UNIT-1

NETWORK THEOREMS

➢ Superposition Theorem Statement

The superposition theorem states that in any linear bilateral network that consisting of two or more independent sources, current through (or voltage across) an element is the algebraic sum of the currents through (voltages across) that element caused by each independent source acting alone with all other sources are replaced by their internal resistances.

Therefore, if the circuit consists of N independent sources, we have to analyse N circuits, each will produce a result with respect to each individual source. And finally, these individual results must be added to get the whole analysis of the circuit. Therefore, this require more work however, this theorem will be very useful in analysing the various parts of a complex circuit.

➢ Steps to Analyse Superposition Theorem

1. Consider the various independent sources in a given circuit.

2. Select and retain one of the independent sources and replace all other sources with their internal resistances or else replace the current sources with open circuits and voltage sources with short circuits.

3. To avoid confusion re-label the voltage and current notations suitably.

4. Find out the desired voltage/currents due to the one source acting alone using various circuit reduction techniques.

5. Repeat the steps 2 to 4 for each independent source in the given circuit.

6. Algebraically add all the voltages/currents that are obtained from each individual source (Consider the voltage signs and current directions while adding).

➢ Limitations of Superposition Theorem

1. For power calculations superposition theorem cannot be used as this theorem works based on the linearity. Because the power equation is not linear as it is the product of voltage and current or square of the current or square of the voltage.

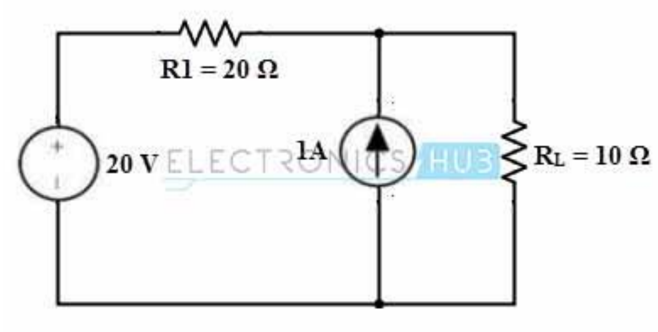
Thus the power consumed by the element in a given circuit with superposition theorem is not possible.

2. If the choice of the load is variable or the load resistance changes frequently, then it is required to perform every source contribution of current or voltage and their sum for every change in load resistance. So this very complex procedure for analysing complex circuits.

3. This theorem applicable for only linear circuits and for non linear circuits (Having transistors and diodes) we can not apply.

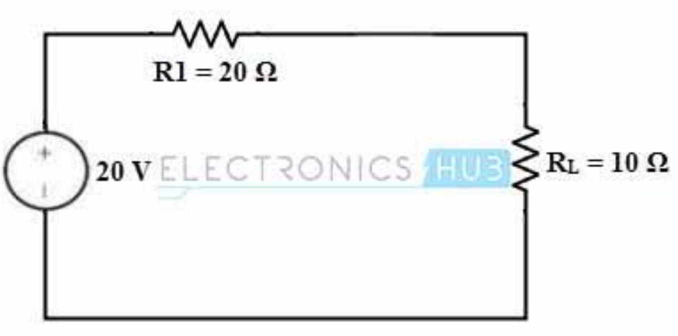
4. This theorem is applicable only if the circuit has more than one source.

Example 1:



1. Let us consider the below simple DC circuit to apply the superposition theorem such that we will obtain the voltage across the resistance 10 Ohms (load terminals). Consider that in a given circuit there are two independent sources as voltage and current sources as shown in figure.

2. First, we retain one source at a time that means , only voltage source is acting in the circuit and the current source is replaced with internal resistance (infinite) so it becomes open circuited as shown in figure.



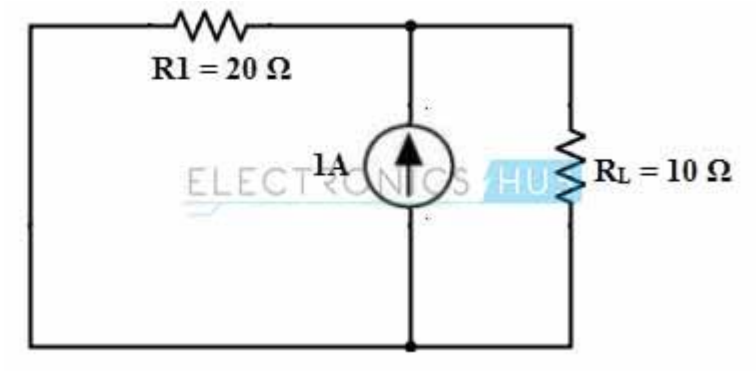
Consider V_{L1} is the voltage across the load terminals with voltage source acting alone, then

$V_{L1} = Vs \times R_L / (R_L + R_1)$

$= 20 \times 10 / (10 + 20)$

$= 6.66$ Volts

3. Retain the current source alone and replace the voltage source with its internal resistance (zero) so it becomes a short circuited as shown in figure.



Consider that V_{L2} is the voltage across the load terminals when current source acting alone. Then $V_{L2} = I_L \times R_L$

$I_L = I \times R_1 / (R_1 + R_L)$

$= 1 \times 20 / (20 + 30) \rightarrow \quad = 0.4$ Amps

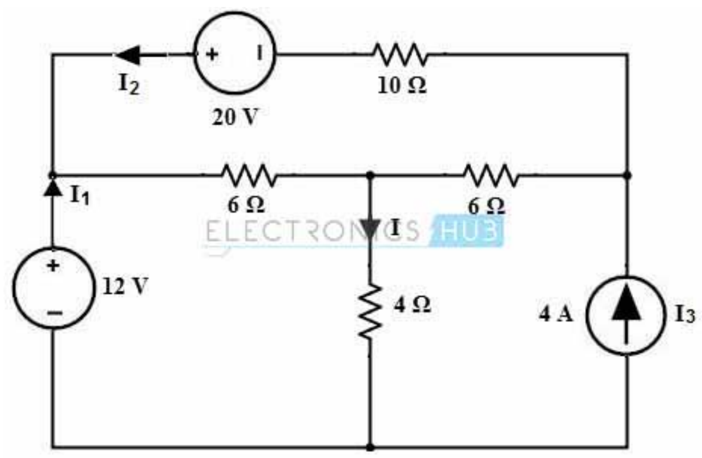
$$V_{L2} = 0.4 \times 10 = 4 \text{ Volts}$$

Therefore, according the superposition theorem, the voltage across the load is the sum of V_{L1} and V_{L2}

$$V_L = V_{L1} + V_{L2} = 6.66 + 4 = 10.66 \text{ Volts}$$

Example 2:

Consider the below circuit to which we are going to determine the current I through the 4 ohm resistor using superposition theorem.



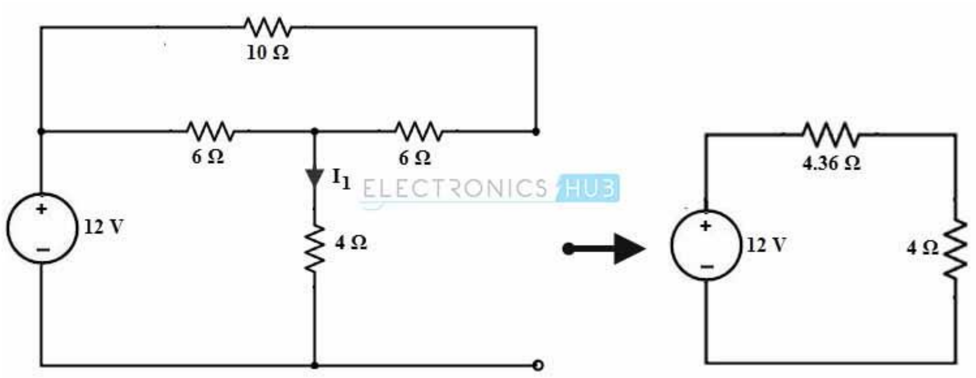
Consider I1, I2 and I3 are the currents due to sources 12v, 20V and 4A sources respectively. Then, based superposition theorem

$I = I_1 + I_2 + I_3$. So let's determine these currents with each individual source.

Only with 12V Voltage source:

Consider the below circuit where only 12V source is retained in the circuit and other sources are replaced by their internal resistances.

By combining the resistance 6 ohm with 10 ohm we get 16 ohm resistance which is parallel with 6 ohm resistance. Then this combination produce, 16 × 6 / (16 + 6) = 4.36 ohm. Therefore the equivalent circuit will be as shown in figure.

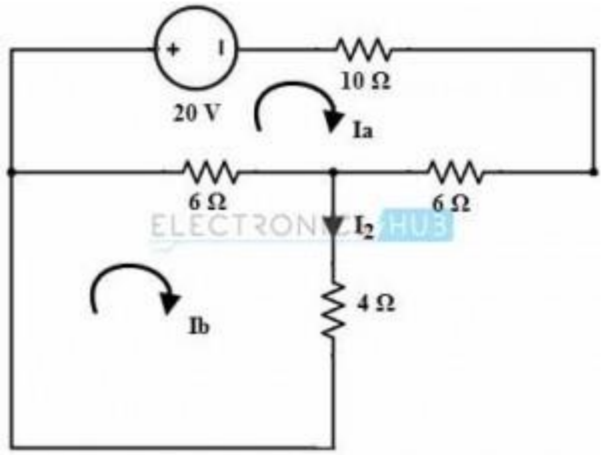


Then the current through 4 ohms resistance,

$$I_1 = 12 / 8.36 = 1.43 \text{ A}$$

Only with 20 V Voltage Source:

Retain only 20 V voltage source and replace other sources with their internal resistance, then the circuit becomes as shown below.



Apply the mesh analysis to the loop a, we get=

$22I_a – 6I_b + 20 = 0$

$22I_a – 6I_b = -20$ ……………….(1)

For Loop b, we get

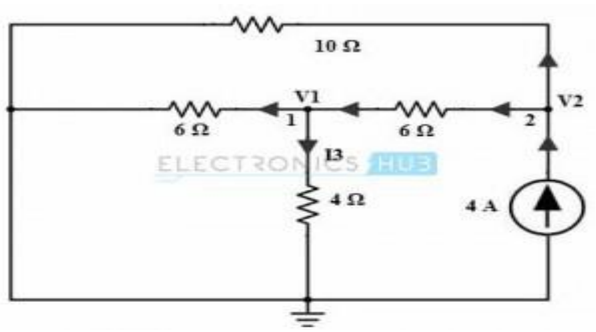
$10I_b – 6I_a = 0 \rightarrow I_a = 10I_b/6$

Substituting I_b in equation 1

$22 (10I_b/6) – 6I_b = -20 \rightarrow I_b = – 0.65$

Therefore, $I_2 = I_b = -0.65$

Only with 4A Current Source



Consider the below circuit where only current source is retained and other sources are replaced with their internal resistances.

By applying nodal analysis at node 2 we get,

$4 = (V_2/10) + (V_2 – V_1)/6$ ………………..(2)

At node1,

$(V_1/6) + (V_1/4) = (V_2 – V_1)/ 6 \rightarrow V_2 = 3.496\ V_1$

Substituting V_2 in equation 2, we get $V_1 = 0.766$ Volts.

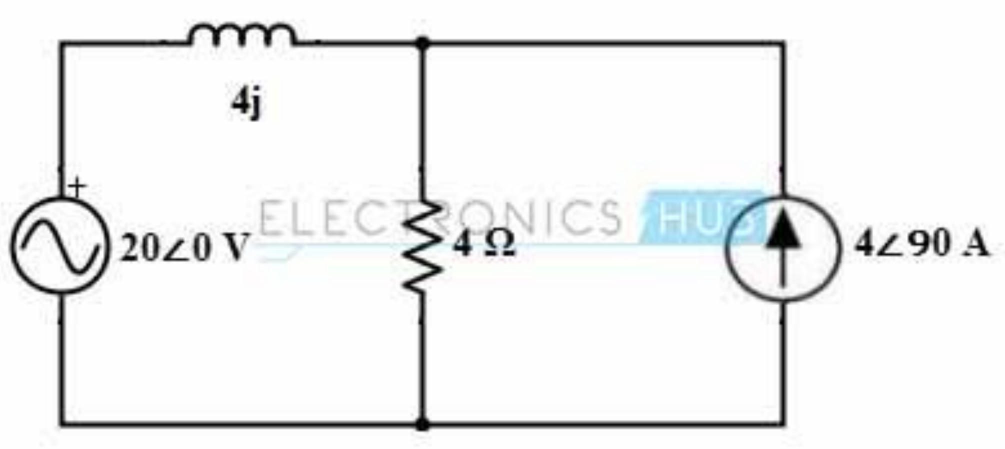
Therefore $I_3 = V_1/4 = 0.766/4 = 0.19$ Amps.

Thus, as per the superposition theorem, $I = I_1 + I_2 + I_3$

$= 1.43 – 0.65 + 0.19 = 0.97$Amps.

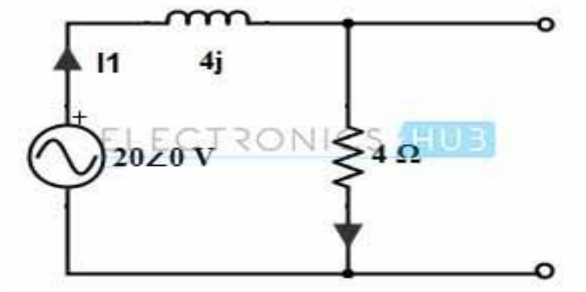
- **Superposition Example Using AC Circuit:**

Consider the below AC circuit, to which we are going to determine the value of current in the 4 ohm resistor using superposition theorem.



