LECTURE NOTES

ON

SIGNALS AND SYSTEMS

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UNIT-I

SIGNALS & SYSTEMS

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Signal : A signal is defined as a time varying physical phenomenon which is intended to convey information. (or) Signal is a function of time. (or) Signal is a function of one or more independent variables, which contain some information.

Example: voice signal, video signal, signals on telephone wires , EEG, ECG etc.

Signals may be of continuous time or discrete time signals.

System : System is a device or combination of devices, which can operate on signals and produces corresponding response. Input to a system is called as excitation and output from it is called as response.

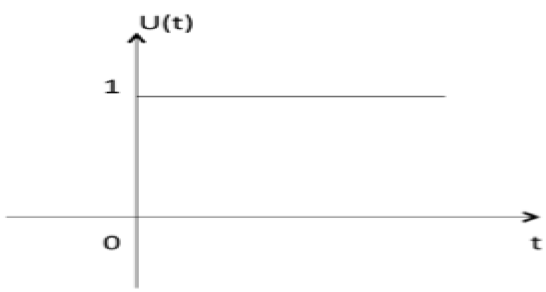
For one or more inputs, the system can have one or more outputs.

Example: Communication System

Elementary Signals or Basic Signals:

Unit Step Function

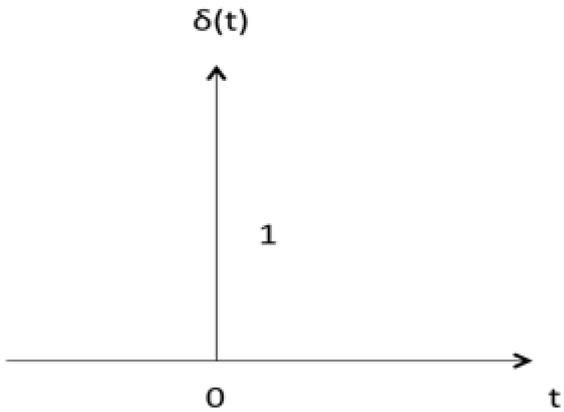
Unit step function is denoted by u(t). It is defined as u(t) = 1 when $t \geq 0$ and
0 when $t < 0$



- It is used as best test signal.
- Area under unit step function is unity.

Unit Impulse Function

Impulse function is denoted by δ(t). and it is defined as δ(t) = $\begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

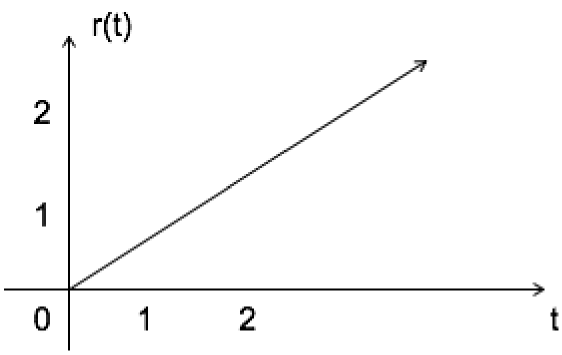


$$\int_{-\infty}^{\infty} \delta(t)dt = u(t)$$

$$\delta(t) = \frac{du(t)}{dt}$$

Ramp Signal

Ramp signal is denoted by r(t), and it is defined as r(t) = $\begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



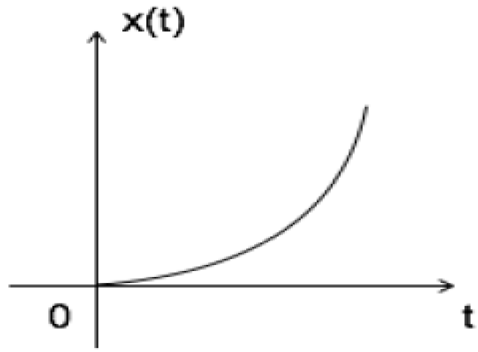
$$\int u(t) = \int 1 = t = r(t)$$

$$u(t) = \frac{dr(t)}{dt}$$

Area under unit ramp is unity.

Parabolic Signal

Parabolic signal can be defined as x(t) = $\begin{cases} t^2/2 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



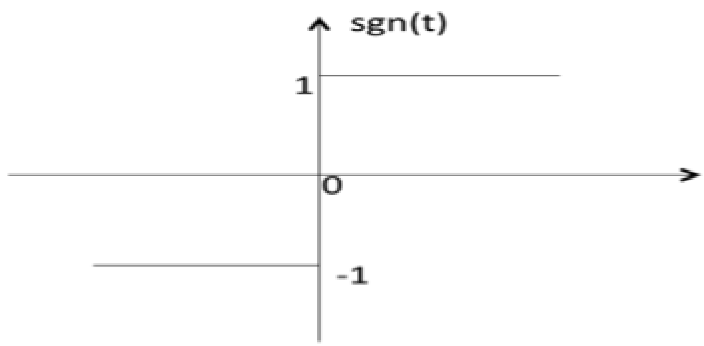
$$\iint u(t)dt = \int r(t)dt = \int tdt = \frac{t^2}{2} = parabolicsignal$$

$$\Rightarrow u(t) = \frac{d^2x(t)}{dt^2}$$

$$\Rightarrow r(t) = \frac{dx(t)}{dt}$$

Signum Function

Signum function is denoted as sgn(t). It is defined as sgn(t) = $\begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$



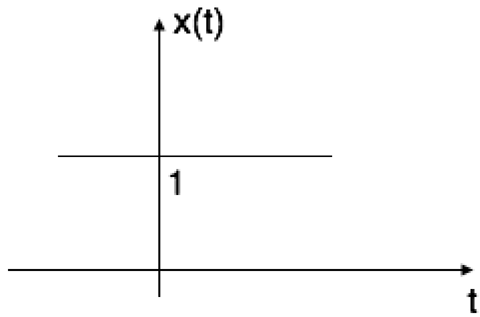
sgn(t) = 2u(t) – 1

Exponential Signal

Exponential signal is in the form of x(t) = $e^{\alpha t}$

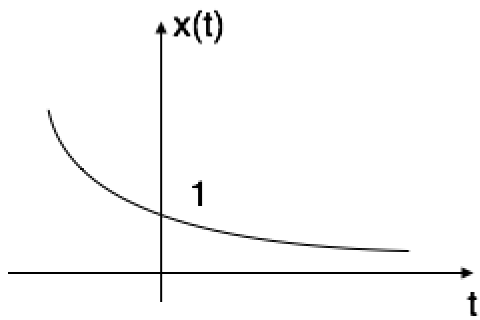
.The shape of exponential can be defined by α

Case i: if $\alpha = 0 \longrightarrow$ x(t) = e^{0}= 1



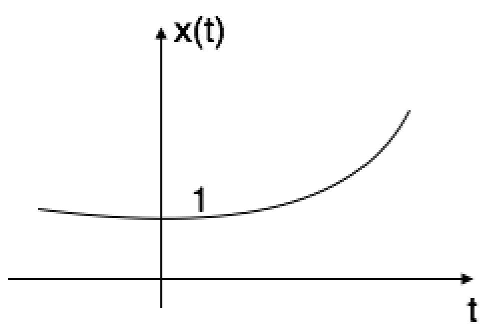
Case ii: if α< 0 i.e. -ve then x(t) = $e^{-\alpha t}$

. The shape is called decaying exponential.



