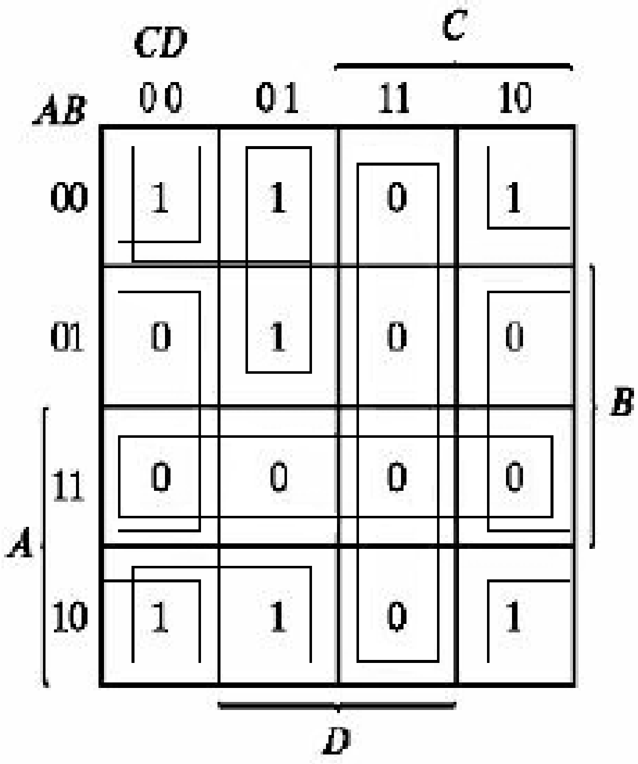
(b) POSs

F’= AB + CD + BD’

By DeMorgan’s thm

F= (A’+B’) .(C’+D’)

.(B’+D)



AB \ CD	00	01	11	10
00	1	1	0	1
01	0	1	0	0
11	0	0	0	0
10	1	1	0	1

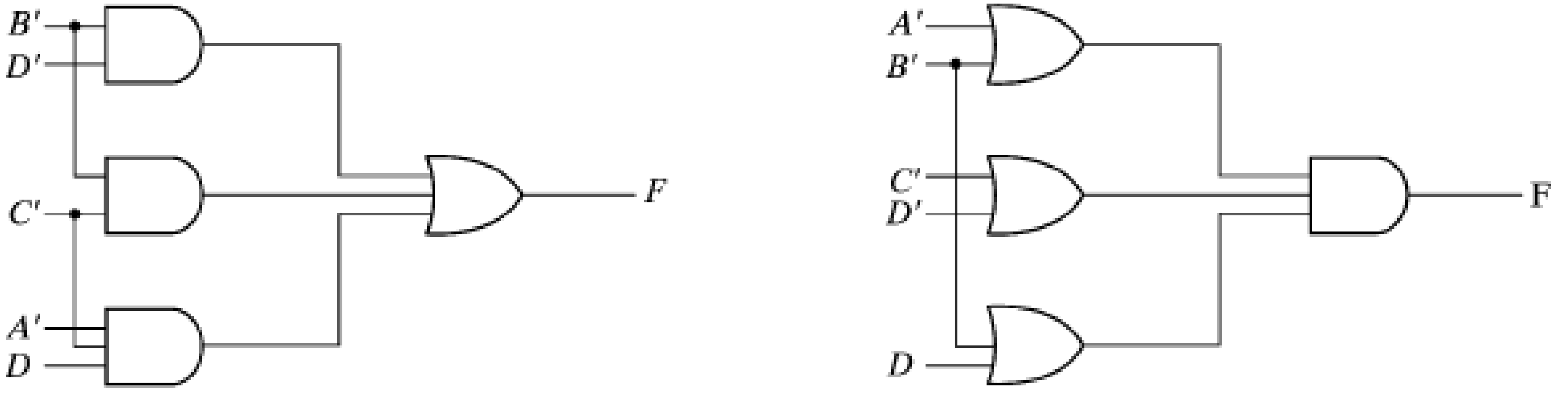
Fig. 3-14 Map for Example 3-8; $F(A,B,C,D) = \Sigma(0,1,2,5,8,9,10$
$= B'D' + B'C' + A'C'D = (A'+B')(C'+D')(B'+D)$

Two Gate Implementations

- Sometimes product-of-sums representations may have smaller implementations

7 literals, 4 gates

6 literals, 4 gates



(a) $F = B'D' + B'C' + A'C'D$

(b) $F = (A' + B')(C' + D')(B' + D)$

Fig. 3-15 Gate Implementation of the Function of Example 3-8

Exchange minterm and maxterm

- Consider the truth table that defines the function F in Table 3-2.

Sum of minterms

$$F(x, y, z) = \Sigma(1, 3, 4, 6)$$

Product of maxterms

$$F(x, y, z) = \Pi(0, 2, 5, 7)$$

- In the other words, the 1's of the function represent the minterms, and the 0's represent the maxterms.