Case 1: Only with 20∠0 Voltage Source

By retaining the voltage source alone in the circuit, the current flow through the circuit is determined as



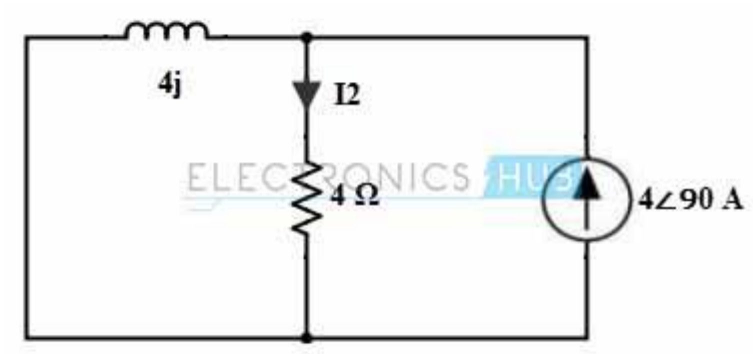
I_1 = 20∠0/ (4 + j4)

= 20∠0 / (5.65∠45)

= 3.53∠- 45 or 2.49 –j2.49 A

Case 2: Only with 4∠90 Current Source

By retaining the current source alone in the circuit, the current I_2 through the circuit is determined as



By current division method, I_2 = 4∠90 × 4j/ (4 + j4)

= 4∠90 × 4∠90 (5.65∠45)

= 4∠90 × 0.707∠45

= 2.828∠135 or -1.99 + j1.99 A

The resultant current through the resistor 4 ohms is I = I_1 + I_2

= 3.53∠- 45 + 2.828∠135 → = 0.785∠45 or 0.56 + j0.56 A

- **Reciprocity Theorem Statement**

Reciprocity Theorem states that – In any branch of a network or circuit, the current due to a single source of voltage (V) in the network is equal to the current through that branch in which the source was originally placed when the source is again put in the branch in which the current was originally obtained.This theorem is used in the bilateral linear network which consists bilateral components.

- **Steps for Solving a Network Utilizing Reciprocity Theorem**

Step 1 – Firstly, select the branches between which reciprocity has to be established.

Step 2 – The current in the branch is obtained using any conventional network analysis method.

Step 3 – The voltage source is interchanged between the branch which is selected.

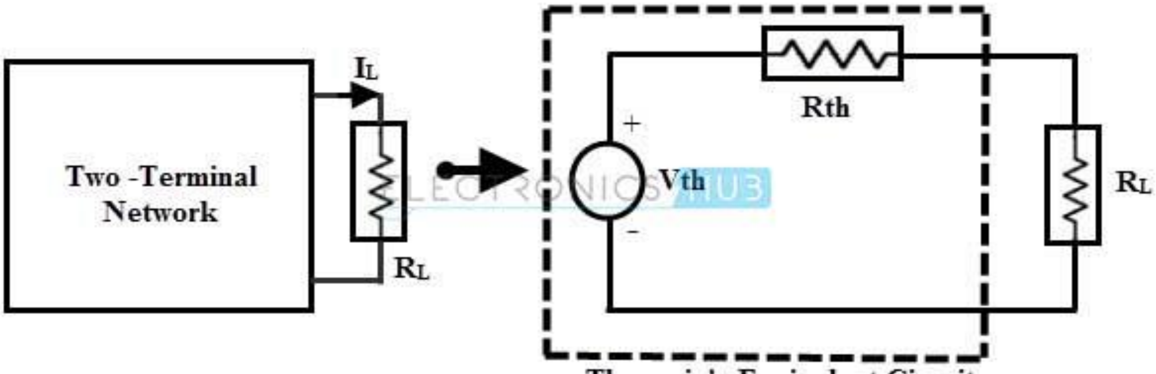
Step 4 – The current in the branch where the voltage source was existing earlier is calculated.

Step 5 – Now, it is seen that the current obtained in the previous connection, i.e., in step 2 and the current which is calculated when the source is interchanged, i.e., in step 4 are identical to each other.

- **Thevenin's Theorem Statement**

The Thevenin's theorem states that any linear two terminal circuit consisting of sources and resistors connected to a given load R_L can be replaced by an equivalent circuit consisting a single voltage source of magnitude Vth with a series resistance R_{th} across the terminal of R_L.

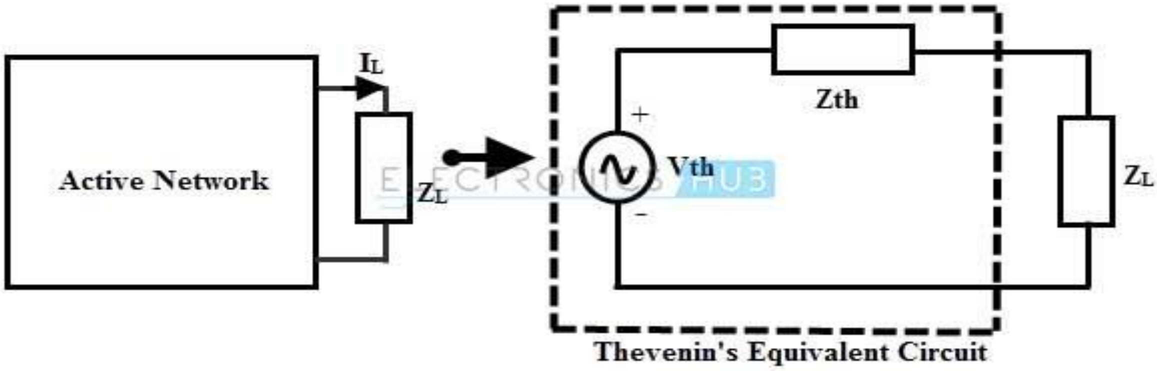
The below figure shows the Thevenin's model of two terminal network where the current through the load is same therefore, these two circuits are equivalent to each other.



Thevenin's Equivalent Circuit

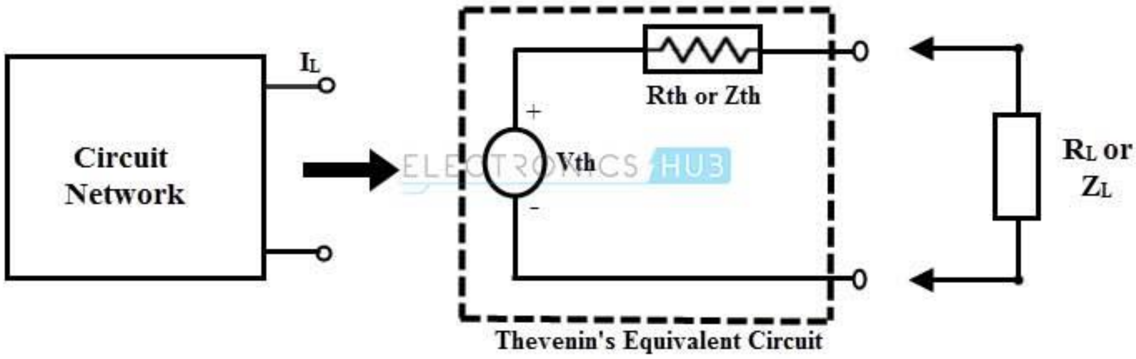
Similar to the DC circuits, this method can be applied to the AC circuits consisting of linear elements like resistors, inductors, capacitors. Like thevinin's equivalent resistance, equivalent thevinin's impedance is obtained by replacing all voltage sources by their internal impedances.

In AC circuits the thevenin's theorem can be stated as any two terminal, linear bilateral circuit consisting of linear elements and active sources connected across the terminal of ZL can be replaced by a single equivalent voltage source of Vth with a single impedance Zth across the two terminals of ZL.



- **Steps to Analyse the Thevenin's Theorem**

The following are the steps to simplify the circuit so that the load current is determined using Thevenin's theorem.



1. Consider the given circuit and disconnect the load resistance RL (load impedance ZL) or branch resistance (branch impedance in AC circuit) through which current flow is to be calculated.

2. Determine the open circuit voltage Vth across the load after disconnecting RL. For finding Vth, one can apply any methods from available circuit reduction techniques like mesh analysis, nodal voltage method, superposition, etc. Or simply, we can measure the voltage at the load terminals using a voltmeter.

3. Redraw the circuit by replacing all the sources with its internal resistances (Internal impedances in case of AC circuit) and make sure that voltage sources are to be short circuited and current sources to be open circuited (for ideal sources).

4. Calculate the total resistance Rth (or Zth) that exist between the load terminals.

5. Insert this equivalent resistance Rth (or Zth) in series with voltage Vth and this circuit is referred as the Thevenin's equivalent circuit.

6. Now reconnect the load resistance (load impedance ZL) across the load terminals and calculate the current, voltage and power of the load by simple calculations.

In DC circuit,

Load current, $I_L = V_{th}/ (R_L + R_{th})$

Voltage across the load, $V_L = R_L \times V_{th}/ (R_L + R_{th})$

Power dissipated in the load resistance, $P_L = R_L \times I_L^2$

In case of AC circuit Load current, $I_L = V_{th}/ (Z_L + Z_{th})$

Voltage acorss the load, $V_L = Z_L \times V_{th}/ (Z_L + Z_{th})$

Power dissipated in the load resistance, $P_L = V_L \times I_L^2$

- **Limitations of Thevinen's Theorem**

• If the circuit consists of non linear elements, this theorem is not applicable.

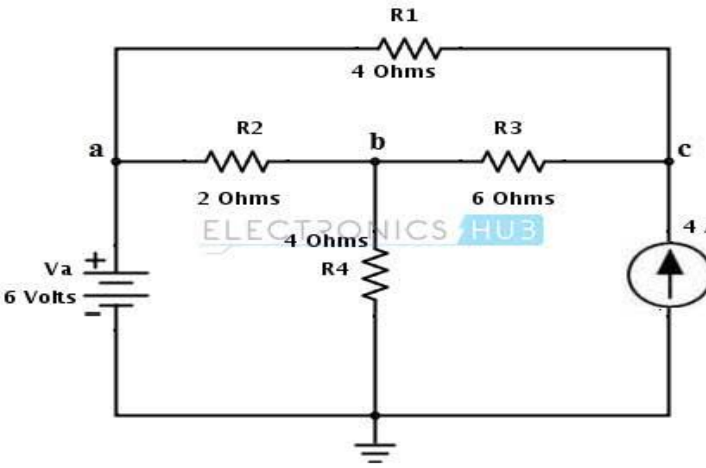
• Also to the unilateral networks it is not applicable.

• There should not be magnetic coupling between the load and circuit to be replaced with the thevinen's equivalent.

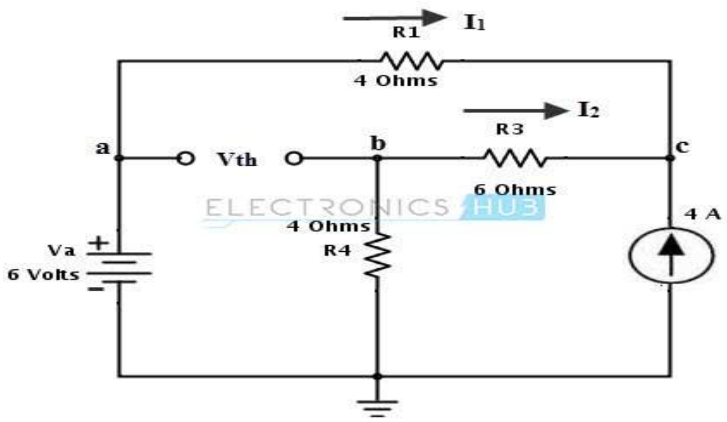
• There should not be controlled sources on the load side which care controlled from some other parts of the network.

➢ Example of Finding Equivalent Circuit to DC circuit

Consider the DC circuit shown in below. We are going to find the current through the resistance R_2 = R_L = 2 ohms (connected between terminals a and b) by applying Thevenin's theorem.



1. Remove the load resistance R_2 or R_L and assume the closed path direction towards point C.



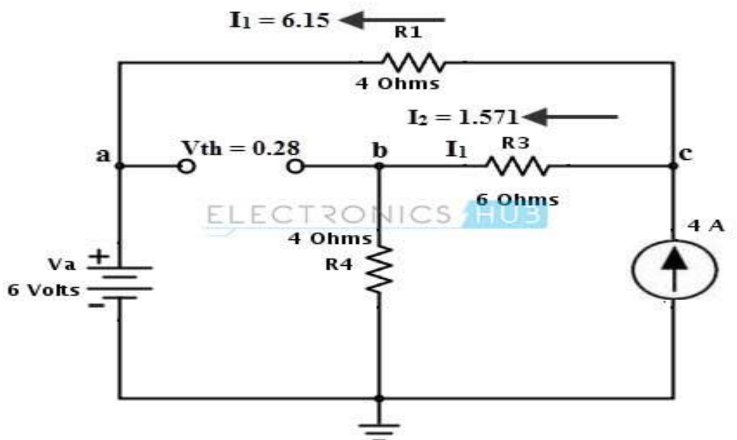
2. Apply nodal analysis at node C to calculate the thevenin's voltage Vth.