Case iii: if α> 0 i.e. +ve then x(t) = $e^{\alpha t}$

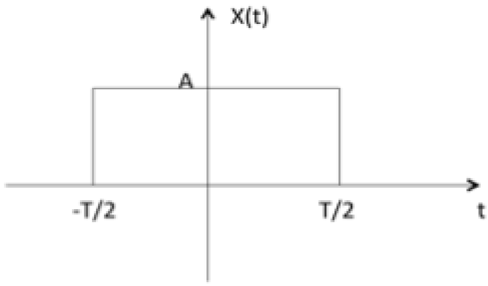
. The shape is called raising exponential.

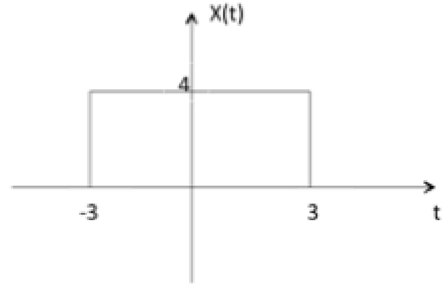


Rectangular Signal

Let it be denoted as x(t) and it is defined as

$$x(t) = A\, rect\left[\frac{r}{T}\right] \qquad \text{ex: } 4\, rect\left[\frac{r}{6}\right]$$

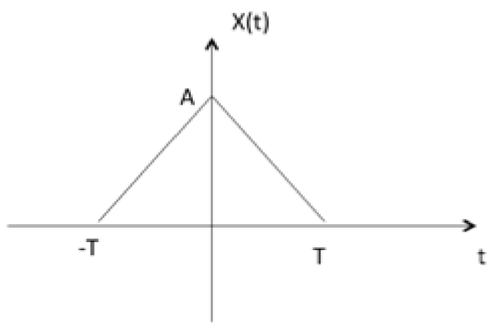


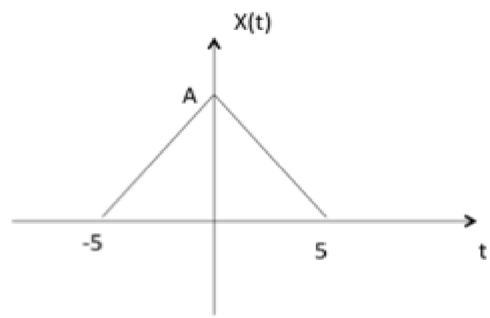


Triangular Signal

Let it be denoted as x(t)

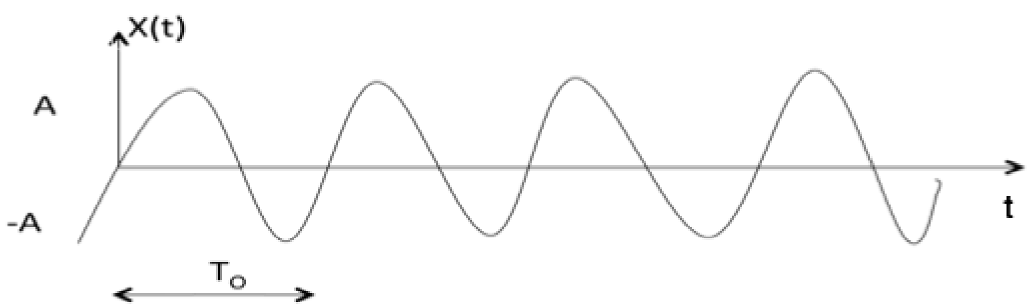
$$x(t) = A\left[1 - \frac{|t|}{T}\right] \qquad \text{ex: } x(t) = A\left[1 - \frac{|t|}{5}\right]$$





Sinusoidal Signal

Sinusoidal signal is in the form of x(t) = A cos($w_0 \pm \phi$) or A sin($w_0 \pm \phi$)



Where $T_0 = 2\pi/w_0$

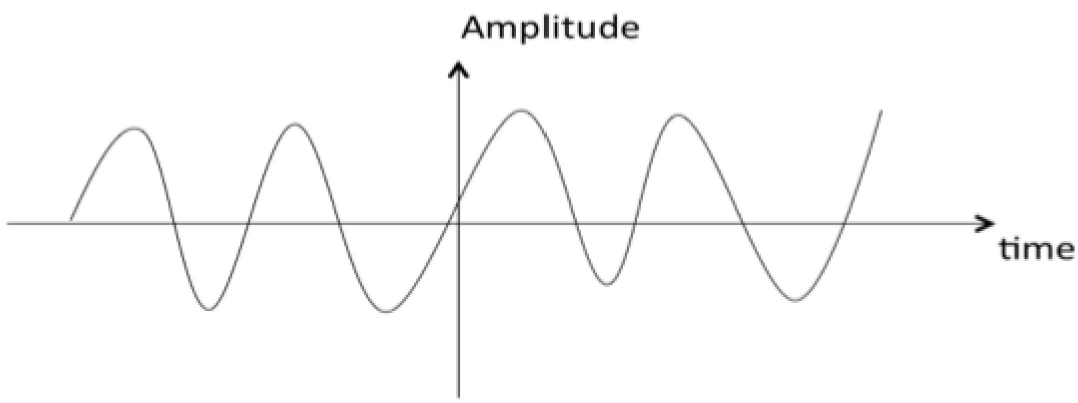
Classification of Signals:

Signals are classified into the following categories:

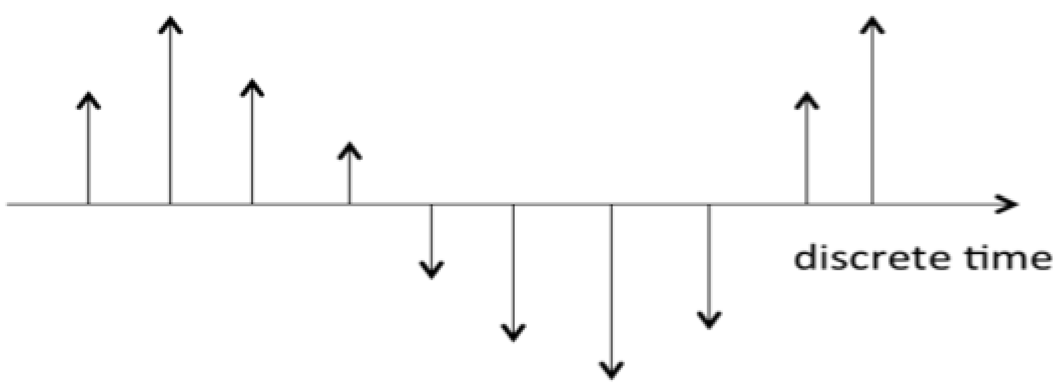
- Continuous Time and Discrete Time Signals
- Deterministic and Non-deterministic Signals
- Even and Odd Signals
- Periodic and Aperiodic Signals
- Energy and Power Signals
- Real and Imaginary Signals

Continuous Time and Discrete Time Signals

A signal is said to be continuous when it is defined for all instants of time.

