UNIT-IV

Number Theory

Definition of Prime numbers:

A prime number is a whole number greater than 1 whose only factors are 1 and itself. A factor is a whole number that can be divided evenly into another number. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.

Relatively Prime Numbers:

When two numbers have no common factors other than 1, they are said to be relatively prime. In other words, no number other than 1 can divide them both exactly (without any remainder). Relatively prime numbers are also called "coprime numbers" or "mutually prime numbers."

Ex: (2, 3) ; (11, 12) ; (21, 22) ; (100, 101) and so on. Any two prime numbers are relatively prime to each other. As every prime number has only two factors 1 and the number itself, the only common factor of two prime numbers will be 1.

fermat's theorem:

Fermat's theorem, also known as Fermat's little theorem and Fermat's primality test, in number theory, the statement, first given in 1640 by French mathematician Pierre de Fermat, that for any prime number p and any integer a such that p does not divide a (the pair are relatively prime), p divides exactly into

- $a^{p-1} = 1 \pmod p$
  - where p is prime and gcd(a,p)=1
- also known as Fermat's Little Theorem
- also have: $a^p = a \pmod p$
- useful in public key and primality testing

EX:1

P = an integer Prime number

a = an integer which is not multiple of P

Let a = 2 and P = 17

According to Fermat's little theorem

$2^{17-1} \equiv 1 \bmod(17)$

we got $65536 \% 17 \equiv 1$

that mean (65536-1) is an multiple of 17

Euler's theorems:

Euler's theorem is a generalization of Fermat's little theorem handling with powers of integers modulo positive integers. It increase in applications of elementary number theory, such as the theoretical supporting structure for the RSA cryptosystem.

$a\phi(n) \equiv 1 (\text{mod} n)$

R = $\{x_1, x_2, \ldots x_{\phi(n)}\}$, i.e., each element xi of R is unique positive integer less than n with ged(x_i, n) = 1. Then multiply each element by a and modulo n −

S = $\{(ax_1 \bmod n), (ax_2 \bmod n), \ldots (ax_{\phi(n)} \bmod n)\}$

Because a is relatively prime to n and x_i is relatively prime to n, ax_i must also be relatively prime to n. Therefore, all the members of S are integers that are less than n and that are relatively prime to n.

There are no duplicates in S.

If $ax_i \bmod n$ and $n = ax_j \bmod n$ then $x_i = x_j$

Therefore,

$\Pi\phi(n)i=1(aximodn)=\Pi\phi(n)i=1xi$
$\Pi\phi(n)i=1axi\equiv\Pi\phi(n)i=1xi(modn)$
$a\phi(n)x[\Pi\phi(n)i=1xi]=\Pi\phi(n)i=1xi(modn)$
$a\phi(n)\equiv1(modn)$
EX: Let $a=4a=4$ and $n=7n=7$, a and n are co-prime as their greatest common divisor is 11. Now plug the values into Euler's equation above resulting in the equation as follows:

$$4\varphi(7)\cong1(mod7)$$

As $\phi(7)=6\varphi(7)=6$, we plug this value into the equation above resulting in the equation as follows:
$46\cong1(mod7)$
$4096\cong1(mod7)$
This equation suggests if we divide 40964096 by 77, we will get a reminder of 11. This proves that Euler's theorem is valid on the given values.

Testing for Primality:

A primality test is an algorithm for determining whether an input number is prime. Among other fields of mathematics, it is used for cryptography. Unlike integer factorization, primality tests do not generally give prime factors, only stating whether the input number is prime or not.

Public Key Cryptography

Introduction to Public key Cryptography:

- Public key cryptography also called as asymmetric cryptography.
- It was invented by whitfield Diffie and Martin Hellman in 1976. Sometimes this cryptography also called as Diffie-Helman Encryption.
- Public key algorithms are based on mathematical problems which admit no efficient solution that are inherent in certain integer factorization, discrete logarithm and Elliptic curve relations.

Public key Cryptosystem Principles:

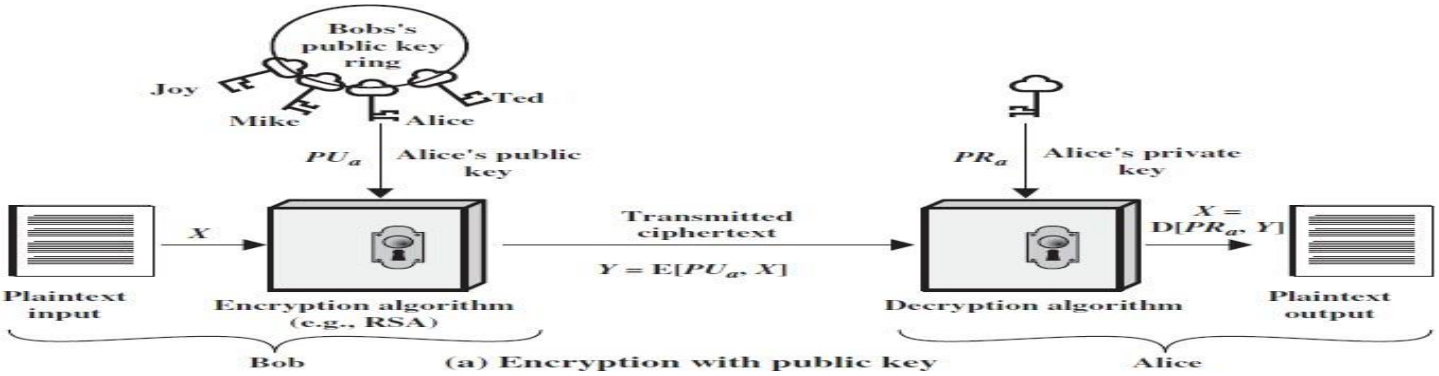
- The concept of public key cryptography in invented for two most difficult problems of Symmetric key encryption.
    - The Key Exchange Problem
    - The Trust Problem

    The Key Exchange Problem: The key exchange problem arises from the fact that communicating parties must somehow share a secret key before any secure communication can be initiated, and both parties must then ensure that the key remains secret. Of course, direct key exchange is not always feasible due to risk, inconvenience, and cost factors.

    The Trust Problem: Ensuring the integrity of received data and verifying the identity of the source of that data can be very important. Means in the symmetric key cryptography system, receiver doesn't know whether the message is coming for particular sender.
- This public key cryptosystem uses two keys as pair for encryption of plain text and Decryption of cipher text.
- These two keys are names as "**Public key**" and "**Private key**". The private key is kept secret where as public key is distributed widely.
- A message or text data which is encrypted with the public key can be decrypted only with the corresponding private-key
- This two key system very useful in the areas of confidentiality (secure) and authentication

A **public-key encryption** scheme has six ingredients		
1	**Plaintext**	This is the readable message or data that is fed into the algorithm as input.
2	**Encryption algorithm**	The encryption algorithm performs various transformations on the plaintext.
3	**Public key**	This is a pair of keys that have been selected so that if one is used for encryption, the other is used for decryption. The exact transformations performed by the algorithm depend on the public or private key that is provided as input
4	**Private key**	
5	**Ciphertext**	This is the scrambled message produced as output. It depends on the plaintext and the key. For a given message, two different keys will produce two different ciphertexts.
6	**Decryption algorithm**	This algorithm accepts the ciphertext and the matching key and produces the original plaintext.

Public key cryptography for providing confidentiality (secrecy)



The essential steps are the following.

1. Each user generates a pair of keys to be used for the encryption and decryption of messages.
2. Each user places one of the two keys in a public register or other accessible file. This is the public key. The companion key is kept private. As Figure 9.1a suggests, each user maintains a collection of public keys obtained from others.
3. If Bob wishes to send a confidential message to Alice, Bob encrypts the messageusing Alice's public key.
4. When Alice receives the message, she decrypts it using her private key. No other recipient can decrypt the message because only Alice knows Alice's private key.

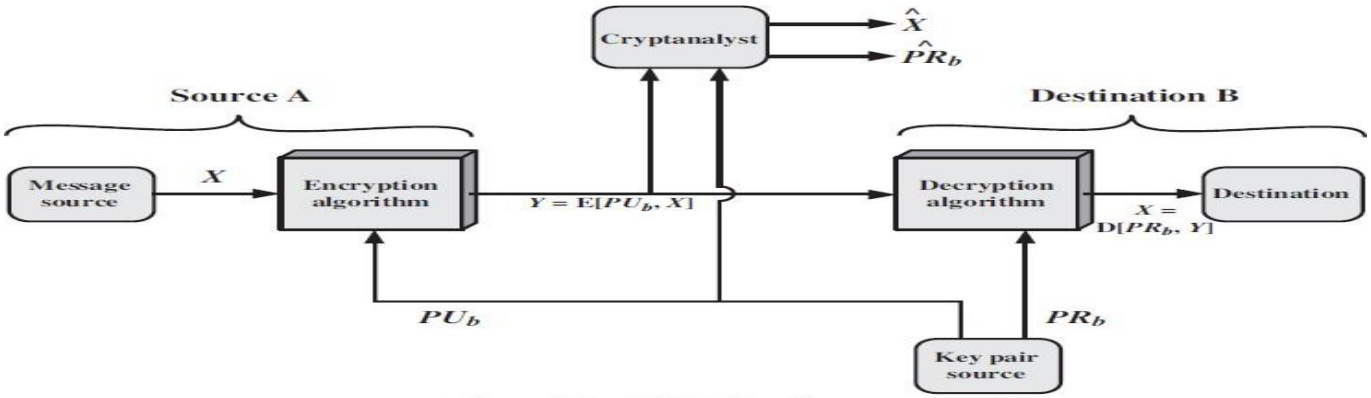


Figure 9.2 Public-Key Cryptosystem: Secrecy

There is some source A that produces a message in plaintext ***X* = [*X*1, *X*2, . . . ,*XM*].**