By applying KCL at node C we get

$$4 + I_1 + I_2 = 0$$

$$4 + (6 – Vc)/ 4 + (0 – Vc)/ 10 = 0$$

$$Vc = 15.714 \text{ Volts}$$



Then, the currents in each branch can be determined as

I_1 = Va – Vc/ 4 = 6 – 15.714/4 = 2.0715 Amps

$I_2 = 0 – Vc/ 10 = – 15.714/ 10 = – 1.571$ Amps

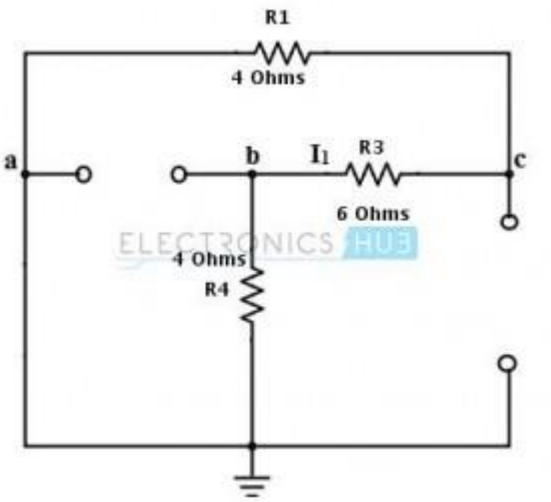
The negative symbol indicates that current is flowing from node C to their respective point (as ‘a’ and ground points for I1 and I2 respectively).

By redrawing the circuit with these currents and by applying KVL, the voltage across the terminal ab is determined as,

$V_{th} = V_a – V_b$ (with respect to ground terminals)

$$= V_a – (I_2 \times R_4) = 6 – (1.571 \times 4) = 0.28 \text{ Volts}$$

3. Next step is to replace all the sources with their internal sources. Consider that voltage source is an ideal source so the internal resistance is zero, therefore it is short circuited and the current source is an ideal current source therefore it has infinite resistance hence it is open circuited. Then the equivalent Thevenin’s resistance circuit is shown below.

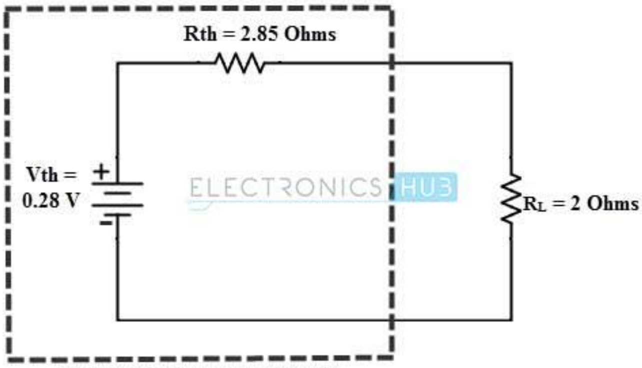


4. Next, we have to find the Thevenin’s equivalent resistance Rth by looking at the terminals a and b (load terminals).

$R_{th} = [(R_1 + R_3) \times R_4] / [(R_1 + R_3) + R4]$ (parallel resistors)

$$= 10 \times 4 / 10 + 4 = 2.85 \text{ Ohms}$$

5. By placing the above calculated voltage source in series with equivalent resistance forms Thevenin’s equivalent circuit as shown in below figure.



Thevenin's Equivalent Circuit

By reconnecting the load resistance across the terminals a and b, we calculate the current flowing through the load as

$I_L = V_{th} / (R_{th} + R_L)$ = 0.28/ (2.85 + 2) = 0.057 Amps

And below figure shows the original circuit where the current through the load resistance is indicated.

We can also find the currents through the load by changing the value of the load resistor as

When R_L = 8 ohms

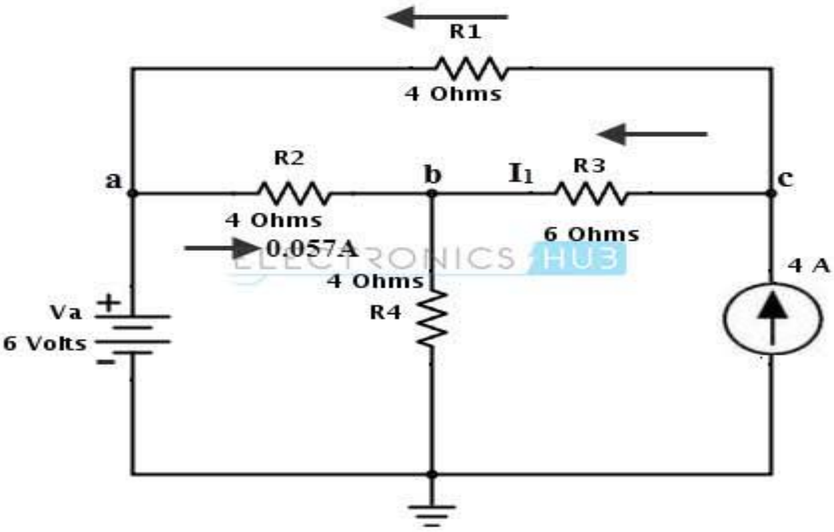
I_L = 0.28 / (2.85 + 8)

= 0.02Amps

When R_L = 12 ohms

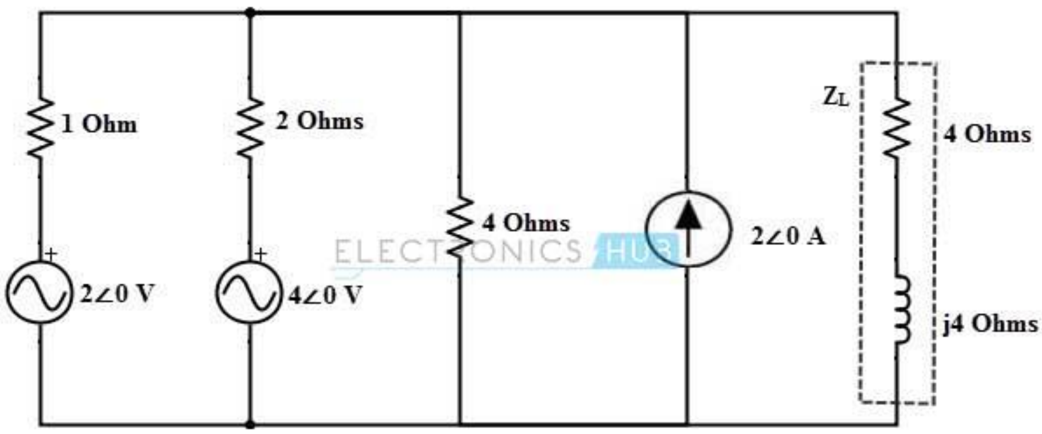
= 0.28 / (2.85 + 12)

= 0.01 Amps

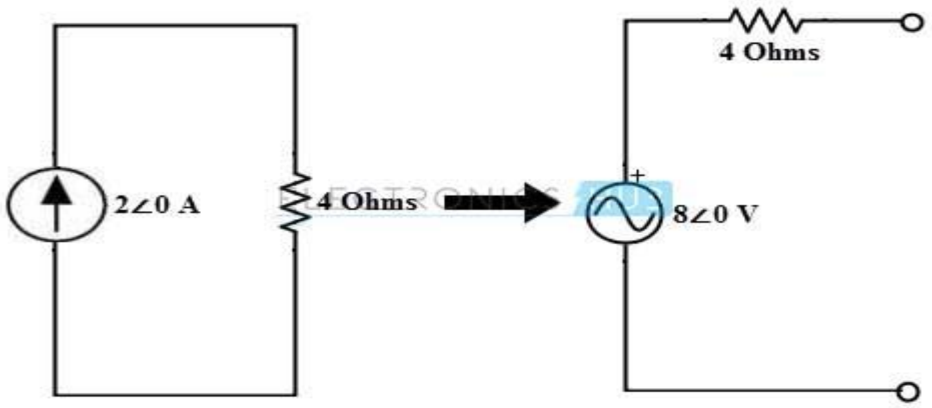


- **Example of Finding Equivalent Circuit to AC circuit**

Consider the below AC circuit to which we are going to find the current through the impedance 4+ 4j ohm using thevenin's theorem.



In the above circuit, the current source of $2\angle 0$ in parallel with 4 ohm resistor. And hence, this can be converted into a voltage source $8\angle 0$ with a series resistance of 4 ohms as shown in figure.

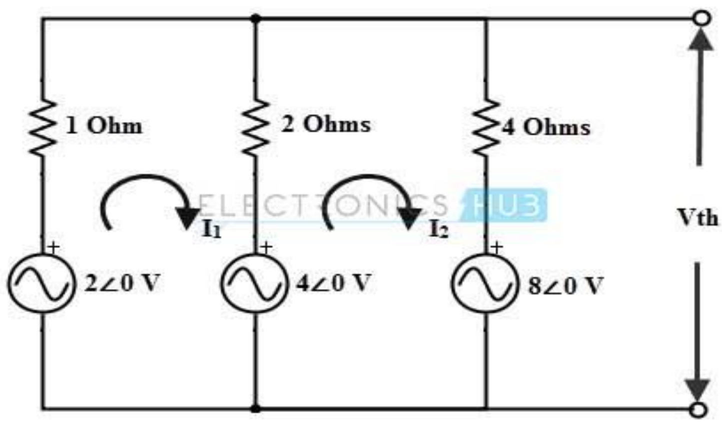


After making the above change, the circuit is redrawn by disconnect the load terminals as shown in below figure.

Assume the mesh currents as shown in modified diagram and KVL equation of the meshes are

$2\angle 0 - I_1 - 2(I_1 - I_2) - 4\angle 0 = 0$

$-3I_1 + 2I_2 = 2$ …….(1)



For mesh 2

$4\angle 0 - 2(I_2 - I_1) - 4I_2 - 8\angle 0 = 0$

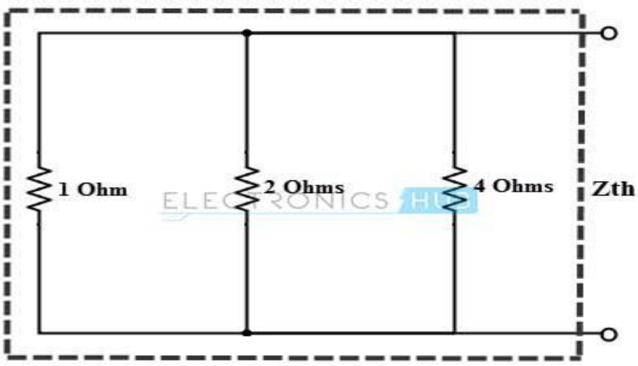
$2I_1 - 6I_2 = 4$ …….(2)

By solving above two equations, we get

$I_2 = -1.142\angle 0$ A

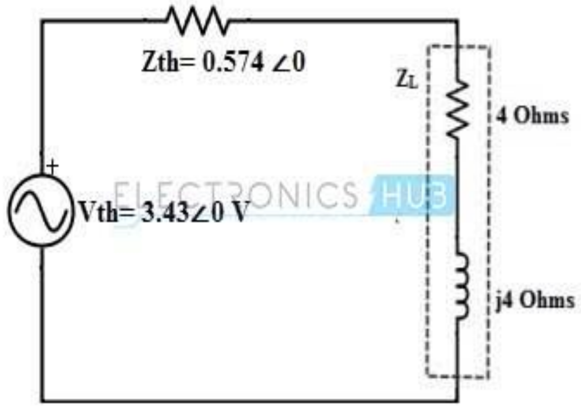
Therefore, $V_{th} = 8\angle 0 - 4\times(1.142\angle 0) = 3.43\angle 0$ V