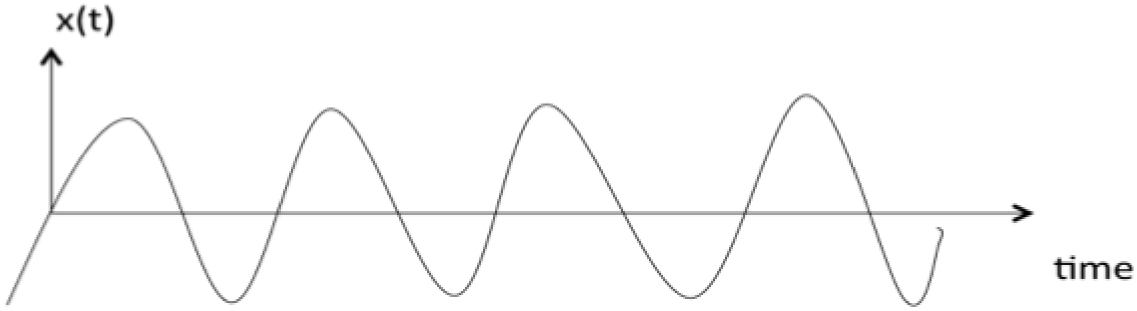


A signal is said to be discrete when it is defined at only discrete instants of time/

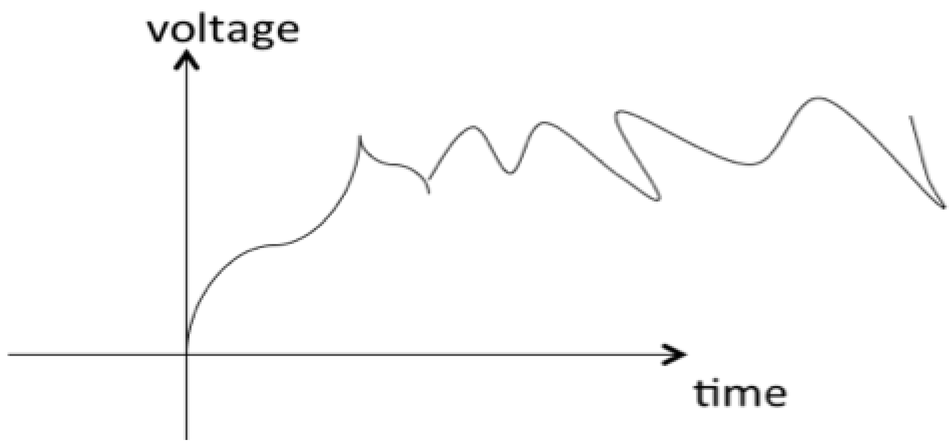


Deterministic and Non-deterministic Signals

A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula are known as deterministic signals.



A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic terms.



Even and Odd Signals

A signal is said to be even when it satisfies the condition x(t) = x(-t)

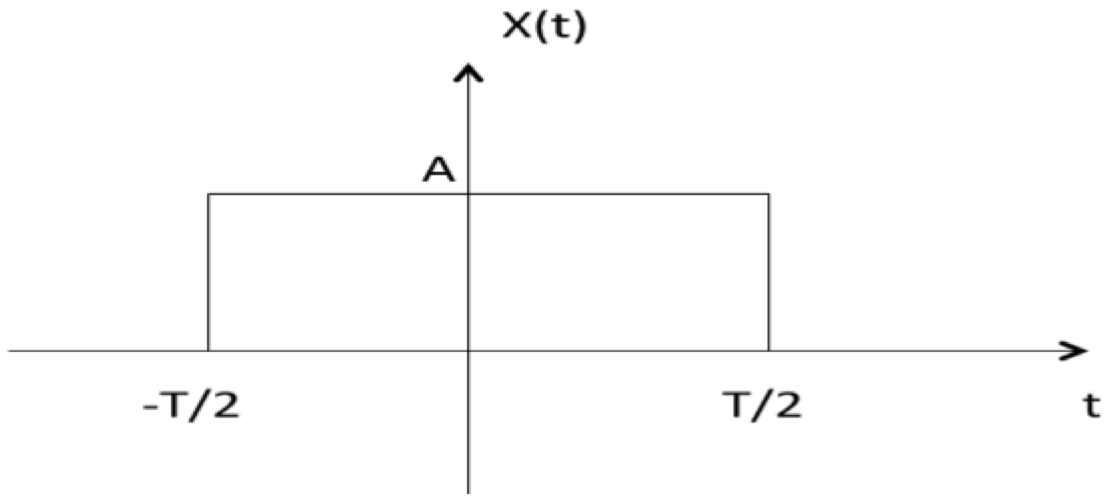
Example 1: t^2, t^4… cost etc.

Let $x(t) = t^2$

$x(-t) = (-t)^2 = t^2 = x(t)$

$\therefore$ t^2 is even function

Example 2: As shown in the following diagram, rectangle function x(t) = x(-t) so it is also even function.



A signal is said to be odd when it satisfies the condition x(t) = -x(-t)

Example: t, t^3 ... And sin t

Let x(t) = sin t

x(-t) = sin(-t) = -sin t = -x(t)

∴ sin t is odd function.

Any function $f(t)$ can be expressed as the sum of its even function $f_e(t)$ and odd function $f_o(t)$.

$f(t) = f_e(t) + f_0(t)$

where

$f_e(t) = ½[f(t) + f(-t)]$

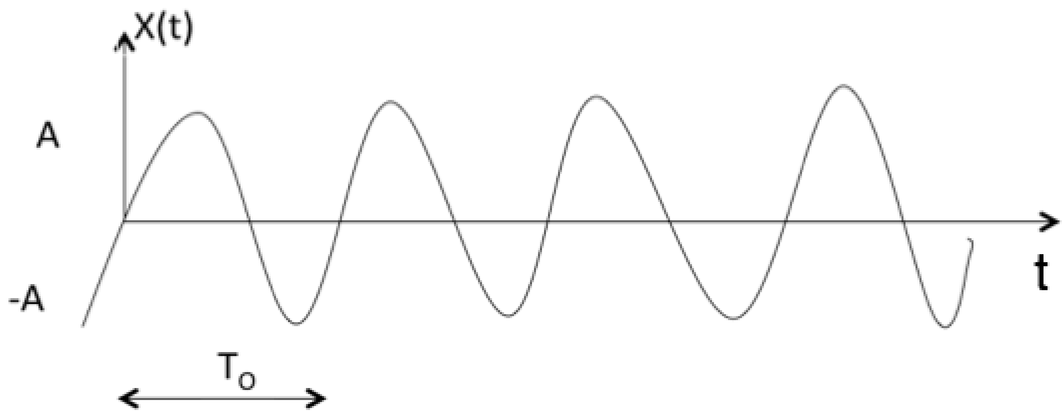
Periodic and Aperiodic Signals

A signal is said to be periodic if it satisfies the condition x(t) = x(t + T) or x(n) = x(n + N).