Table 3-2
Truth Table of Function F

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

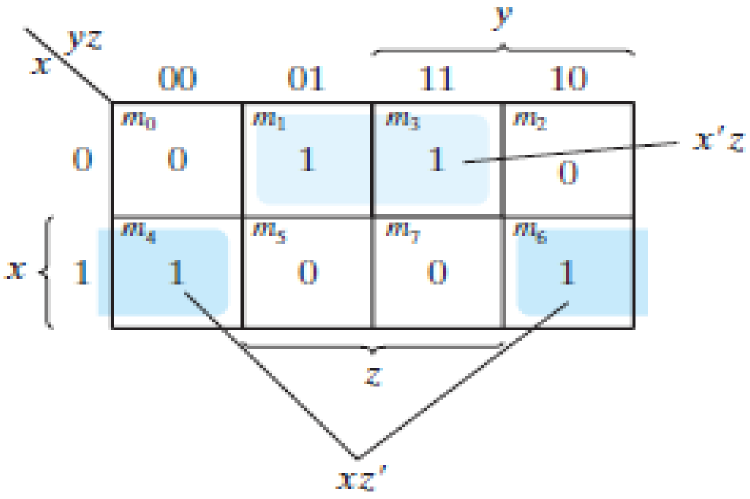


Fig. 3-16 Map for the Function of Table 3-2

Exchange minterm and maxterm

- For the sum of products, we combine the 1’s to obtain $F = x'z + xz'$
- For the product of sums, we combine the 0’s to obtain the simplified complemented function $F' = xz + x'z'$
- Taking the complement of F, we obtain the simplified function in product-of-sums form:

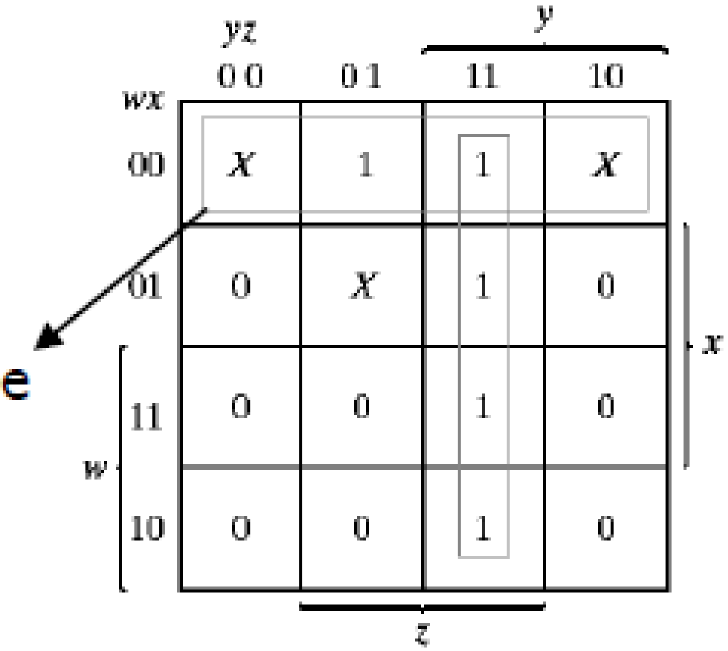
$$F = (x' + z')(x + z)$$

DON’T-CARE CONDITIONS

- X = don't care (can be 0 or 1)
- Don’t cares can be included to form a larger cube, but not necessary to be completely covered
- Ex: $F(w,x,y,z) = \sum(1,3,7,11,15)$ $d(w,x,y,z) = \sum(0,2,5)$

In part (a) with minterms 0 and 2→ F = yz + w’x’

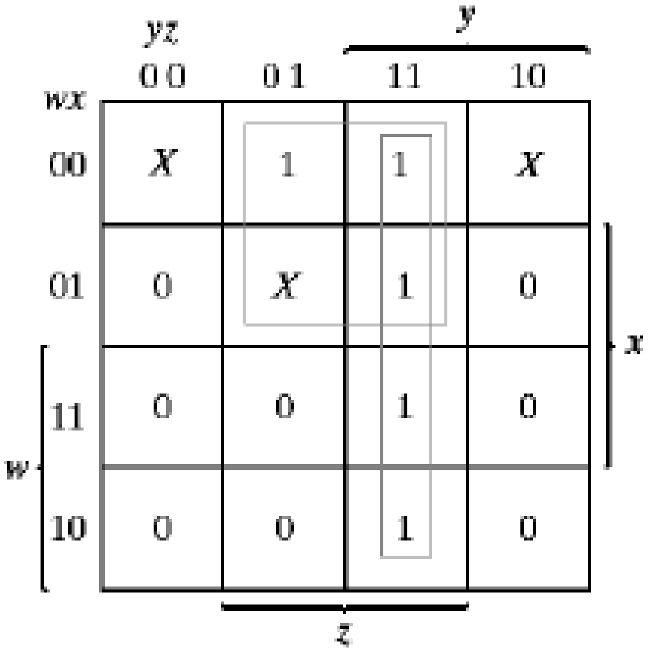
In part (b) with minterm 5 → F = yz + w’z



wx \ yz	00	01	11	10
00	X	1	1	X
01	0	X	1	0
11	0	0	1	0
10	0	0	1	0

larger cube with don’t cares

(a) $F = yz + w'x'$



wx \ yz	00	01	11	10
00	X	1	1	X
01	0	X	1	0
11	0	0	1	0
10	0	0	1	0

(a) $F = yz + w'z$