The equivalent impedance,



$Z_{th} = 1/ (1 + (1/2) + (1/4)) \quad = 0.574 \angle 0 \Omega$

Therefore, the current through the 2 + 2j impedance is given as,



$I_{AB} = V_{th}/ Z_{th} + Z_L$

$= 3.43\angle 0/ (0.574 \angle 0 + 4 + 4j)$

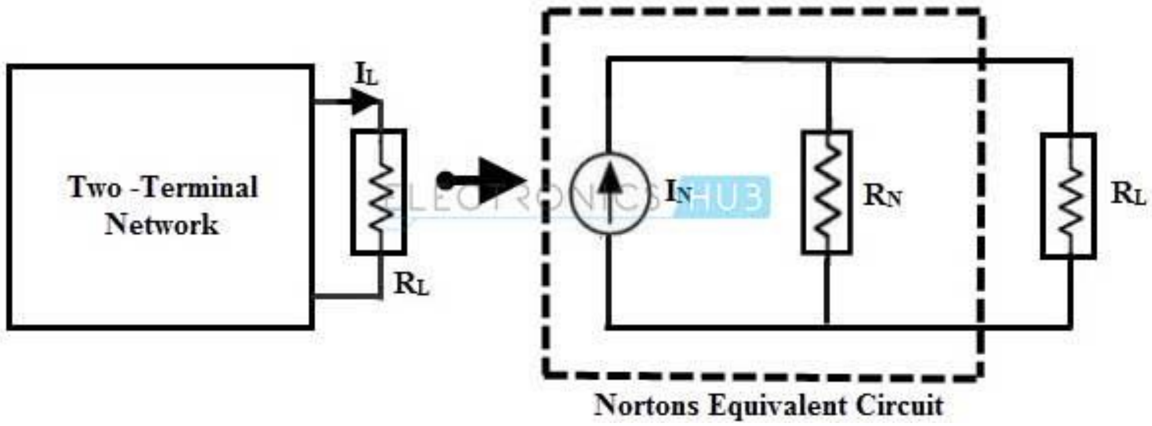
$= 3.43\angle 0 / (6.07 \angle 41.17)$

$= 0.56\angle -41.17$ A

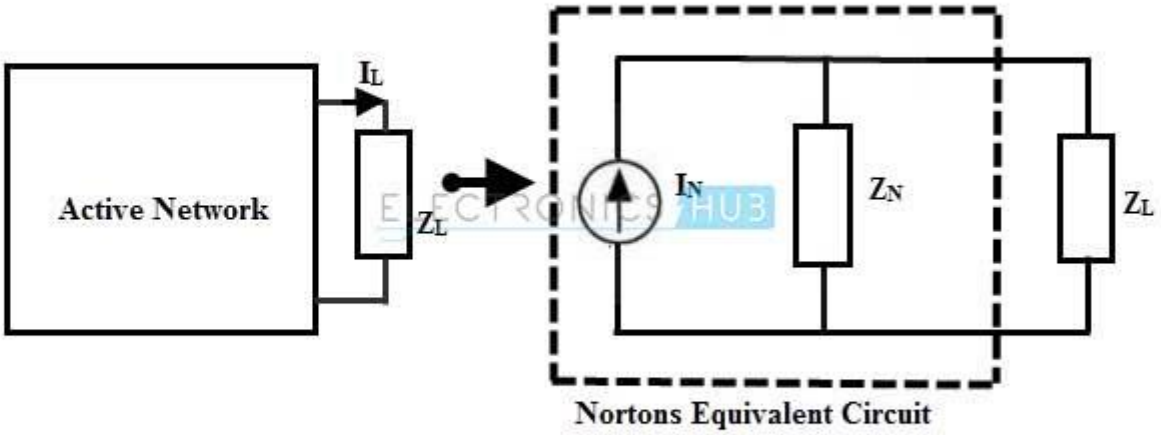
- **Norton's Theorem Statement**

Norton's theorem states that, any two terminal linear network that constitute independent sources and linear resistances can be replaced with an equivalent circuit , consisting of a current source with a parallel resistor. Magnitude of this equivalent current source is equal to the short circuit current flowing through the load terminals and the equivalent resistance is the resistance at the load terminals , when all the sources in a given circuit are replaced by their internal resistances.

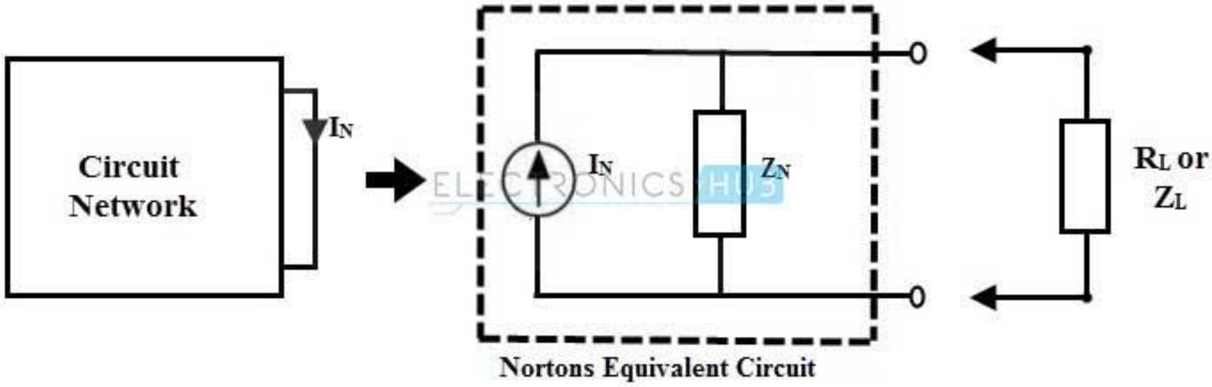
In below figure a part of a network , constituting of sources (either voltage or current or both) and resistances is replaced with a current source and a parallel resistor such that current flowing through the load is same in both cases.



For an AC circuit it can be stated as , any active two terminal network consisting of independent sources and impedances can be replaced with an equivalent circuit consisting of a constant current source with a parallel impedance. The value of the current source is equal to the current flow through the short circuited terminals of the network. And the parallel impedance is the equivalent impedance viewed from the short circuited terminals when all the sources are replaced with their internal impedances.



- **Steps to Analyse the Norton's Theorem**



To find the load variables using Norton's theorem, Norton's equivalent parameters have to be determined. Those are Norton current or magnitude of equivalent current source and Norton resistance Rn or impedance ZN. The following steps are required to determine them.

1. Consider the given circuit and short the load terminals after disconnecting the load resistance (or impedance in case of AC circuit) from output or load terminals.

2. Determine the short circuit current IN, through the shorted terminals by applying any of the circuit reduction techniques like mesh analysis or nodal analysis or superposition theorem. Or simply measure the load current using the ammeter experimentally.

3. Redraw the given circuit by replacing all the practical sources in the circuit with their internal voltages or simply short circuit voltages sources and open circuit the current sources . And also make sure to open or remove the short circuited terminals of the load.

4. Calculate the resistance (or impedance) that exists between the load terminals by looking from the load terminals. This resistance is equivalent Norton's resistance RN or (impedance ZN).

5. Insert the resistance (or impedance) in parallel with a current source IN which forms a Norton's equivalent circuit.

6. Now reconnect the load to the Norton's equivalent circuit and calculate the current, voltage and power associated with the load as

In DC circuit,

Load current, $I_L = I_N \times [R_N / (R_L + R_N)]$

Load Voltage, $V_L = I_L \times R_L$

Power dissipated at the load, $P = I_L^2 \times R_L$

In AC circuit,

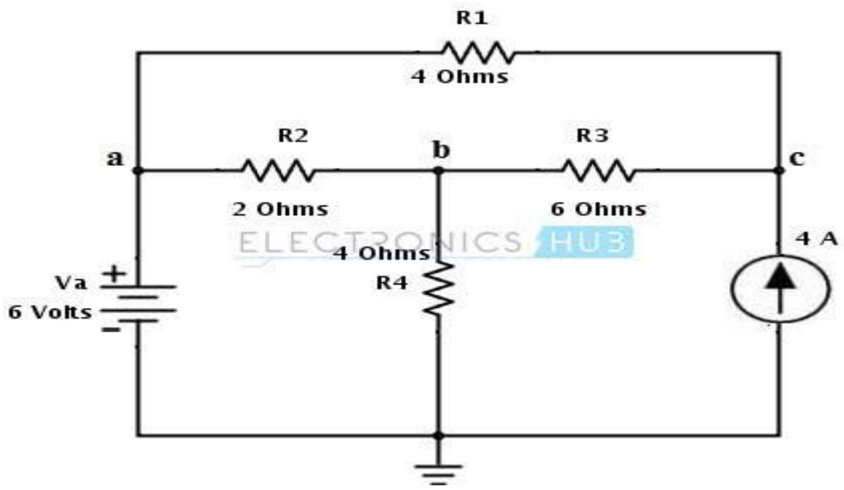
Load current, $I_L = I_N \times [Z_N / (Z_L + Z_N)]$

Load Voltage, $V_L = I_L \times Z_L$

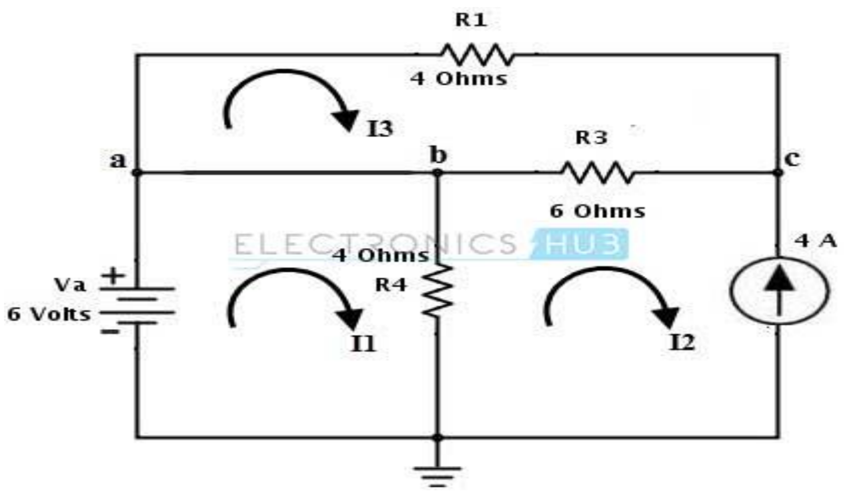
Power dissipated at the load, $P = I_L^2 \times Z_L$

Example for Finding Equivalent Circuit to DC Circuit

Let us consider the same DC circuit in Thevenin's theorem example to apply the Norton's theorem to find the current flow through branch ab i.e., through the load resistance RL = R2 = 2 ohms.



1. Disconnect the load resistance and short the load terminals a and b. Represent the current flow direction in every loop as shown in figure.



2. Apply the mesh analysis for each loop to find the current flow IN through the shorted terminals.

By applying KVL to loop 1 we get $6 – (I_1 – I_2) R_4 = 0$

Substituting $I_2 = -4A$

$I_1 = 6 – 16 / 4 = – 2.5$ A

By applying KVL to Loop 3 we get $– I_3R_1 – (I_3 – I_2)R_3 = 0$

$$-4I_3 – 6 (I_3 + 4) = 0$$

$$– 10I_3 = 24 \rightarrow I_3 = – 2.4 \text{ A}$$

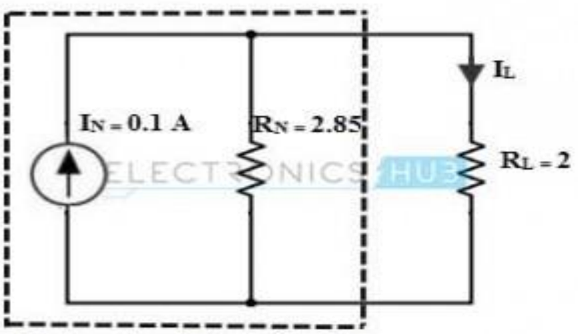
Therefore $I_n = I_1 – I_3 = -2.5 + 2.4 = 0.1$A which is flowing from a to b.

3. Next step is to determine the equivalent resistance RN. To compute this resistance all sources have to be replaced with their internal resistances by removing the short terminals of the load .

Then the total resistance across the terminals a and b, $R_N = 10 \times 4 / 10 + 4$

= 2.85 Ohms

4. By placing above calculated current In in parallel with resistance Rn forms a Norton's equivalent circuit as shown in figure. To determine the load variables we reconnect the load resistance across the load terminals.

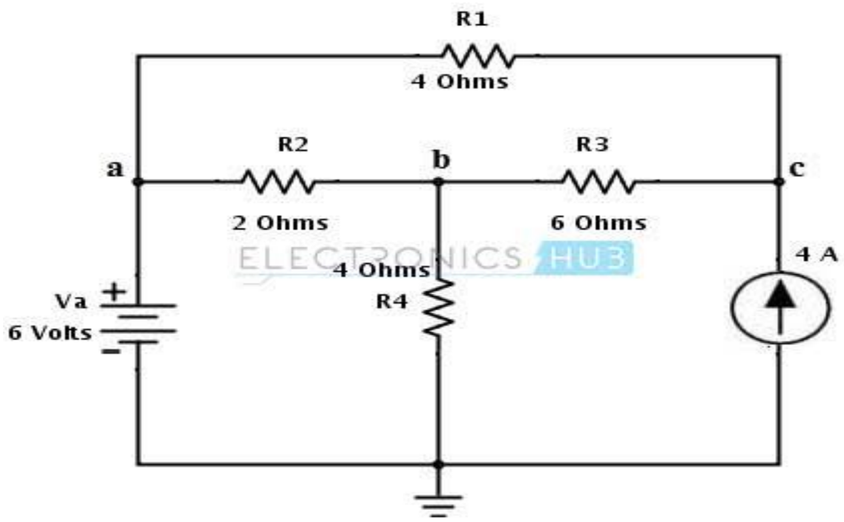


Nortons Equivalent Circuit

Then Load current $I_L = I_N \times [R_N / (R_L + R_N)]$

$= 0.1 \times [2.85/ (2 + 2.85)] = 0.05$ Amps

With the above calculated values, the original circuit is similar to the below shown figure with the representation of load current at branch ab.



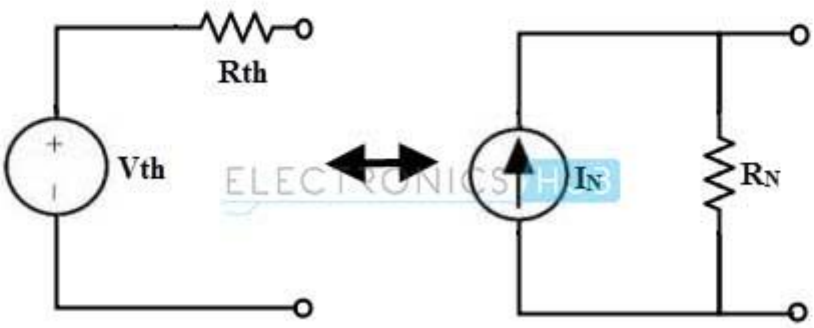
For different values of the load resistance the current flow is determined as

When R_L = 8 ohms $I_L = 0.1 \times [2.85 / (8 + 2.85)] = 0.02$ A

When R_L = 12 ohms $I_L = 0.1 \times [2.85 / (12+ 2.85)] = 0.01$ A.