Where

T = fundamental time period,

1/T = f = fundamental frequency.



The above signal will repeat for every time interval T_0 hence it is periodic with period T_0.

Energy and Power Signals

A signal is said to be energy signal when it has finite energy.

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2\,(t)dt$$

A signal is said to be power signal when it has finite power.

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

NOTE:A signal cannot be both, energy and power simultaneously. Also, a signal may be neither energy nor power signal.

Power of energy signal = 0
Energy of power signal = ∞

Real and Imaginary Signals

A signal is said to be real when it satisfies the condition x(t) = x*(t)
A signal is said to be odd when it satisfies the condition x(t) = -x*(t)
Example:

If x(t)= 3 then x*(t)=3*=3 here x(t) is a real signal.
If x(t)= 3j then x*(t)=3j* = -3j = -x(t) hence x(t) is a odd signal.

Note: For a real signal, imaginary part should be zero. Similarly for an imaginary signal, real part should be zero.

Basic operations on Signals:

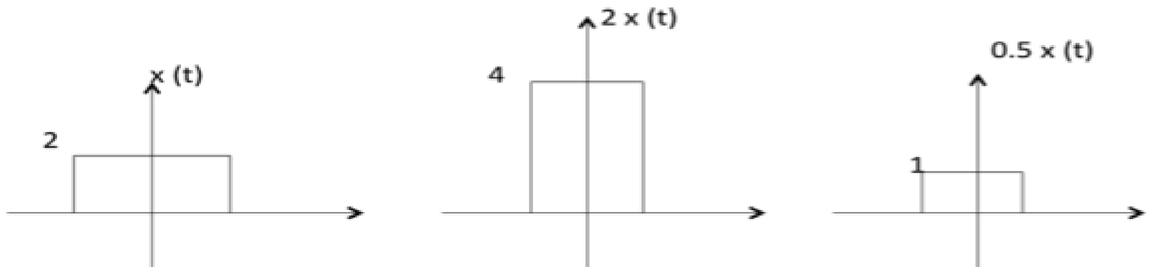
There are two variable parameters in general:

1. Amplitude
2. Time

(1) The following operation can be performed with amplitude:

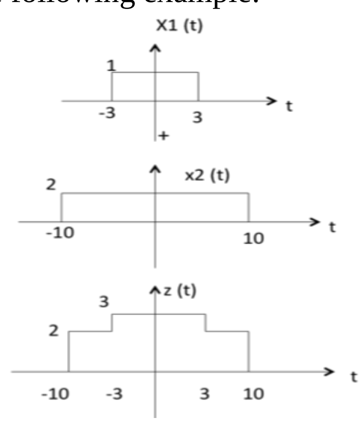
Amplitude Scaling

C x(t) is a amplitude scaled version of x(t) whose amplitude is scaled by a factor C.



Addition

Addition of two signals is nothing but addition of their corresponding amplitudes. This can be best explained by using the following example:



As seen from the previous diagram,

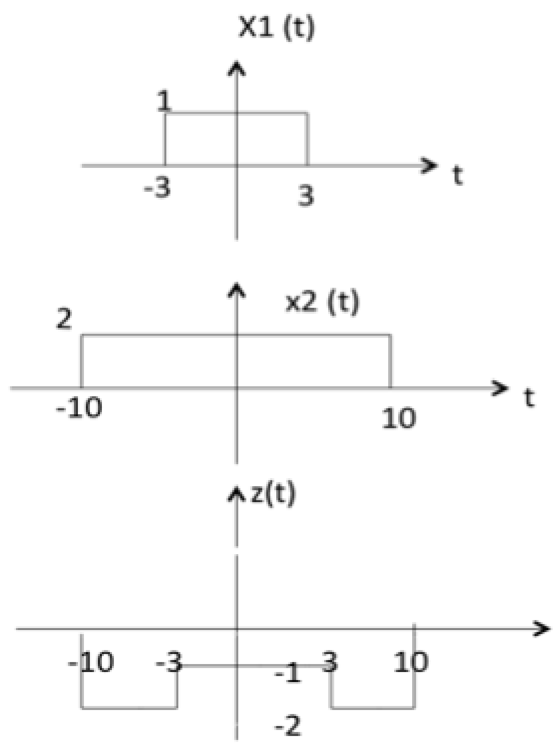
$$-10 < t < -3 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 0 + 2 = 2$$

$$-3 < t < 3 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 1 + 2 = 3$$

$$3 < t < 10 \text{ amplitude of } z(t) = x_1(t) + x_2(t) = 0 + 2 = 2$$

Subtraction

subtraction of two signals is nothing but subtraction of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

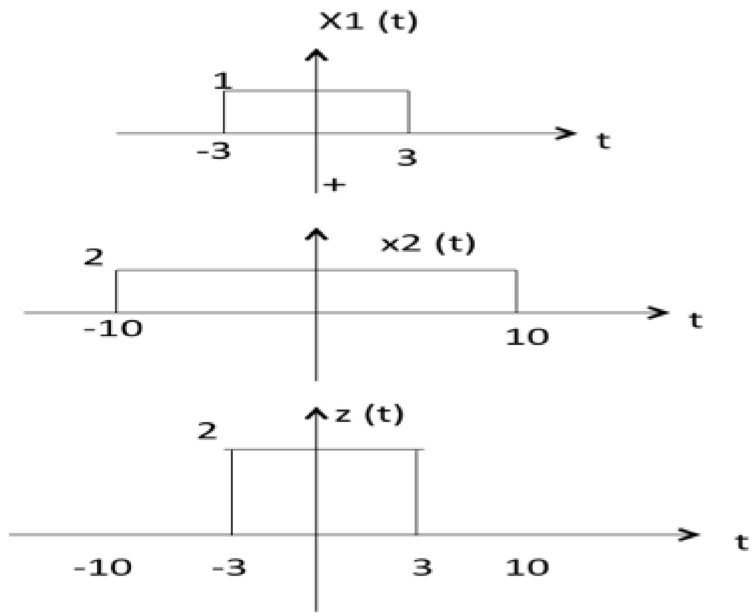
-10 < t < -3 amplitude of z (t) = $x_1(t) - x_2(t) = 0 - 2 = -2$

-3 < t < 3 amplitude of z (t) = $x_1(t) - x_2(t) = 1 - 2 = -1$

3 < t < 10 amplitude of z (t) = $x_1(t) - x_2(t) = 0 - 2 = -2$

Multiplication

Multiplication of two signals is nothing but multiplication of their corresponding amplitudes. This can be best explained by the following example:



As seen from the diagram above,

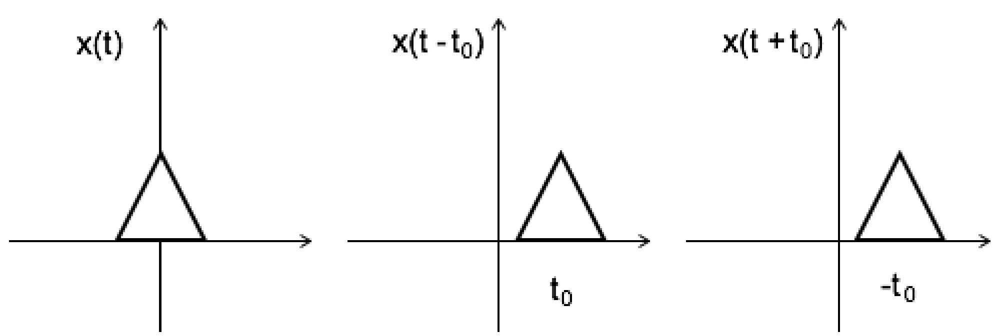
> $-10 < t < -3$ amplitude of z (t) = $x_1(t) \times x_2(t) = 0 \times 2 = 0$
> $-3 < t < 3$ amplitude of z (t) = $x_1(t) - x_2(t) = 1 \times 2 = 2$
> $3 < t < 10$ amplitude of z (t) = $x_1(t) - x_2(t) = 0 \times 2 = 0$

(2)The following operations can be performed with time:

<u>Time Shifting</u>

$x(t \pm t_0)$ is time shifted version of the signal x(t).

> $x\ (t + t_0) \longrightarrow$negative shift
> $x\ (t - t_0) \longrightarrow$positive shift

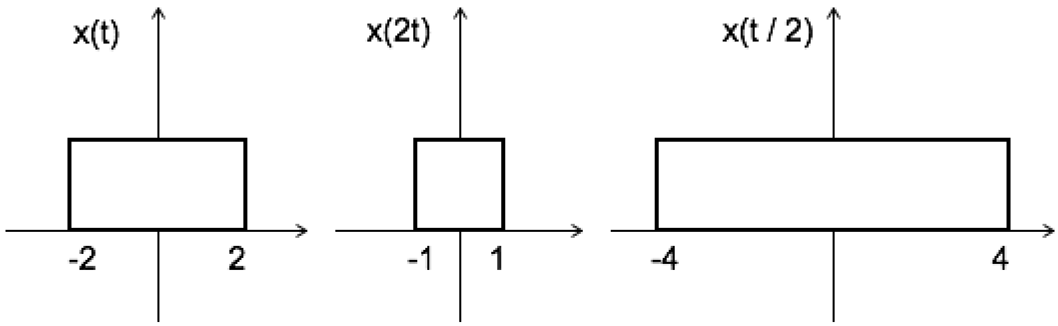


<u>Time Scaling</u>

x(At) is time scaled version of the signal x(t). where A is always positive.

|A| > 1 ⟶ Compression of the signal

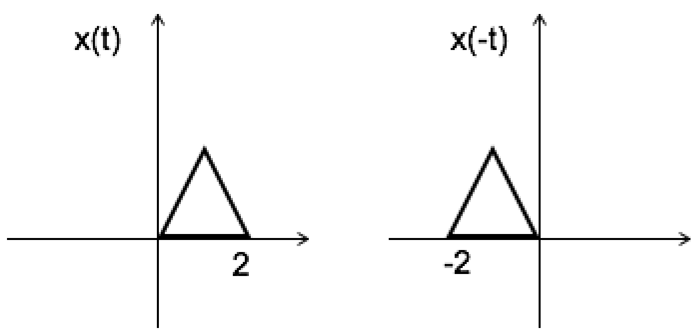
|A| < 1 ⟶ Expansion of the signal



Note: u(at) = u(t) time scaling is not applicable for unit step function.

<u>Time Reversal</u>

x(-t) is the time reversal of the signal x(t).



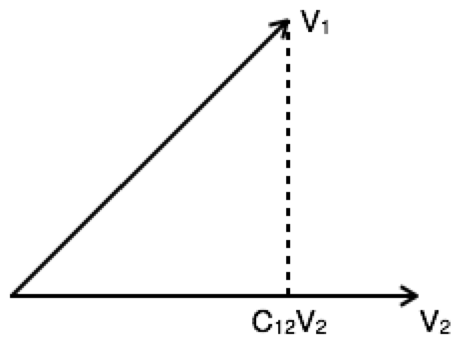
Analogy Between Vectors and Signals:

There is a perfect analogy between vectors and signals.

Vector

A vector contains magnitude and direction. The name of the vector is denoted by bold face type and their magnitude is denoted by light face type.

Example: V is a vector with magnitude V. Consider two vectors V_1 and V_2 as shown in the following diagram. Let the component of V_1 along with V_2 is given by $C_{12}V_2$. The component of a vector V_1 along with the vector V_2 can obtained by taking a perpendicular from the end of V_1 to the vector V_2 as shown in diagram:



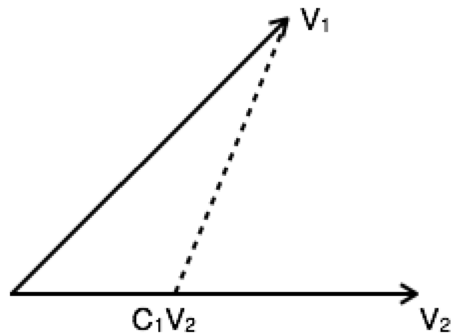
The vector V_1 can be expressed in terms of vector V_2

$$V_1 = C_{12}V_2 + V_e$$

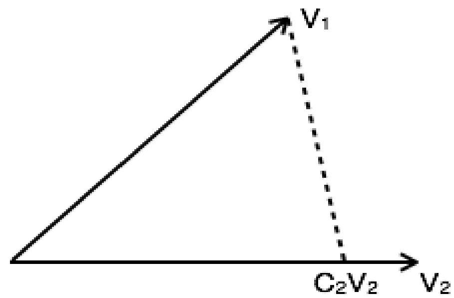
Where Ve is the error vector.

But this is not the only way of expressing vector V_1 in terms of V_2. The alternate possibilities are:

$V_1=C_1V_2+V_{e1}$



$V_2=C_2V_2+V_{e2}$



The error signal is minimum for large component value. If C_{12}=0, then two signals are said to be orthogonal.

Dot Product of Two Vectors

$V_1 . V_2 = V_1.V_2 \cos\theta$

θ = Angle between V1 and V2

$V_1. V_2 = V_2.V_1$

From the diagram, components of V_1 a long $V_2 = C_{12} V_2$

$$\frac{V_1 . V_2}{V_2 = C_1 2\, V_2}$$

$$\Rightarrow C_{12} = \frac{V_1 . V_2}{V_2}$$

Signal

The concept of orthogonality can be applied to signals. Let us consider two signals $f_1(t)$ and $f_2(t)$. Similar to vectors, you can approximate $f_1(t)$ in terms of $f_2(t)$ as

$f_1(t) = C_{12}\, f_2(t) + f_e(t)$ for $(t_1 < t < t_2)$

$\Rightarrow f_e(t) = f_1(t) - C_{12}\, f_2(t)$

One possible way of minimizing the error is integrating over the interval t_1 to t_2.