The ***M*** elements of ***X*** are letters in some finite alphabet. The message is intended fordestination **B**.
B generates a related pair of keys: a public key, ***PUb,*** and a private key, ***PRb***.

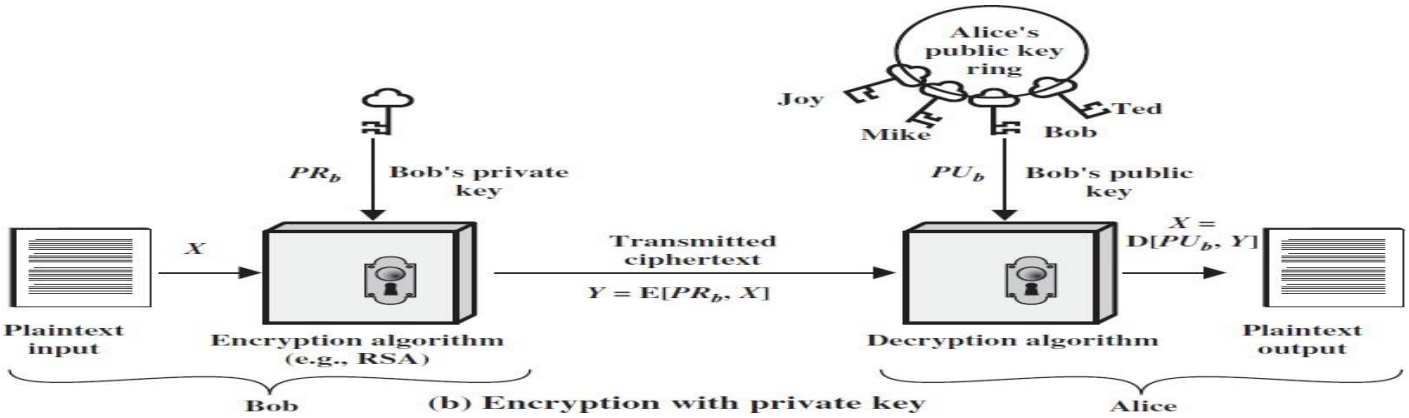
PRb is known only to B, whereas ***PUb*** is publicly available and therefore accessible by A. With the message *X* and the encryption key ***PUb*** as input, A forms the ciphertext *Y* = [*Y*1, *Y*2, . . . , *YN*]:

$$Y = \mathrm{E}(PU_b, X)$$

The intended receiver, in possession of the matching private key, is able to invert the transformation:

$$X = \mathrm{D}(PR_b, Y)$$

Public key cryptography for proving Authentication:



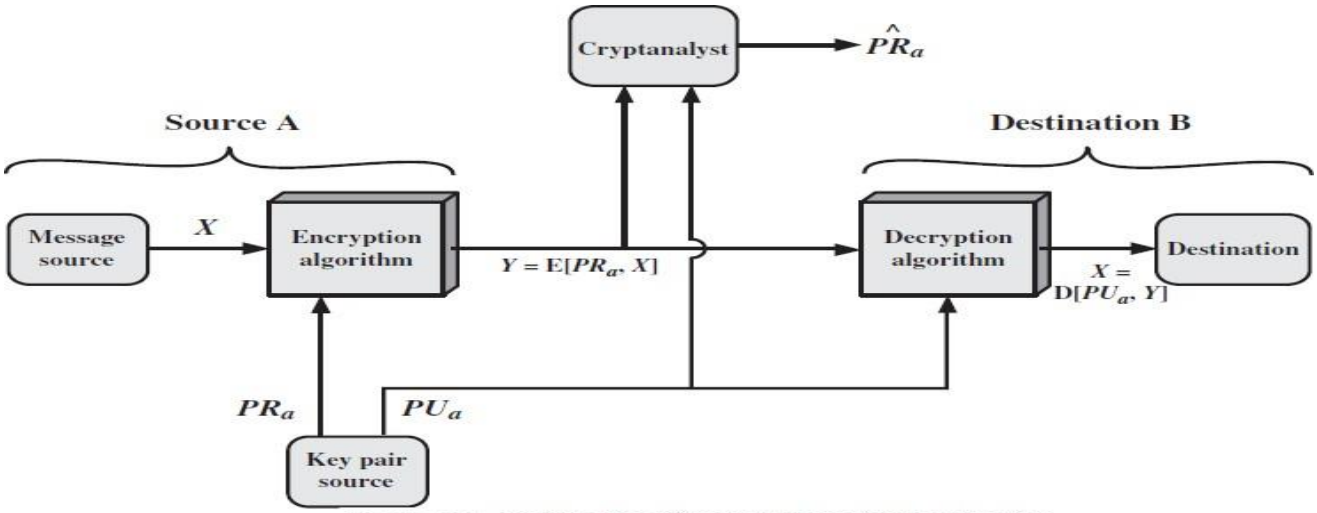


Figure 9.3 Public-Key Cryptosystem: Authentication

The above diagrams show the use of public-key encryption to provide authentication:

$$Y = \mathrm{E}(PR_a, X)$$

$$X = \mathrm{D}(PU_a, Y)$$

- In this case,A prepares a message to B and encrypts it using A's private key before transmitting it. B can decrypt the message using A's public key. Because the message was encrypted using A's private key, only A could have prepared the message. Therefore, the entire encrypted message serves as a **digital signature.**

- It is impossible to alter the message without access to A's private key, so the message is authenticated both in terms of source and in terms of data integrity.

Public key cryptography for both authentication and confidentiality (Secrecy)

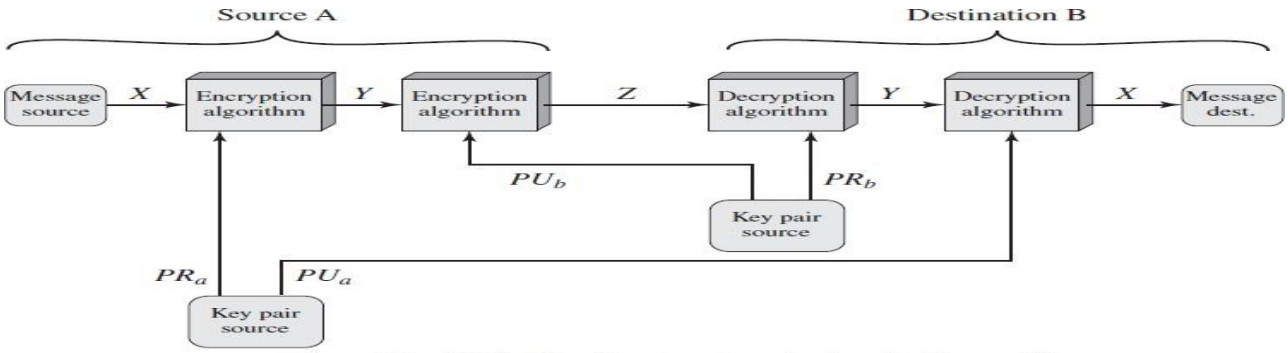


Figure 9.4 Public-Key Cryptosystem: Authentication and Secrecy

It is, however, possible to provide both the authentication function and confidentiality by adouble use of the public-key scheme (above figure):

$$Z = \mathrm{E}(PU_b, \mathrm{E}(PR_a, X))$$

$$X = \mathrm{D}(PU_a, \mathrm{D}(PR_b, Z))$$

In this case, we begin as before by encrypting a message, using the sender's private key. This provides the digital signature. Next, we encrypt again, using the receiver's public key. The final ciphertext can be decrypted only by the intended receiver, who alone has the matching private key. Thus, confidentiality is provided.

Applications for Public-Key Cryptosystems:

Public-key systems are characterized by
the use of a cryptographic algorithm with two keys, one held private and one available publicly. Depending on the application, the sender uses either the sender's private key or the receiver's public key, or both, to perform some type of cryptographic function.
the use of **public-key cryptosystems**
into three categories

- Encryption /decryption: The sender encrypts a message with the recipient'spublic key.
- Digital signature: The sender "signs" a message with its private key. Signing isachieved by a cryptographic algorithm applied to the message or to a small block of data that is a function of the message.
- Key exchange: Two sides cooperate to exchange a session key. Several differentapproaches are possible, involving the private key(s) of one or both parties.

Applications for Public-Key Cryptosystems

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

Public-Key Cryptanalysis

As with symmetric encryption, a public-key encryption scheme is vulnerable to a brute-force attack. The countermeasure is the same: Use large keys. However, there is a tradeoff to be considered. Public-key systems depend on the use of some sort of invertible mathematical function. The complexity of calculating these functions may not scale linearly with the number of bits in the key but grow more rapidly than that. Thus, the key size must be large enough to make brute-force attack impractical but small enough for practical encryption and decryption. In practice, the key sizes that have been proposed do make brute-force attack impractical but result in encryption/decryption speeds that are too slow for general-purpose use. Instead, as was mentioned earlier, public-key encryption is currently confined to key management and signature applications.