- **Relation between Norton's and Thevinin's Theorems**

By comparing the above example with example of Thevenin's example problem, we can observe that Norton's equivalent circuit of a linear network constitute a Norton current source IN in parallel with a Thevenin's resistance Rth. Therefore it is possible to perform a source transformation of Thevenin's equivalent circuit to get the Norton's equivalent circuit or vice-versa.



Norton's Equivalent Source Transformation

The magnitude of the voltage source (Vth) and a series resistance (Rth) from Norton's equivalent circuit using source transformation are determined as

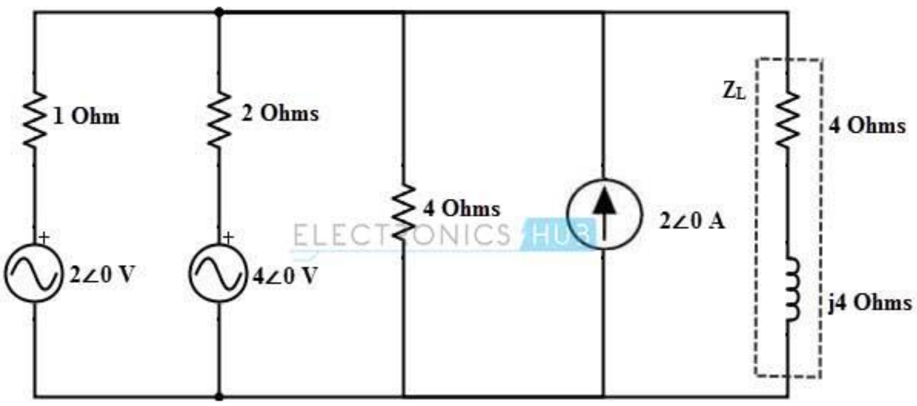
$$V_{th} = R_N \times I_N \text{ and } R_{th} = R_N$$

For above example $V_{th} = 2.85 \times 0.1 = 0.28$ Volts.

Therefore, we can use any of these two methods to analyse the circuit in a simple way. However the advantages of Thevenin's theorem are also applicable to Norton's theorem as well. By using these methods one can find the current and voltage values of different load resistance values without doing any complex calculations again and again. Hence the Norton's theorem aids the designing much easier based on the application. The use of these two theorems is decided by the application where these equivalents are required, such as current follower circuits (use Norton's equivalent) and voltage amplifiers (Thevenin's equivalent).

- **Example of Finding Equivalent Circuit to AC Circuit**

Consider the below AC circuit which was already analyzed using thevenin's theorem. In this circuit we are going to find the current through the impedance 4+ 4j ohm using Norton's theorem.



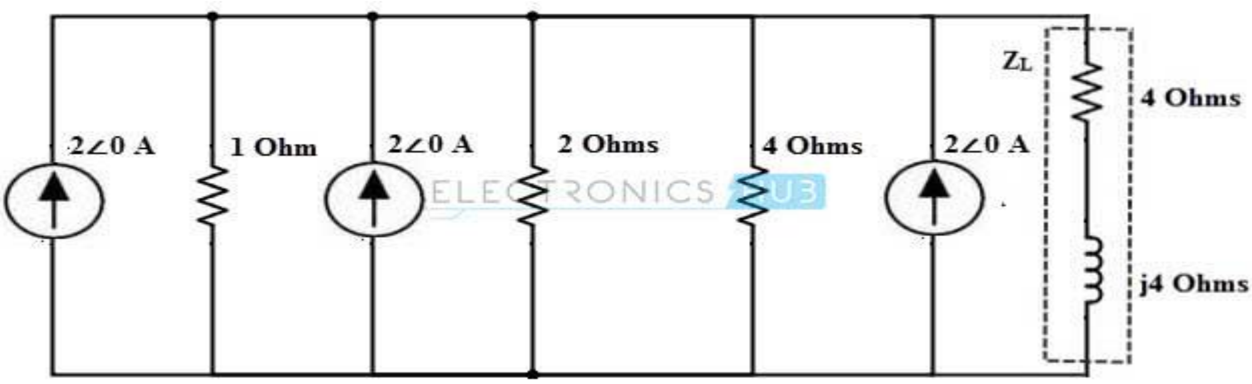
The above circuit consists of two voltage sources which can be transformed into the current source as

$$I_{S1} = V_{S1}/ R_{S1} = 2\angle 0 / 1 = 2\text{ A}$$

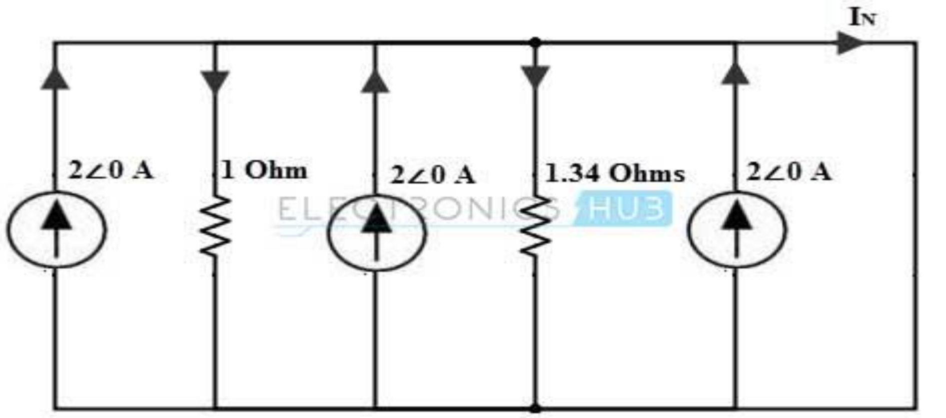
Similarly

$$I_{S2} = V_{S2}/ R_{S2} = 4\angle 0/ 2 = 2\text{ A}$$

Then the circuit becomes



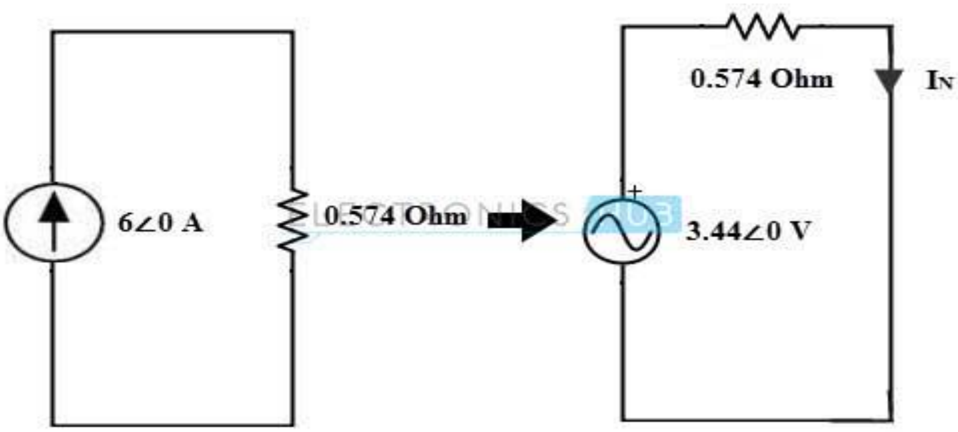
For applying Norton’s theorem, we disconnect the load impedance and short the load terminals as shown in figure. Assume the current directions as represented in figure.



Consider the above figure as single node and the total current becomes 6 amperes and the total parallel combination of resistance is 0.574 ohms. This can be transformed into a voltage source for ease of finding the Norton’s current is given as

$$V_s = 6\angle 0 \times 0.574 = 3.44\angle 0$$

Therefore, $I_N = V_N / 0.574 = 3.44\angle 0 / 0.574 = 5.97\angle 0$ A



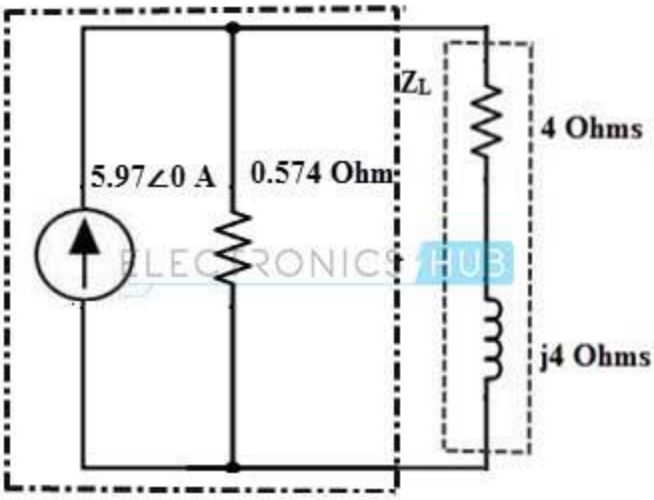
The Norton’s equivalent impedance is equal to the circuit equivalent impedance, $Z_N = 0.574$

Therefore the Load current across the 4 + j4 impedance is,

$I_L = I_N \times [Z_N / (Z_L + Z_N)]$

$= 5.97\angle 0 \times [0.574 / (4 + j4 + 0.574)]$
$= 3.42 / 6.07\angle 41.17 = 0.56 \angle -41.17$ A

This value is identical to the value obtained in the case thevenin's example of AC circuit. And hence Norton's theorem is the dual of thevenin's theorem.

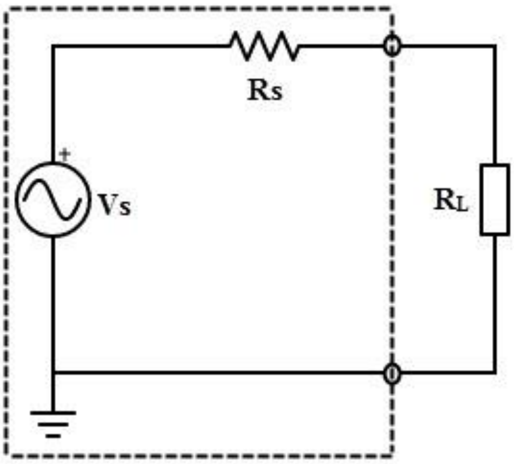


Norton's Equivalent Circuit

Limitations of Thevenin's theorem are also applicable for norton's theorem.

- **Maximum Power Transfer Theorem Statement**

The maximum power transfer theorem states that in a linear, bilateral DC network , maximum power is delivered to the load when the load resistance is equal to the internal resistance of a source.

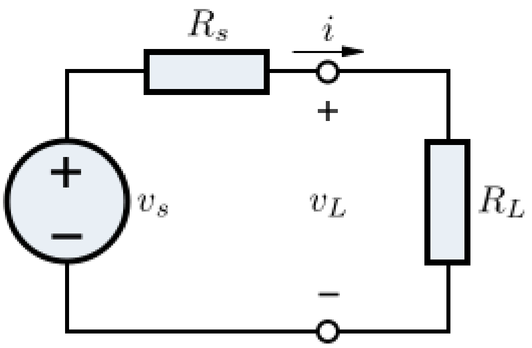


If it is an independent voltage source, then its series resistance (internal resistance R_s) or if it is independent current source, then its parallel resistance (internal resistance R_s) must equal to the load resistance RL to deliver maximum power to the load.

- **Proof of Maximum Power Transfer Theorem**

The maximum power transfer theorem ensures the value of the load resistance , at which the maximum power is transferred to the load.

Let us consider a source modelled by a Thévenin equivalent (a Norton equivalent will lead to the same result, as the two are directly equivalent), with a load resistance, R_L. The source resistance is R_s and the open circuit voltage of the source is v_s:



The current in this circuit is found using Ohm's Law: $i = \frac{V_L}{R_s + R_L}$

The voltage across the load resistor, v_L, is found using the voltage divider rule:

$$V_L = V_S \frac{R_L}{R_S + R_L}$$

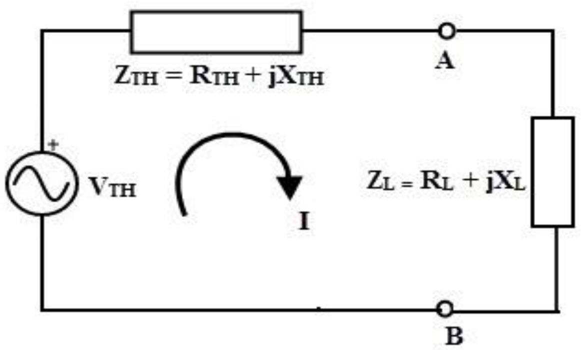
We can now find the power dissipated in the load, P_L as follows:

$$P_L = V_L i = \frac{R_L V_s^2}{(R_S + R_L)^2}$$

$$\boxed{\mathbf{R_L = R_{TH}}}$$

- **Maximum Power Transfer Theorem for AC Circuits**

It can be stated as in an active network, the maximum power is transferred to the load when the load impedance is equal to the complex conjugate of an equivalent impedance of a given network as viewed from the load terminals.