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [f_e(t)]dt$$

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [f_1(t) - C_{12}f_2(t)]dt$$

However, this step also does not reduce the error to appreciable extent. This can be corrected by taking the square of error function.

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [f_e(t)]^2 dt$$

$$\Rightarrow \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [f_e(t) - C_{12}f_2]^2 dt$$

Where ε is the mean square value of error signal. The value of C_{12} which minimizes the error, you need to calculate $d\varepsilon/dC12=0$

$$\Rightarrow \frac{d}{dC_{12}}\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [f_1(t) - C_{12}f_2(t)]^2 dt\right] = 0$$

$$\Rightarrow \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left[\frac{d}{dC_{12}} f_1^2(t) - \frac{d}{dC_{12}} 2 f_1(t) C_{12} f_2(t) + \frac{d}{dC_{12}} f_2^2(t) C_{12}^2\right] dt = 0$$

Derivative of the terms which do not have C12 term are zero.

$$\Rightarrow \int_{t_1}^{t_2} -2 f_1(t) f_2(t) dt + 2C_{12}\int_{t_1}^{t_2} [f_2^2(t)] dt = 0$$

If $C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t) dt}{\int_{t_1}^{t_2} f_2^2(t) dt}$ component is zero, then two signals are said to be orthogonal.

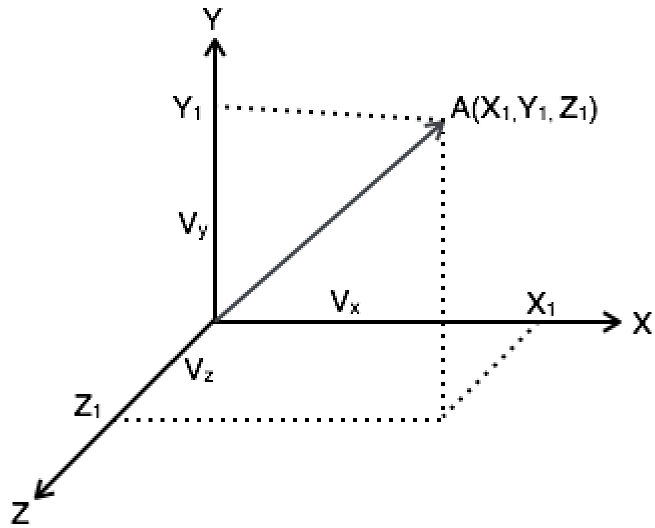
Put $C_{12} = 0$ to get condition for orthogonality.

$$0 = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t) dt}{\int_{t_1}^{t_2} f_2^2(t) dt}$$

$$\int_{t_1}^{t_2} f_1(t) f_2(t) dt = 0$$

Orthogonal Vector Space

A complete set of orthogonal vectors is referred to as orthogonal vector space. Consider a three dimensional vector space as shown below:



Consider a vector A at a point (X_1, Y_1, Z_1). Consider three unit vectors (V_X, V_Y, V_Z) in the direction of X, Y, Z axis respectively. Since these unit vectors are mutually orthogonal, it satisfies that

$$V_X . V_X = V_Y . V_Y = V_Z . V_Z = 1$$
$$V_X . V_Y = V_Y . V_Z = V_Z . V_X = 0$$

We can write above conditions as

$$V_a \,.\, V_b = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

The vector A can be represented in terms of its components and unit vectors as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Any vectors in this three dimensional space can be represented in terms of these three unit vectors only.

If you consider n dimensional space, then any vector A in that space can be represented as

$$A = X_1 V_X + Y_1 V_Y + Z_1 V_Z + \ldots + N_1 V_N \ldots\ldots (2)$$

As the magnitude of unit vectors is unity for any vector A

The component of A along x axis = $A.V_X$
The component of A along Y axis = $A.V_Y$
The component of A along Z axis = $A.V_Z$

Similarly, for n dimensional space, the component of A along some G axis

$$= A.V_G \ldots\ldots\ldots\ldots\ldots (3)$$

Substitute equation 2 in equation 3.

$$\Rightarrow CG = (X_1 V_X + Y_1 V_Y + Z_1 V_Z + \ldots + G_1 V_G \ldots + N_1 V_N) V_G$$

$$= X_1 V_X V_G + Y_1 V_Y V_G + Z_1 V_Z V_G + \ldots + G_1 V_G V_G \ldots + N_1 V_N V_G$$

$$= G_1 \quad \text{since } V_G V_G = 1$$

$$If V_G V_G \neq 1 \text{ i.e.} V_G V_G = k$$

$$A V_G = G_1 V_G V_G = G_1 K$$

$$G_1 = \frac{(A V_G)}{K}$$

Orthogonal Signal Space

Let us consider a set of n mutually orthogonal functions $x_1(t)$, $x_2(t)$... $x_n(t)$ over the interval t_1 to t_2. As these functions are orthogonal to each other, any two signals $x_j(t)$, $x_k(t)$ have to satisfy the orthogonality condition. i.e.

$$\int_{t_1}^{t_2} x_j(t)x_k(t)dt = 0 \ \text{ where } j \neq k$$

$$\text{Let} \int_{t_1}^{t_2} x_k^2(t)dt = k_k$$

Let a function f(t), it can be approximated with this orthogonal signal space by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t) + f_e(t)$$

$$= \Sigma_{r=1}^{n} C_r x_r(t)$$

$$f(t) = f(t) - \Sigma_{r=1}^{n} C_r x_r(t)$$

Mean sqaure error $\varepsilon = \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f_e(t)]^2 dt$

$$= \frac{1}{t_2 - t_2} \int_{t_1}^{t_2} [f[t] - \sum_{r=1}^{n} C_r x_r(t)]^2 dt$$

The component which minimizes the mean square error can be found by

$$\frac{d\varepsilon}{dC_1} = \frac{d\varepsilon}{dC_2} = \ldots = \frac{d\varepsilon}{dC_k} = 0$$

Let us consider $\frac{d\varepsilon}{dC_k} = 0$

$$\frac{d}{dC_k}[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt] = 0$$

All terms that do not contain C_k is zero. i.e. in summation, r=k term remains and all other terms are zero.

$$\int_{t_1}^{t_2} -2f(t)x_k(t)dt + 2C_k\int_{t_1}^{t_2}[x_k^2(t)]dt = 0$$

$$\Rightarrow C_k = \frac{\int_{t_1}^{t_2} f(t)x_k(t)dt}{int_{t_1}^{t_2} x_k^2(t)dt}$$

$$\Rightarrow \int_{t_1}^{t_2} f(t)x_k(t)dt = C_k K_k$$

<u>Mean Square Error:</u>

The average of square of error function $f_e(t)$ is called as mean square error. It is denoted by ε (epsilon).

$$\varepsilon = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f_e(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f_e(t) - \Sigma_{r=1}^{n} C_r x_r(t)]^2 dt$$

$$= \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2}[f_e^2(t)]dt + \Sigma_{r=1}^{n} C_r^2\int_{t_1}^{t_2} x_r^2(t)dt - 2\Sigma_{r=1}^{n} C_r\int_{t_1}^{t_2} x_r(t)f(t)dt$$

You know that $C_r^2\int_{t_1}^{t_2} x_r^2(t)dt = C_r\int_{t_1}^{t_2} x_r(t)f(d)dt = C_r^2 K_r$

$$\varepsilon = \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2}[f^2(t)]dt + \Sigma_{r=1}^{n} C_r^2 K_r - 2\Sigma_{r=1}^{n} C_r^2 K_r]$$

$$= \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2}[f^2(t)]dt - \Sigma_{r=1}^{n} C_r^2 K_r]$$

$$\therefore \varepsilon = \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2}[f^2(t)]dt + (C_1^2 K_1 + C_2^2 K_2 + \ldots + C_n^2 K_n)]$$

The above equation is used to evaluate the mean square error.