RSA (Rivest, Shamir, Adleman):

- It is the most common public key algorithm.
- This RSA name is get from its inventors first letter (Rivest (R), Shamir (S) andAdleman (A)) in the year 1977.
- The RSA scheme is a block cipher in which the plaintext & ciphertext are integersbetween 0 and n-1 for some 'n'.
- A typical size for 'n' is 1024 bits or 309 decimal digits. That is, n is less than 2^{1024}

Description of the Algorithm:

- RSA algorithm uses an expression with exponentials.
- In RSA plaintext is encrypted in blocks, with each block having a binary value lessthan some number n. that is, the block size must be less than or equal to **log2(n)**
- **RSA** uses two exponents 'e' and 'd' where e□public and d□private.
- Encryption and decryption are of following form, for some PlainText 'M' andCipherText block 'C'

$$C = M^e \bmod n$$

$$M = C^d \bmod n$$

$$M = C^d \bmod = (M^e \bmod n)^d \text{ mon } n = (M^e)^d \bmod n = M^{ed} \bmod n$$

Both sender and receiver must know the value of n.
The sender knows the value of 'e' & only the reviver knows the value of 'd' thus thisis a public key encryption algorithm with a
Public key PU={e, n} Private key
PR={d, n}

Requirements:

The RSA algorithm to be satisfactory for public key encryption, the following requirementsmust be met:

1. It is possible to find values of e, d n such that " $\mathbf{M^{ed} \bmod n = M}$ " for all M<n
2. It is relatively easy to calculate " $\mathbf{M^e \bmod n}$ " and " $\mathbf{C^d \bmod n}$ "for M<n
3. It is infeasible to determine "d" given 'e' & 'n'. The " $\mathbf{M^{ed} \bmod n = M}$ " relationshipholds if 'e' & 'd' are multiplicative inverses modulo Ø(n).
   Ø(n)□ Euler Totient function
   For p,q primes where p*q and p≠q.Ø(n)=
   Ø(pq)=(p-1)(q-1)

Then the relation between 'e' & 'd' can be expressed as " $\mathbf{ed \bmod Ø(n)=1}$ "
this is equivalent to saying

$$\mathbf{ed \equiv 1 \bmod Ø(n)}$$

$$\mathbf{d \equiv e^{-1} \bmod Ø(n)}$$

That is 'e' and 'd' are multiplicative inverses mod Ø(n).
Note: according to the rules of modular arithmetic, this is true only if 'd' (and 'e') isrelatively prime to Ø(n).
Equivalently **gcd(Ø(n), d)=1.**

Steps of RSA algorithm:

Step 1 Select 2 prime numbers p & qStep
2 Calculate n=pq
Step 3 Calculate Ø(n)=(p-1)(q-1)
Step 4 Select or find integer e (public key) which is relatively prime to Ø(n).ie., e with gcd (Ø(n), e)=1 where 1<e< Ø(n).

Step 5 Calculate "d" (private key) by using following condition.d< Ø(n). $ed \equiv 1 \bmod \emptyset(n)$

Step 6 Perform encryption by using $C = M^e \bmod n$

Step 7 perform Decryption by using $M = C^d \bmod n$

Example:

1. Select two prime numbers, $p = 17$ and $q = 11$.
2. Calculate $n = pq = 17 \times 11 = 187$.
3. Calculate $\emptyset(n) = (p - 1)(q - 1) = 16 \times 10 = 160$.
4. Select e such that e is relatively prime to $\emptyset(n) = 160$ and less than $\emptyset(n)$; we choose $e = 7$.
5. Determine d such that $de \equiv 1 \pmod{160}$ and $d < 160$.The correct value is $d = 23$, because $23 * 7 = 161 = (1 \times 160) + 1$; d can be calculated using the extended Euclid's algorithm

The resulting keys are public key $PU = \{7, 187\}$ and private key $PR = \{23, 187\}$.
The example shows the use of these keys for a plaintext input of $M = 88$. For encryption, we need to calculate $C = 88^7 \bmod 187$. Exploiting the properties of modular arithmetic, wecan do this as follows.

$$88^7 \bmod 187 = [(88^4 \bmod 187) \times (88^2 \bmod 187) \times (88^1 \bmod 187)] \bmod 187$$
$$88^1 \bmod 187 = 88$$
$$88^2 \bmod 187 = 7744 \bmod 187 = 77$$
$$88^4 \bmod 187 = 59{,}969{,}536 \bmod 187 = 132$$
$$88^7 \bmod 187 = (88 \times 77 \times 132) \bmod 187 = 894{,}432 \bmod 187 = 11$$