Consider the above Thevenin's equivalent circuit across the load terminals in which the current flowing through the circuit is given as

$$I = V_{TH} / Z_{TH} + Z_L$$

$$\text{Where } Z_L = R_L + jX_L$$

$$Z_{TH} = R_{TH} + jX_{TH}$$

$$\text{Therefore, } I = V_{TH} / (R_L + jX_L + R_{TH} + jX_{TH})$$

$$= V_{TH} / ((R_L + R_{TH}) + j(X_L + X_{TH}))$$

The power delivered to the load, $P_L = I^2 R_L$

$$P_L = V^2_{TH} \times R_L / ((R_L + R_{TH})^2 + (X_L + X_{TH})^2) \quad \ldots\ldots(1)$$

For maximum power, the derivative of the above equation must be zero, after simplification we get

$$X_L + X_{TH} = 0$$

$$X_L = - X_{TH}$$

Putting the above relation in equation 1, we get

$$P_L = V^2_{TH} \times R_L / ((R_L + R_{TH})^2$$

Again, for maximum power transfer, derivation of above equation must be equal to zero, after simplification we get

$$R_L + R_{TH} = 2 R_L$$

$$R_L = R_{TH}$$

Hence, the maximum power will transferred to the load from source, if $R_L = R_{TH}$ and XL = – XTH in an AC circuit. This means that the load impedance should be equal to the complex conjugate of equivalent impedance of the circuit,

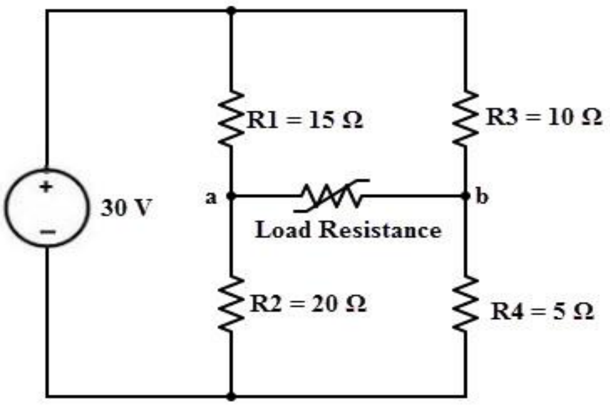
$$\boxed{\mathbf{Z_L = Z_{TH}}}$$

Where Z_{TH} is the complex conjugate of the equivalent impedance of the circuit.

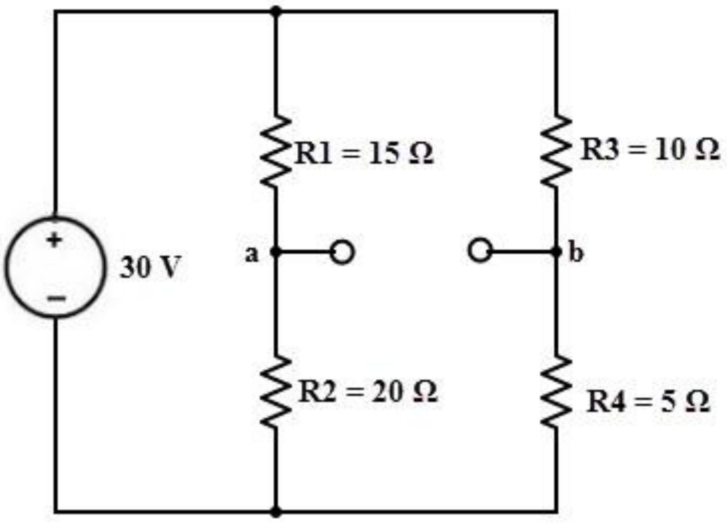
This maximum power transferred, $P_{max} = V^2_{TH} / 4 R_{TH}$ or $V^2_{TH} / 4 R_L$

Applying Maximum Power Transfer Example to DC circuit

Consider the below circuit to which we determine the value of the load resistance that receives the maximum power from the supply source and the maximum power under the maximum power transfer condition.



1.Disconnect the load resistance from the load terminals a and b. To represent the given circuit as Thevenin's equivalent, we are to determine the Thevenin's voltage V_{TH} and Thevenin's equivalent resistance R_{TH}.



The Thevenin's voltage or voltage across the terminals ab is $V_{ab} = V_a – V_b$

$$V_a = V \times R2 / (R_1 + R_2) = 30 \times 20 /\times(20 + 15) = 17.14 \text{ V}$$

$$V_b = V \times R_4/ (R_3 + R_4) = 30 \times 5 /(10 + 5) = 10 \text{ V}$$

$$V_{ab} = 17.14 – 10 = 7.14 \text{ V}$$

$$V_{TH} = V_{ab} = 7.14 \text{ Volts}$$

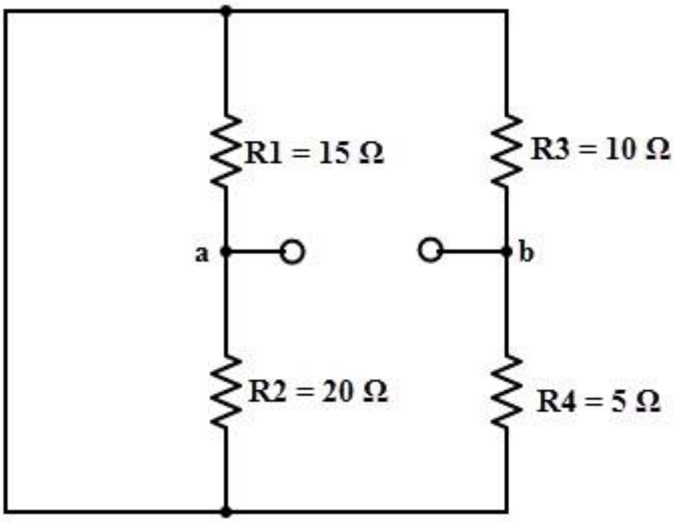
2.Calculate the Thevenin's equivalent resistance R_{TH} by replacing sources with their internal resistances (here assume that voltage source has zero internal resistance so it becomes a short circuited).

Thevenin's equivalent resistance or resistance across the terminals ab is

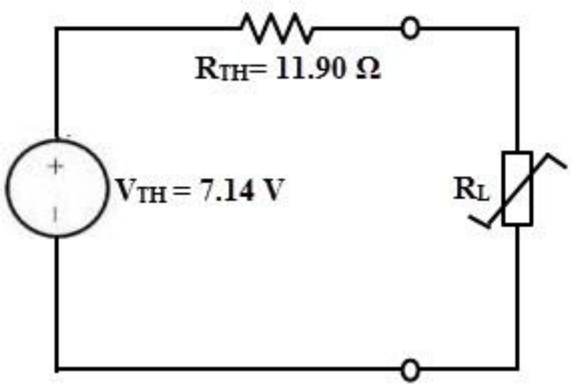
$$R_{TH} = R_{ab} = [R_1R_2 / (R_1 + R_2)] + [R_3R_4 /(R_3 + R_4)]$$

$= [(15 \times 20) / (15 + 20)] + [(10 \times 5) / (10+ 5)] = 8.57 + 3.33$

$R_{TH} = 11.90$ Ohms



3. The Thevenin's equivalent circuit with above calculated values by reconnecting the load resistance is shown below.



From the maximum power transfer theorem, RL value must equal to the RTH to deliver the maximum power to the load.

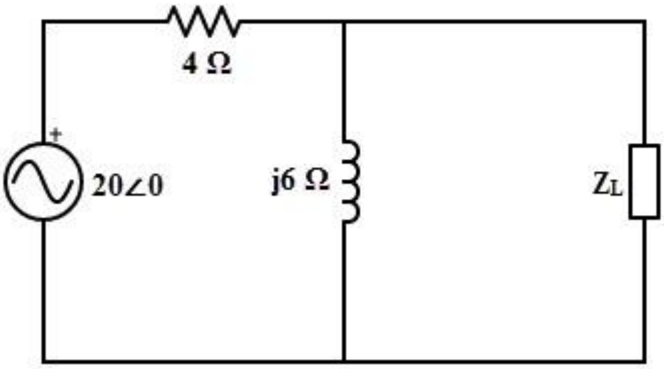
Therefore, $R_L = R_{TH} = 11.90$ Ohms

And the maximum power transferred under this condition is,

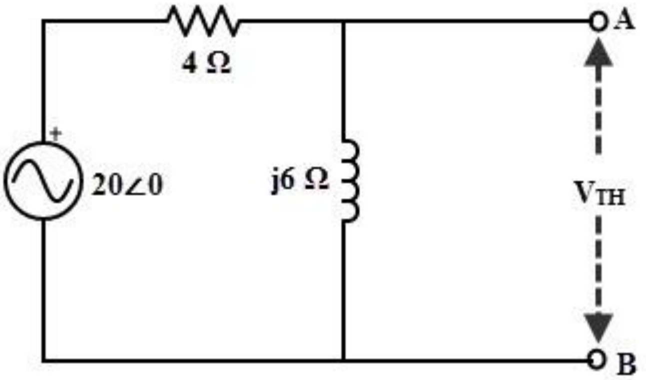
$P_{max} = V^2_{TH} / 4\, R_{TH} = (7.14)^2 / (4 \times 11.90) = 50.97 / 47.6 = 1.07$ Watts

Applying Maximum Power Transfer to AC circuit

The below AC network consists of load impedance ZL of which both reactive and resistive parts can be varied. Hence, we have to determine the load impedance value at which the maximum power delivered from the source and the value of maximum power.



To find the value of load impedance, first, we find the Thevenin's equivalent circuit across the load terminals. For finding Thevenin's voltage, disconnect the load impedance as shown in below figure.

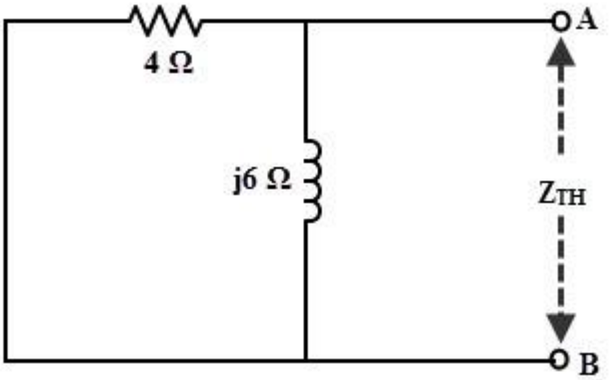


By voltage divider rule,

$V_{TH} = 20\angle 0 \times [j6 / (4 + j6)] = 20\angle 0 \times [6\angle 90 / 7.21\angle 56.3] = 20\angle 0 \times 0.825\angle 33.7$

$V_{TH} = 16.5\angle 33.7$ V

By shorting the voltage source, we calculate the Thevenin's equivalent impedance of the circuit as shown in figure.

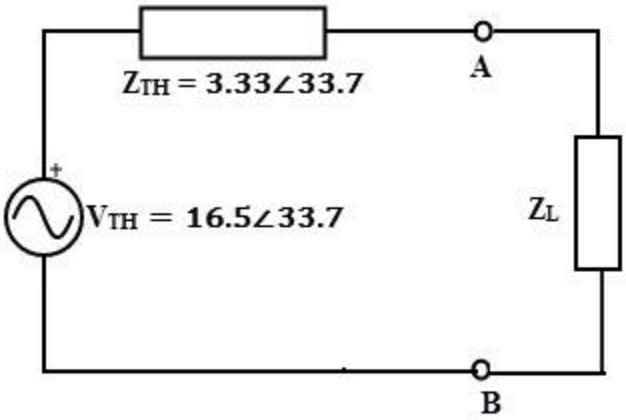


Therefore, $Z_{TH} = (4 \times j6) / (4 + j6)$

$= (4 \times 6\angle 90) / (7.21\angle 56.3)$

$= 3.33\angle 33.7$ 0r $2.77 + j1.85$ Ohms

Hence, the Thevenin's equivalent circuit across the load terminals is shown in below.