Closed and Complete Set of Orthogonal Functions:

Let us consider a set of n mutually orthogonal functions $x_1(t)$, $x_2(t)...x_n(t)$ over the interval t_1 to t_2. This is called as closed and complete set when there exist no function f(t) satisfying the condition

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

If this function is satisfying the equation

$$\int_{t_1}^{t_2} f(t)x_k(t)dt = 0$$

For k=1,2,.. then f(t) is said to be orthogonal to each and every function of orthogonal set. This set is incomplete without f(t). It becomes closed and complete set when f(t) is included.

f(t) can be approximated with this orthogonal set by adding the components along mutually orthogonal signals i.e.

$$f(t) = C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t) + f_e(t)$$

If the infinite series $C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t)$ converges to ft then mean square error is zero.

Orthogonality in Complex Functions:

If $f_1(t)$ and $f_2(t)$ are two complex functions, then $f_1(t)$ can be expressed in terms of $f_2(t)$ as

$f_1(t) = C_{12}f_2(t)$.. with negligible error

$$\text{Where } C_{12} = \frac{\int_{t_1}^{t_2} f_1(t)f_2^*(t)dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt}$$

Where $f_2^*(t)$ is the complex conjugate of $f_2(t)$

If $f_1(t)$ and $f_2(t)$ are orthogonal then $C_{12} = 0$

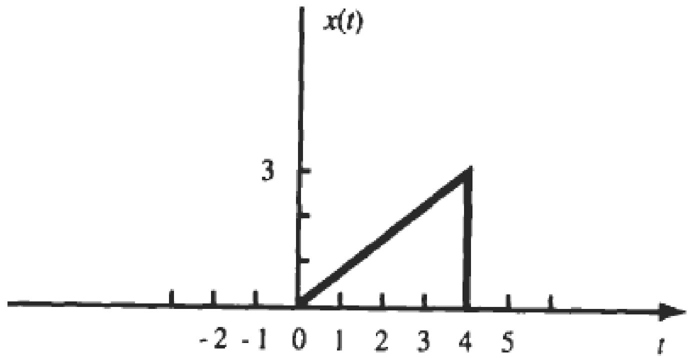
$$\frac{\int_{t_1}^{t_2} f_1(t) f_2^*(t) dt}{\int_{t_1}^{t_2} |f_2(t)|^2 dt} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} f_1(t) f_2^*(dt) = 0$$

The above equation represents orthogonality condition in complex functions.

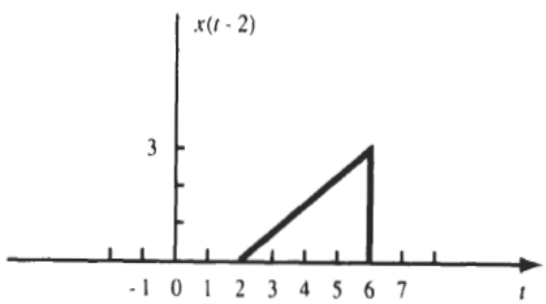
Problems

1. **A continuous-time signal *x (t)* is shown in the following figure. Sketch and label each of the following signals.**

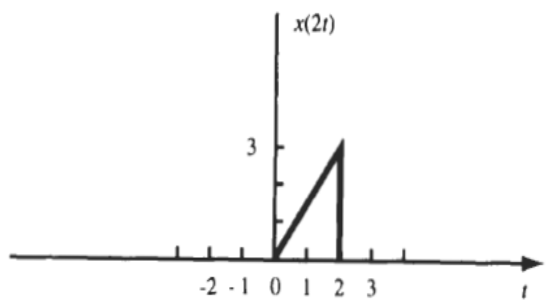


(a) x(t - 2); (b) x(2t); (c) x(t/2); (d) x (- t)

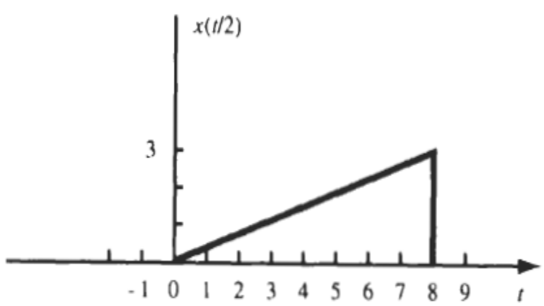
Sol:



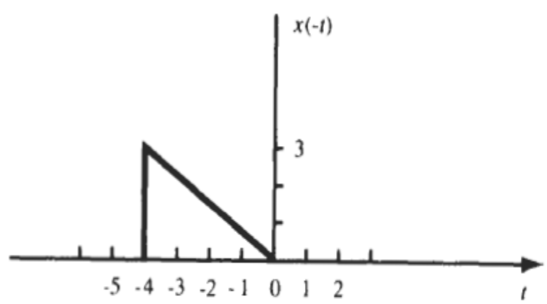
(*a*)



(*b*)



(*c*)



(*d*)

2. **Determine whether the following signals are energy signals, power signals, or neither.**

(a) $x(t) = e^{-at}u(t), \quad a > 0$

(b) $x(t) = A\cos(\omega_0 t + \theta)$

(c) $x(t) = tu(t)$

(d) $x[n] = (-0.5)^n u[n]$

(e) $x[n] = u[n]$

(f) $x[n] = 2e^{j3n}$

(a) $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{2a} < \infty$

Thus, $x(t)$ is an energy signal.

(b) The sinusoidal signal $x(t)$ is periodic with $T_0 = 2\pi/\omega_0$. Then by the result from Prob. 1.18, the average power of $x(t)$ is

$$P = \frac{1}{T_0}\int_0^{T_0}[x(t)]^2 dt = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} A^2\cos^2(\omega_0 t + \theta)\, dt$$

$$= \frac{A^2\omega_0}{2\pi}\int_0^{2\pi/\omega_0}\frac{1}{2}[1 + \cos(2\omega_0 t + 2\theta)]\, dt = \frac{A^2}{2} < \infty$$

Thus, $x(t)$ is a power signal. Note that periodic signals are, in general, power signals.