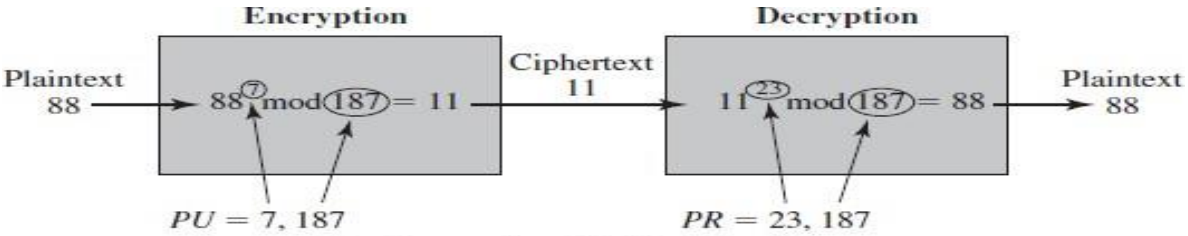


Figure 9.6 Example of RSA Algorithm

For decryption, we calculate $M = 11^{23} \bmod 187$:

$$11^{23} \bmod 187 = [(11^1 \bmod 187) \times (11^2 \bmod 187) \times (11^4 \bmod 187) \times (11^8 \bmod 187) \times (11^8 \bmod 187)] \bmod 187$$
$$11^1 \bmod 187 = 11$$
$$11^2 \bmod 187 = 121$$
$$11^4 \bmod 187 = 14{,}641 \bmod 187 = 55$$
$$11^8 \bmod 187 = 214{,}358{,}881 \bmod 187 = 33$$
$$11^{23} \bmod 187 = (11 \times 121 \times 55 \times 33 \times 33) \bmod 187 = 79{,}720{,}245 \bmod 187 = 88$$

The Security of RSA:

Four possible approaches to attacking the RSA algorithm are

- **Brute force:** This involves trying all possible private keys.
- **Mathematical attacks:** There are several approaches, all equivalent in effort to factoringthe product of two primes.
- **Timing attacks:** These depend on the running time of the decryption algorithm.
- **Chosen ciphertext attacks:** This type of attack exploits properties of the RSA algorithm.

Diffie-Hellman Key Exchange:

- Diffie-Hellman key exchange is the first published public key algorithm
- This Diffie-Hellman key exchange protocol is also known as exponential key agreement. And it is based on mathematical principles.
- The purpose of the algorithm is to enable two users to exchange a key securely that can then be used for subsequent encryption of messages.
- This algorithm itself is limited to exchange of the keys.
- This algorithm depends for its effectiveness on the difficulty of computing discrete logarithms.
- The discrete logarithms are defined in this algorithm in the way of define a primitive root of a prime number.
  - Primitive root: we define a primitive root of a prime number P as one whose power generate all the integers form 1 to P-1 that is if 'a' is a primitive root of the prime number P, then the numbers

    $a \bmod P, a^2 \bmod P, a^3 \bmod P, \ldots\ldots a^{P-1} \bmod P$ are distinct and consist of the integers form 1 through P-1 in some permutation.

    For any integer 'b' and 'a', here 'a' is a primitive root of prime number P, then

    $b \equiv a^i \bmod P$ $\quad 0 \le i \le (P-1)$

  The exponent i is refer as discrete logarithm or index of b for the base a, mod P.The value denoted as **ind a,p(b)**

Algorithm for Diffie-Hellman Key Exchange:

Step 1 two public known numbers q, α

q Prime number

α primitive root of q and α< q.

Step 2 if A & B users wish to exchange a key

a) User A select a random integer XA<q and computes $Y_A = \alpha^{X_A} \bmod q$

b) User B independently select a random integer XB <q and computes $Y_B = \alpha^{X_B} \bmod q$

c) Each side keeps the X value private and Makes the Y value available publicly tothe outer side.

Step 3 User A Computes the key asUser B $\boxed{K = (Y_B)^{X_A} \bmod q}$

Computes the key as $\boxed{K = (Y_A)^{X_B} \bmod q}$

Step 4 two calculation produce identical results

$$K = (Y_B)^{X_A} \bmod q$$

$$K = (\alpha^{X_B} \bmod q)^{X_A} \bmod q$$ (We know that $Y_B = \alpha^{X_B} \bmod q$)