Therefore to transfer the maximum power to the load, the value of the load impedance should be

$$Z_L = Z^*_{TH}$$

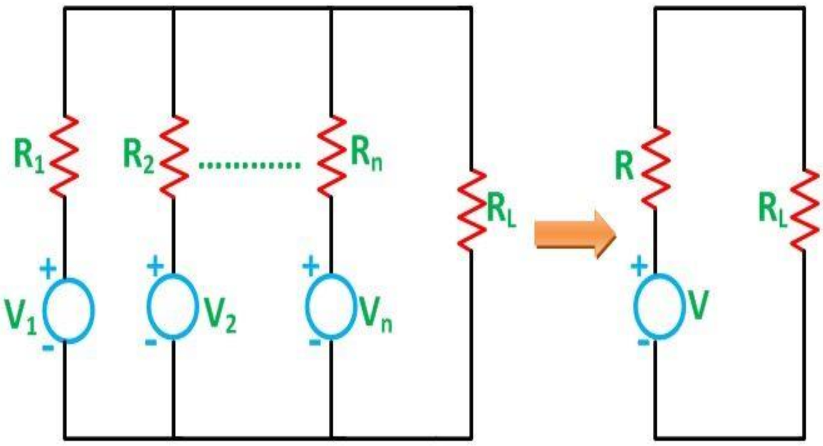
$$= 2.77 – j1.85 \text{ ohms}$$

The maximum power delivered, P_{max}

$$= V^2_{TH} / 4 R_{TH} = (16.5)^2/4(2.77) = 272.25 / 11.08 = 24.5 \text{ W}$$

- **Millman's Theorem statement**

The **Millman's Theorem** states that – when a number of voltage sources (V_1, V_2, V_3……… V_n) are in parallel having internal resistance (R_1, R_2, R_3………….R_n) respectively, the arrangement can replace by a single equivalent voltage source V in series with an equivalent series resistance R. In other words; it determines the voltage across the parallel branches of the circuit, which have more than one voltage sources, i.e., reduces the complexity of the electrical circuit.This Theorem is given by Jacob Millman. The utility of **Millman's Theorem** is that the number of parallel voltage sources can be reduced to one equivalent source. It is applicable only to solve the parallel branch with one resistance connected to one voltage source or current source.As per Millman's Theorem

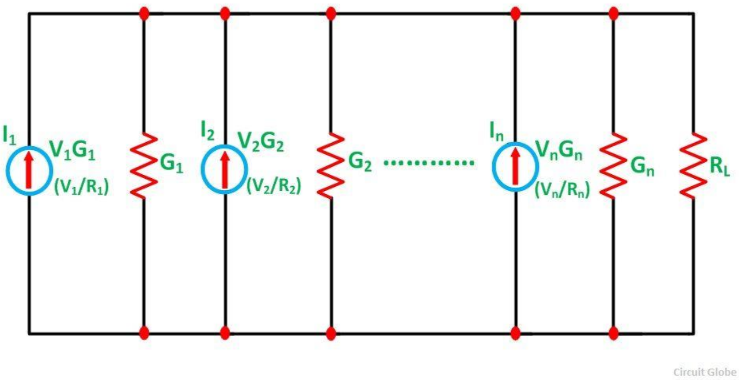


$$V = \frac{\pm V_1 G_1 \pm V_2 G_2 \pm \dots\dots\dots \pm V_n G_n}{G_1 + G_2 + \dots\dots . + G_n} \quad \text{and}$$

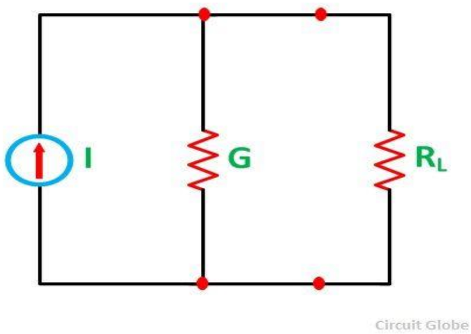
$$R = \frac{1}{G} = \frac{1}{G_1 + G_2 + ,,,,,,, + G_n}$$

Explanation of Millman's Theorem

Assuming a DC network of numerous parallel voltage sources with internal resistances supplying power to a load resistance RL as shown in the figure below



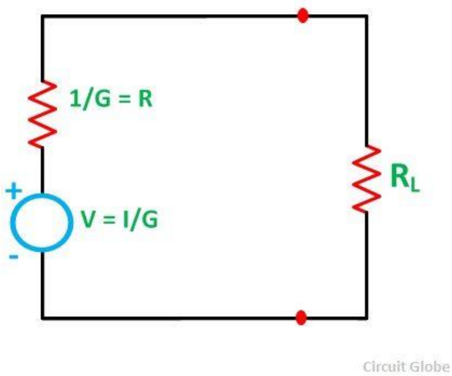
Let I represent the resultant current of the parallel current sources while G the equivalent conductance as shown in the figure below



$$I = I_1 + I_2 + I_3 + \dots\dots\dots . \quad \text{and}$$

$$G = G_1 + G_2 + G_3 + \dots\dots ..$$

Next, the resulting current source is converted to an equivalent voltage source as shown in the figure below



Thus,

$$V = \frac{I}{G} = \frac{\pm I_1 \pm I_2 \pm \dots\dots \pm I_n}{G_1 + G_2 + \dots\dots\dots + G_n}$$

Positive (+) and negative (-) sign appeared to include the cases where the sources may not be supplying current in the same direction.

Also,

$$R = \frac{1}{G} = \frac{1}{G_1 + G_2 + ,,,,,,,, + G_n}$$

And as we know,

I = V/R, and we can also write R = I/G as G = I/R

So the equation can be written as

$$V = \frac{\pm \frac{V_1}{R_1} \pm \frac{V_2}{R_2} \pm \dots \dots \dots \dots .. \pm \frac{V_n}{R_n}}{\frac{1}{R_1} + \frac{1}{R_2} + \dots \dots \dots . + \frac{1}{R_n}}$$

Where R is the equivalent resistance connected to the equivalent voltage source in series.

Thus, the final equation becomes

$$V = \frac{\pm V_1 G_1 \pm V_2 G_2 \pm \dots \dots \dots \pm V_n G_n}{G_1 + G_2 + \dots \dots . + G_n} \quad \text{or}$$

$$V = \frac{\sum_{K=1}^{n} V_K G_K}{\sum_{K=1}^{n} G_K} \quad \text{and} \quad G_K = \frac{1}{R_K}$$

Steps for Solving Millman’s Theorem

Following steps are used to solve the network by Millman’s Theorem

Step 1 – Obtain the conductance (G_1, G_2,….) of each voltage source (V_1, V_2,….).

Step 2 – Find the value of equivalent conductance G by removing the load from the network.

Step 3 – Now, apply Millman's Theorem to find the equivalent voltage source V by the equation shown below

$$V = \frac{\pm V_1 G_1 \pm V_2 G_2 \pm \dots \dots \dots \pm V_n G_n}{G_1 + G_2 + \dots \dots . + G_n}$$

Step 4 – Determine the equivalent series resistance (R) with the equivalent voltage sources (V) by the equation

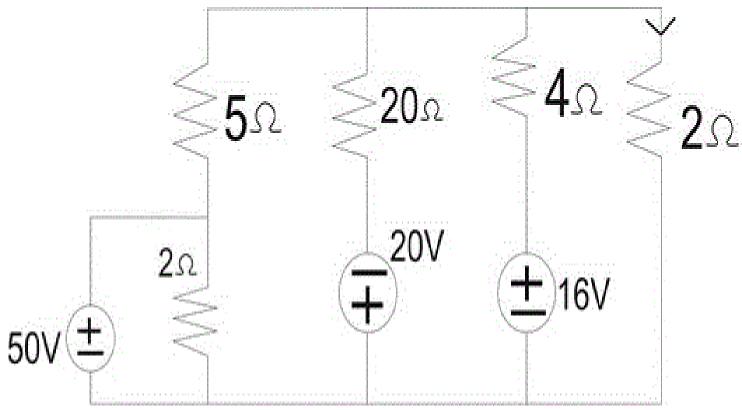
$$R = \frac{1}{G}$$

Step 5 – Find the current I_L flowing in the circuit across the load resistance R_L by the equation

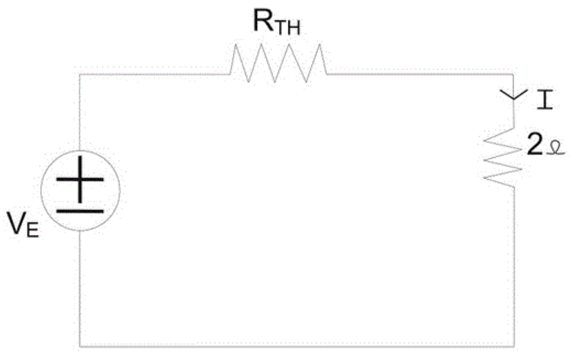
$$I_L = \frac{V}{R + R_L}$$

Example – 1

A circuit is given as shown in fig. Find out the voltage across 2 Ohm resistance and current through the 2 ohm resistance.



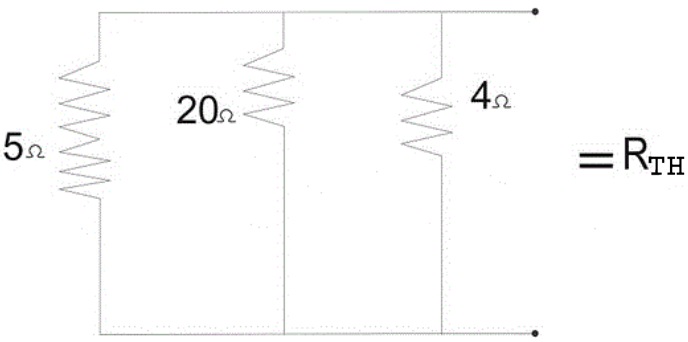
Given circuit can be reduced to a circuit shown in fig where equivalent voltage V_E can be obtained by millman's theorem and that is



$$V_E = \frac{\frac{50}{5} - \frac{20}{20} + \frac{16}{4}}{\frac{1}{5} + \frac{1}{20} + \frac{1}{4}} = 26\ V$$

Equivalent resistance or Thevenin resistance can be found by shorting the voltage sources as shown in fig

$$R_{TH} = \frac{1}{\frac{1}{5} + \frac{1}{20} + \frac{1}{4}} = 2\ \Omega$$



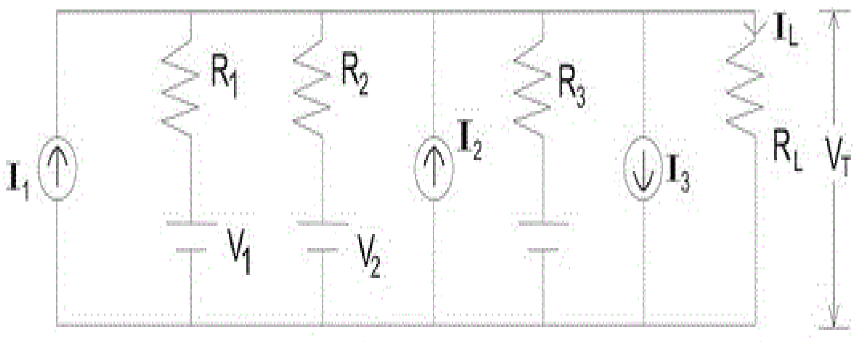
Now we can easily found the required current through 2 Ohm load resistance by Ohm's law.

$$I_{2\Omega} = \frac{26}{2+2} = 6.5\ A$$

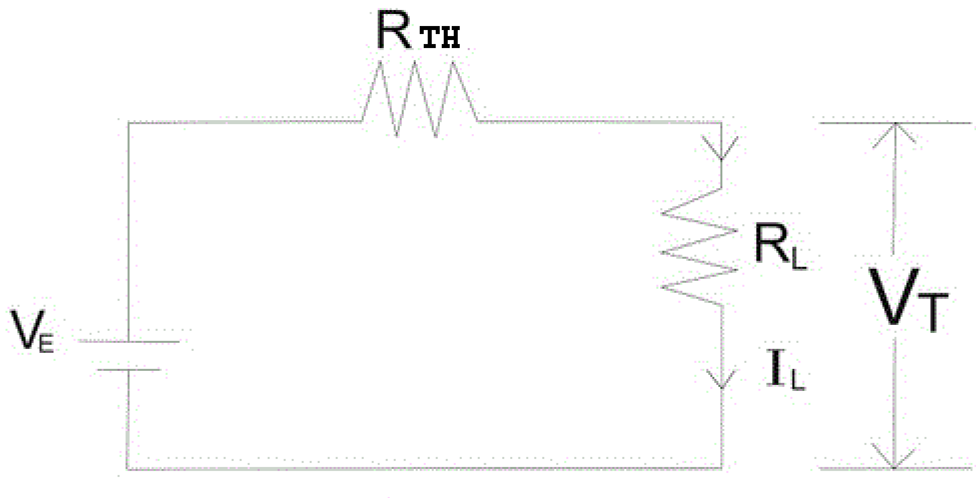
Voltage across load is, $V_L = I_{2\Omega} \times 2 = 6.5 \times 2 = 13\ V$

- **Circuit is Consisting Mixture of Voltage and Current Source**
-

Millman's Theorem is also helpful to reduce a mixture of voltage and current source connected in parallel to a single equivalent voltage or current source. Let's have a circuit as shown in below figure.



Here all letters are implying their conventional representation.This circuit can be reduced to a circuit as shown in figure.



Here V_E which is nothing but thevenin voltage which will be obtained as per Millman's theorem and that is

$$V_E = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3} + I_1 + I_2 - I_3}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{\sum \frac{V}{R} + \sum I}{\sum \frac{1}{R}}$$

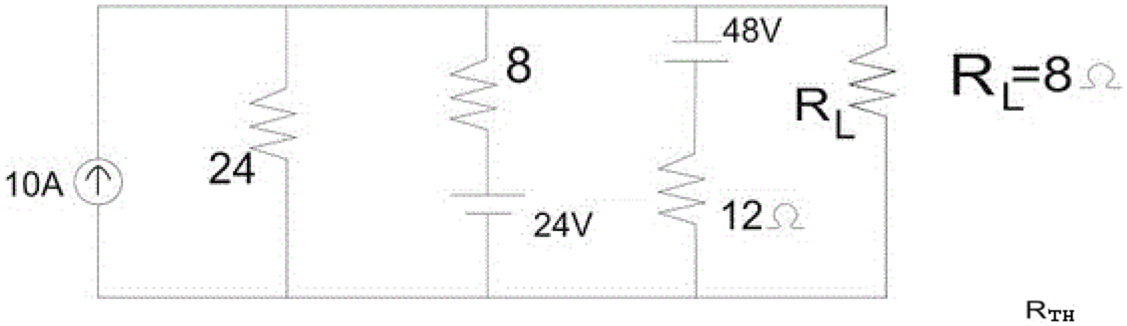
And R_{TH} will be obtained by replacing current sources with open circuits and voltage sources with short circuits.