(c) $E = \lim_{T\to\infty}\int_{-T/2}^{T/2}|x(t)|^2 dt = \lim_{T\to\infty}\int_0^{T/2} t^2\, dt = \lim_{T\to\infty}\frac{(T/2)^3}{3} = \infty$

$P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} t^2\, dt = \lim_{T\to\infty}\frac{1}{T}\frac{(T/2)^3}{3} = \infty$

Thus, $x(t)$ is neither an energy signal nor a power signal.

(d) we know that energy of a signal is

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

And by using

$$\sum_{n=0}^{\infty}\alpha^n = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

we obtain

$$E = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

Thus, ***x [n]*** is a power signal.

(e) By the definition of power of signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$
$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1^2 = \lim_{N\to\infty} \frac{1}{2N+1}(N+1) = \frac{1}{2} < \infty$$

Thus, $x[n]$ is a power signal.

(*f*) Since $|x[n]| = |2e^{j3n}| = 2|e^{j3n}| = 2$,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 2^2$$
$$= \lim_{N\to\infty} \frac{1}{2N+1} 4(2N+1) = 4 < \infty$$

Thus, $x[n]$ is a power signal.

3. Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(*a*) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$ (*b*) $x(t) = \sin \frac{2\pi}{3} t$

(*c*) $x(t) = \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t$ (*d*) $x(t) = \cos t + \sin \sqrt{2}\, t$

(*e*) $x(t) = \sin^2 t$ (*f*) $x(t) = e^{j[(\pi/2)t - 1]}$

(*g*) $x[n] = e^{j(\pi/4)n}$ (*h*) $x[n] = \cos \frac{1}{4} n$

(*i*) $x[n] = \cos \frac{\pi}{3} n + \sin \frac{\pi}{4} n$ (*j*) $x[n] = \cos^2 \frac{\pi}{8} n$

Sol:

(*a*) $x(t) = \cos\left(t + \frac{\pi}{4}\right) = \cos\left(\omega_0 t + \frac{\pi}{4}\right) \longrightarrow \omega_0 = 1$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 2\pi$.

(*b*) $x(t) = \sin \frac{2\pi}{3} t \longrightarrow \omega_0 = \frac{2\pi}{3}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 3$.

(*c*) $x(t) = \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t = x_1(t) + x_2(t)$

where $x_1(t) = \cos(\pi/3)t = \cos \omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 6$ and $x_2(t) = \sin(\pi/4)t = \sin \omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = 8$. Since $T_1/T_2 = \frac{6}{8} = \frac{3}{4}$ is a rational number, $x(t)$ is periodic with fundamental period $T_0 = 4T_1 = 3T_2 = 24$.

(*d*) $x(t) = \cos t + \sin\sqrt{2}\,t = x_1(t) + x_2(t)$

where $x_1(t) = \cos t = \cos\omega_1 t$ is periodic with $T_1 = 2\pi/\omega_1 = 2\pi$ and $x_2(t) = \sin\sqrt{2}\,t = \sin\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \sqrt{2}\,\pi$. Since $T_1/T_2 = \sqrt{2}$ is an irrational number, $x(t)$ is nonperiodic.

(*e*) Using the trigonometric identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we can write

$$x(t) = \sin^2 t = \tfrac{1}{2} - \tfrac{1}{2}\cos 2t = x_1(t) + x_2(t)$$

where $x_1(t) = \frac{1}{2}$ is a dc signal with an arbitrary period and $x_2(t) = -\frac{1}{2}\cos 2t = -\frac{1}{2}\cos\omega_2 t$ is periodic with $T_2 = 2\pi/\omega_2 = \pi$. Thus, $x(t)$ is periodic with fundamental period $T_0 = \pi$.

(*f*) $x(t) = e^{j[(\pi/2)t-1]} = e^{-j}e^{j(\pi/2)t} = e^{-j}e^{j\omega_0 t} \longrightarrow \omega_0 = \dfrac{\pi}{2}$

$x(t)$ is periodic with fundamental period $T_0 = 2\pi/\omega_0 = 4$.

(*g*) $x[n] = e^{j(\pi/4)n} = e^{j\Omega_0 n} \longrightarrow \Omega_0 = \dfrac{\pi}{4}$

Since $\Omega_0/2\pi = \frac{1}{8}$ is a rational number, $x[n]$ is periodic, and by Eq. (*1.55*) the fundamental period is $N_0 = 8$.

(*h*) $x[n] = \cos\frac{1}{4}n = \cos\Omega_0 n \longrightarrow \Omega_0 = \frac{1}{4}$

Since $\Omega_0/2\pi = 1/8\pi$ is not a rational number, $x[n]$ is nonperiodic.

(*i*) $x[n] = \cos\dfrac{\pi}{3}n + \sin\dfrac{\pi}{4}n = x_1[n] + x_2[n]$

where

$$x_1[n] = \cos\frac{\pi}{3}n = \cos\Omega_1 n \longrightarrow \Omega_1 = \frac{\pi}{3}$$

$$x_2[n] = \sin\frac{\pi}{4}n = \cos\Omega_2 n \longrightarrow \Omega_2 = \frac{\pi}{4}$$

Since $\Omega_1/2\pi = \frac{1}{6}$ (= rational number), $x_1[n]$ is periodic with fundamental period $N_1 = 6$, and since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, from the result of Prob. 1.15, $x[n]$ is periodic and its fundamental period is given by the least common multiple of 6 and 8, that is, $N_0 = 24$.

(*j*) Using the trigonometric identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we can write

$$x[n] = \cos^2\frac{\pi}{8}n = \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{4}n = x_1[n] + x_2[n]$$

where $x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n$ is periodic with fundamental period $N_1 = 1$ and $x_2[n] = \frac{1}{2}\cos(\pi/4)n = \frac{1}{2}\cos\Omega_2 n \longrightarrow \Omega_2 = \pi/4$. Since $\Omega_2/2\pi = \frac{1}{8}$ (= rational number), $x_2[n]$ is periodic with fundamental period $N_2 = 8$. Thus, $x[n]$ is periodic with fundamental period $N_0 = 8$ (the least common multiple of N_1 and N_2).

5. Determine the even and odd components of the following signals:

$(a)\quad x(t) = u(t)$

$(b)\quad x(t) = \sin\left(\omega_0 t + \frac{\pi}{4}\right)$

$(c)\quad x[n] = e^{j(\Omega_0 n + \pi/2)}$

$(d)\quad x[n] = \delta[n]$

Ans. $(a)\quad x_e(t) = \frac{1}{2},\ x_o(t) = \frac{1}{2}\operatorname{sgn} t$

$(b)\quad x_e(t) = \frac{1}{\sqrt{2}}\cos\omega_0 t,\ x_o(t) = \frac{1}{\sqrt{2}}\sin\omega_0 t$

$(c)\quad x_e[n] = j\cos\Omega_0 n,\ x_o[n] = -\sin\Omega_0 n$

$(d)\quad x_e[n] = \delta[n],\ x_o[n] = 0$