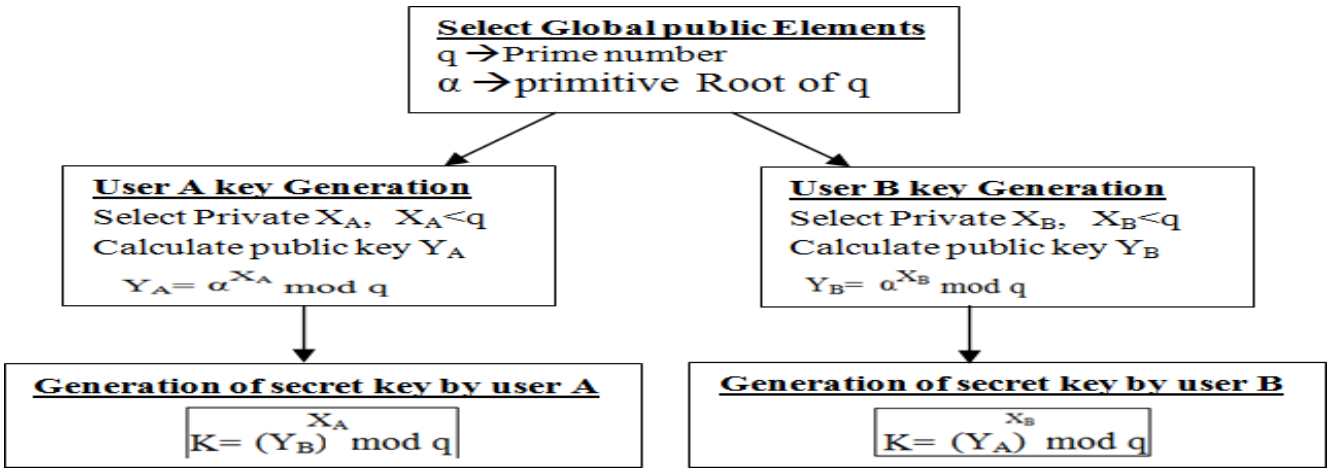
$$= (\alpha^{X_B})^{X_A} \bmod q$$

$$= (\alpha^{X_A})^{X_B} \bmod q$$

$$= (\alpha^{X_A} \bmod q)^{X_B} \bmod q$$

$= (Y_A)^{X_B} \bmod q$ (We know that $Y_A = \alpha^{X_A} \bmod q$)

The result is that the two sides have exchanged a secret key.



Example:

Here is an example. Key exchange is based on the use of the prime number $q = 353$ and a primitive root of 353, in this case $\alpha = 3$. A and B select secret keys $X_A = 97$ and $X_B = 233$, respectively. Each computes its public key:

A computes $Y_A = 3^{97} \bmod 353 = 40$.

B computes $Y_B = 3^{233} \bmod 353 = 248$.

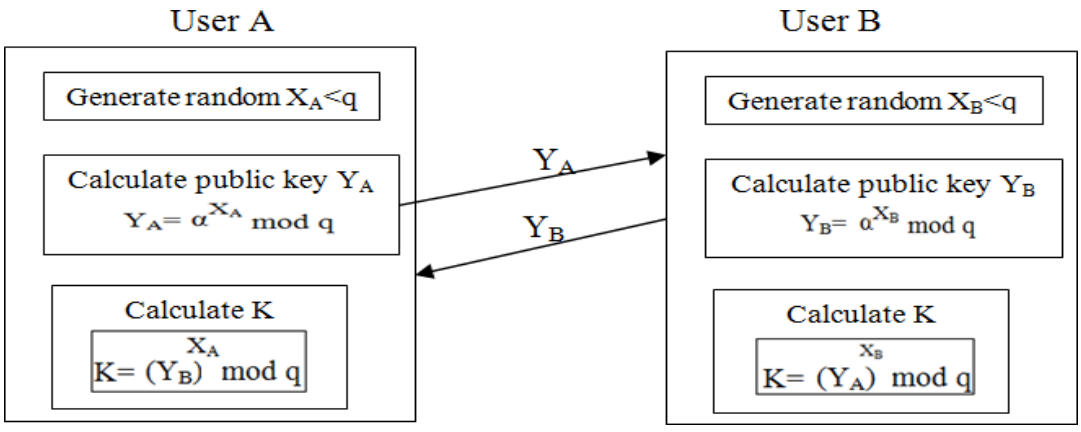
After they exchange public keys, each can compute the common secret key:

A computes $K = (Y_B)^{X_A} \bmod 353 = 248^{97} \bmod 353 = 160$.

B computes $K = (Y_A)^{X_B} \bmod 353 = 40^{233} \bmod 353 = 160$.

We assume an attacker would have available the following information:

$$q = 353; \alpha = 3; Y_A = 40; Y_B = 248$$



MAN-in the Middle Attack (MITM)

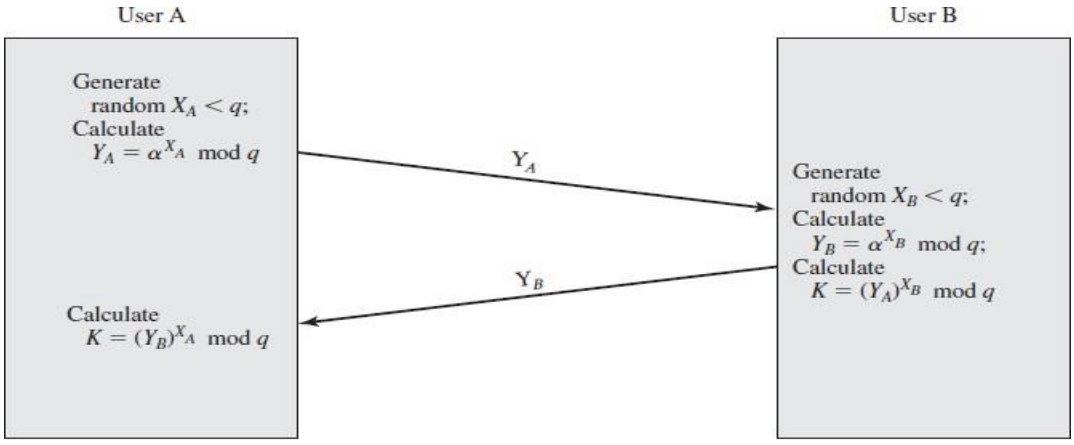


Figure 10.2 Diffie-Hellman Key Exchange

Definition: A man in the middle attack is a form of eavesdropping where communication between two users is monitored and modified by an unauthorized party.

Generally the attacker actively eavesdrops by intercepting (stoping) a public key message exchange.
The Diffie- Hellman key exchange is insecure against a "Man in the middle attack".

Suppose user 'A' & 'B' wish to exchange keys, and D is the adversary (opponent). The attack proceeds as follows.

1. 'D' prepares for the attack by generating two random private keys X_{D1} & X_{D2} and thencomputing the corresponding public keys Y_{D1} and Y_{D2}.
2. 'A' transmits 'Y_A' to 'B'
3. 'D' intercepts Y_A and transmits Y_{D1} to 'B'. and D also calculates $K2 = (Y_A)^{X_{D2}} \bmod q$.
4. 'B' receives Y_{D1} & calculate $K1 = (Y_{D1})^{X_B} \bmod q$.
5. 'B' transmits 'Y_B' to 'A"
6. 'D' intercepts 'Y_B' and transmits Y_{D2} to 'A' and 'D' calculate K1 $K1 = (Y_B)^{X_{D1}} \bmod q$.
7. A receives Y_{D2} and calculates $K2 = (Y_{D2})^{X_A} \bmod q$

At this point, Bob and Alice think that they share a secret key, but instead Bob and Darth share secret key *K*1 and Alice and Darth share secret key *K*2. All future communication between Bob and Alice is compromised in the following way.

1. A sends an encrypted message *M*: E(*K*2, *M*) .
2. D intercepts the encrypted message and decrypts it to recover *M*.
3. D sends B $E(K1, M)$ or $E(K1, M')$, where M' is any message. In the first case, D simply wants to eavesdrop on the communication without altering it. In the second case, D wants to modify the message going to B

The key exchange protocol is vulnerable to such an attack because it does not authenticate theparticipants. This vulnerability can be overcome with the use of digital signatures and public- key certificates.

To encrypt and send a message P_m to B, A chooses a random positive integer k and produces the ciphertext C_m consisting of the pair of points:

$$C_m = \{kG, P_m + kP_B\}$$

Note that A has used B's public key P_B. To decrypt the ciphertext, B multiplies the first point in the pair by B's secret key and subtracts the result from the second point:

$$P_m + kP_B - n_B(kG) = P_m + k(n_BG) - n_B(kG) = P_m$$

A has masked the message P_m by adding kP_B to it. Nobody but A knows the value of k, so even though P_b is a public key, nobody can remove the mask kP_B. However, A also includes a "clue," which is enough to remove the mask if one knows the private key n_B. For an attacker to recover the message, the attacker would have to compute k given G and kG, which is assumed to be hard.

As an example of the encryption process (taken from [KOBL94]), take $p = 751$; $E_p(-1, 188)$, which is equivalent to the curve $y^2 = x^3 - x + 188$; and $G = (0, 376)$. Suppose that A wishes to send a message to B that is encoded in the elliptic point $P_m = (562, 201)$ and that A selects the random number $k = 386$. B's public key is $P_{B = (201, 5)}$. We have $386(0, 376) = (676, 558)$, and $(562, 201) + 386(201, 5) = (385, 328)$. Thus, A sends the cipher text $\{(676, 558), (385, 328)\}$.