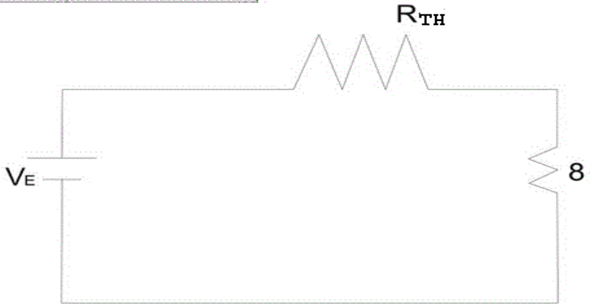
$$R_{TH} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Now we can easily find out load current I_L and terminal voltage V_T by Ohm's law.

$$I_L = \frac{V_E}{R_L + R_{TH}} \quad \& \quad V_T = I_L \times R_L$$

Example 2 :





A circuit is given as shown in fig-h. Find out the current through load resistance where $R_L = 8\ \Omega$.

Answer : The given circuit can be reduced in a circuit as shown in fig. Where, V_E can be obtained with the help of Millman's theorem,

$$V_E = \frac{10 + \frac{0}{24} + \frac{24}{8} - \frac{48}{12}}{\frac{1}{24} + \frac{1}{8} + \frac{1}{12}} = 36\ V \quad \& \quad R_{TH} = 6\ \Omega$$

Therefore, current through load resistance 8 Ω is,

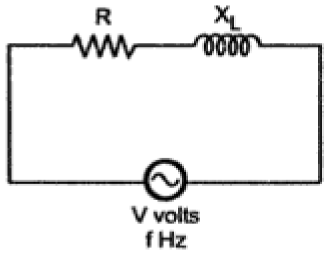
$$I_{8\Omega} = \frac{36}{6+8} = 2.57\ A$$

- **Locus diagram**

In a particular circuit, if one of the circuit elements is variable, then depending upon its value the circuit characteristics varies. As the value of the variable element is changed, the circuit parameters like current, power factor, power consumed also change. The locus of the extremity of the current phasor, obtained for various values of a variable element is called a locus diagram. Such a graphical approach to obtain a locus diagram helps in predetermining the operating characteristics of a.c. circuits, under different conditions.

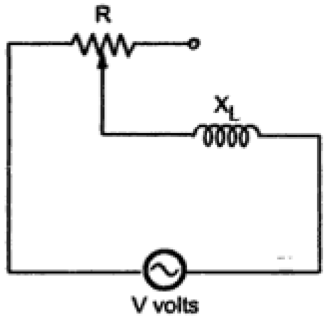
➢ Locus Diagram for R-L Series Circuit

Consider a series R-L circuit, excited by a constant but alternating voltage source V of fixed frequency. To obtain the locus diagram it is necessary to have one element, variable in nature. The element R can be variable or the element X_L can be variable. Let us analyse these two cases independently and the corresponding locus diagrams.

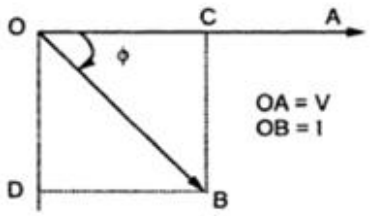


➢ Variable R and Constant XL Circuit

Consider a circuit where resistance is variable and inductance is constant as shown in the Fig. The current flowing through the circuit is say I amperes. The current magnitude can be obtained as,



$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + X_L^2}} \quad \text{... (1)}$$



The current I lags voltage V, by an angle ϕ where,

$$\phi = \tan^{-1}\frac{X_L}{R} \quad \text{... (2)}$$

The phasor sum of the voltage drops across resistance and reactance is the applied voltage.

$$\bar{V} = \bar{IR} + \bar{IX} \quad \text{... (3)}$$

Consider the phasor diagram shown

The current I has two component, x component and y component.

$$x = OC = OB\cos\phi \quad \text{... (4)}$$

and $$y = OD = -OB\sin\phi \quad \text{... (5)}$$

The negative sign indicates that OD is in negative y direction

Now $$\phi = \tan^{-1}\frac{X_L}{R}$$

$$\therefore \quad \tan\phi = \frac{X_L}{R} \quad \text{... (6)}$$

$$\therefore \quad \cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + X_L^2}} \quad \text{... (7)}$$

$$\therefore \quad \sin\phi = \frac{X_L}{Z} = \frac{X_L}{\sqrt{R^2 + X_L^2}} \quad \text{... (8)}$$

$$\therefore \quad x = I\cos\phi = \frac{V}{\sqrt{R^2 + X_L^2}} \cdot \frac{R}{\sqrt{R^2 + X_L^2}}$$

$$\therefore \quad x = \frac{V \cdot R}{R^2 + X_L^2} \quad \text{... (9)}$$

and $$y = -I \sin\phi = \frac{-V}{\sqrt{R^2 + X_L^2}} \cdot \frac{X_L}{\sqrt{R^2 + X_L^2}}$$

$$\therefore \quad y = \frac{-VX_L}{R^2 + X_L^2} \qquad \ldots (10)$$

Squaring and adding (9) and (10), to obtain the equation for the locus of point B which represents extremity of a current phasor.

$$\therefore \quad x^2 + y^2 = \frac{V^2 R^2}{\left(R^2 + X_L^2\right)^2} + \frac{V^2 X_L^2}{\left(R^2 + X_L^2\right)^2} = \frac{V^2\left(R^2 + X_L^2\right)}{\left(R^2 + X_L^2\right)^2}$$

$$\therefore \quad \boxed{x^2 + y^2 = \frac{V^2}{R^2 + X_L^2}} \qquad \ldots (11)$$

Now $$x^2 + y^2 = V \cdot \frac{V}{R^2 + X_L^2}$$

From (10), $$\frac{V}{R^2 + X_L^2} = \frac{-y}{X_L}$$

$$\therefore \quad x^2 + y^2 = \frac{-Vy}{X_L}$$

$$\therefore \quad x^2 + y^2 + \frac{Vy}{X_L} = 0 \qquad \ldots (12)$$

To complete square on left hand side, add $\left(\frac{V}{2X_L}\right)^2$ on both sides.

$$\therefore \quad x^2 + y^2 + \frac{Vy}{X_L} + \left(\frac{V}{2X_L}\right)^2 = \left(\frac{V}{2X_L}\right)^2$$

$$\boxed{x^2 + \left(y + \frac{V}{2X_L}\right)^2 = \left(\frac{V}{2X_L}\right)^2} \qquad \ldots (13)$$

Comparing this with the equation of a circle,

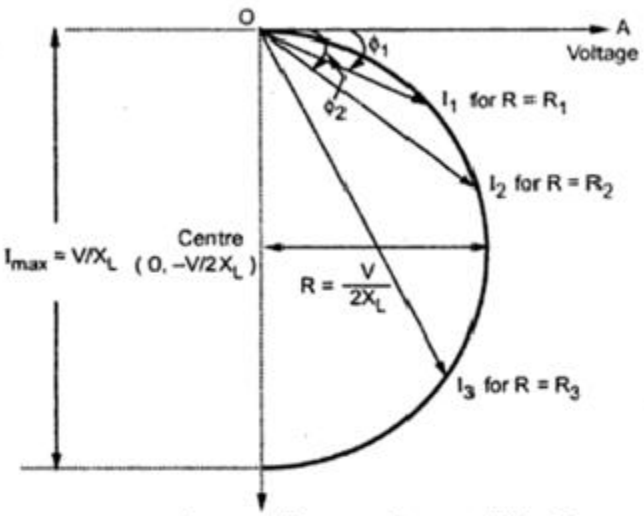
$(x - x_1)^2 + (y - y_1)^2 = R^2$ we can say that equation (13) represents a circle with,

$$\text{Radius } R = \frac{V}{2X_L} \qquad \ldots (14\text{ a})$$

$$\text{Centre } (x_1, y_1) = \left(0, \frac{-V}{2X_L}\right) \qquad \ldots (14\text{ b})$$

Key Point : *Thus the locus of the extremity of a current phasor is a circle.*

But as the maximum ϕ possible is 90° when R = 0, the locus will be a semicircle. Thus the locus diagram for R-L series circuit with a variable R is a semicircle of radius $\frac{V}{2X_L}$ and centre $\left(0, \frac{-V}{2X_L}\right)$



Locus diagram for variable R

When R = 0, we get maximum current through the circuit as,

$$\boxed{I_{max} = \frac{V}{X_L}}$$

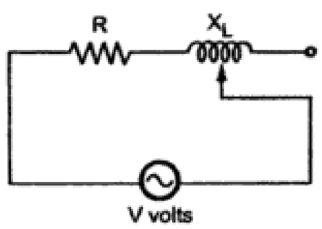
and ϕ will also maximum equal to 90° .

The locus diagram is shown in the Fig.

The power supplied to the circuit is,

$$P = VI \cos \phi$$

Variable X_L and Constant R Circuit



Consider a circuit where inductive reactance acts as a variable element and resistance is constant, as shown in the Fig.

As basically the circuit is R-L series circuit, the current expression remains same.

Similarly the phasor diagram will also remain same. Hence x and y component of the current will also remain same as,

$$x = \frac{V \cdot R}{R^2 + X_L^2}$$

$$y = \frac{-VX_L}{R^2 + X_L^2}$$

and $$x^2 + y^2 = V \cdot \frac{V}{R^2 + X_L^2}$$

From the equation of x we can write,

$$\frac{V}{R^2 + X_L^2} = \frac{x}{R} \qquad \text{... (16)}$$

Substituting in the above equation we get,

$$x^2 + y^2 = \frac{Vx}{R}$$

$\therefore \qquad x^2 - \frac{Vx}{R} + y^2 = 0$ ... (17)

Adding the term $\left(\frac{V}{2R}\right)^2$ to both sides, to complete the square on left hand side,

$$\therefore \qquad x^2 - \frac{Vx}{R} + \left(\frac{V}{2R}\right)^2 + y^2 = \left(\frac{V}{2R}\right)^2$$

$$\boxed{\left(x - \frac{V}{2R}\right)^2 + y^2 = \left(\frac{V}{2R}\right)^2} \qquad \text{... (18)}$$

Comparing with the equation of circle,

$$(x - x_1)^2 + (y - y_1)^2 = R^2$$

It can be concluded that equation (18) represents a circle with,

$$\text{Radius } R = \frac{V}{2R} \qquad \text{... (19 a)}$$

$$\text{Centre } (x_1, y_1) = \left(\frac{V}{2R}, 0\right) \qquad \text{... (19 b)}$$

Key Point : *Thus the locus of the extremity of a current phasor is a semicircle of radius* $\frac{V}{2R}$.

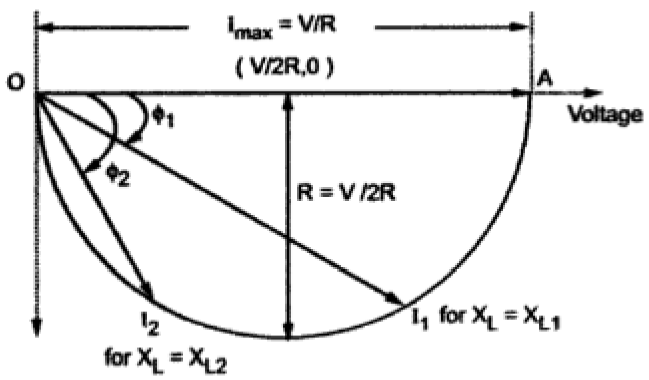


Fig. Locus diagram for variable X_L

When $X_L = 0$, we get a maximum current as,

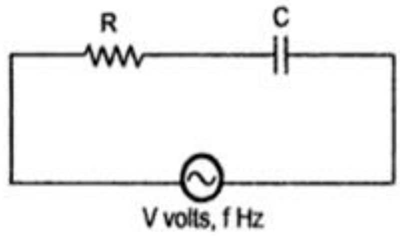
$$I_{max} = \frac{V}{R}$$

with $\phi = 0°$

This I_{max} is in phase with voltage phasor V. The locus diagram is shown in the Fig.

Key Point : *The phasor diagram for both cases remain same for a particular set of values of R and X_L. But the locus diagram for variable R is semicircle on y-axis while the locus diagram for variable X_L is semicircle on x-axis i.e. voltage axis.*

Locus Diagrams for R-C Series Circuit



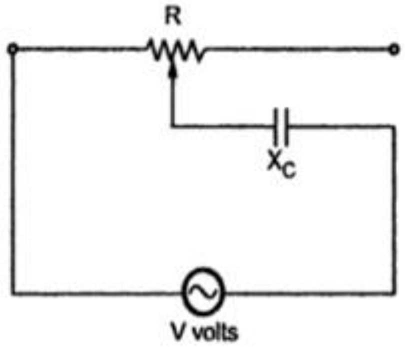
Consider a series R-C circuit excited by a constant but alternating voltage source V of fixed frequency.

To have locus diagram, either R or C should be a variable element.

Let us consider these two cases and corresponding locus diagrams.

Variable R and Constant X_C Circuit

The circuit is shown in the Fig.



The current flowing is I amperes whose magnitude is given by,

$$I = \frac{|V|}{|Z|} = \frac{V}{\sqrt{R^2 + X_C^2}} \quad \ldots (1)$$

The current I leads voltage V by an angle ϕ where,

$$\phi = \tan^{-1}\frac{X_C}{R}$$

The phasor sum of the voltage drops across resistance and the reactance is the applied voltage.

$$\overline{V} = \overline{IR} + \overline{IX_C}$$

As proved for the series R-L circuit, the locus of tips of the current phasor for the various values of R is a semicircle.

When R = 0, $\quad I_{max} = \frac{V}{X_C}$ = Maximum current

This represents diameter of the circle.

$$\therefore \quad \boxed{R = \text{Radius} = \frac{V}{2X_C}}$$

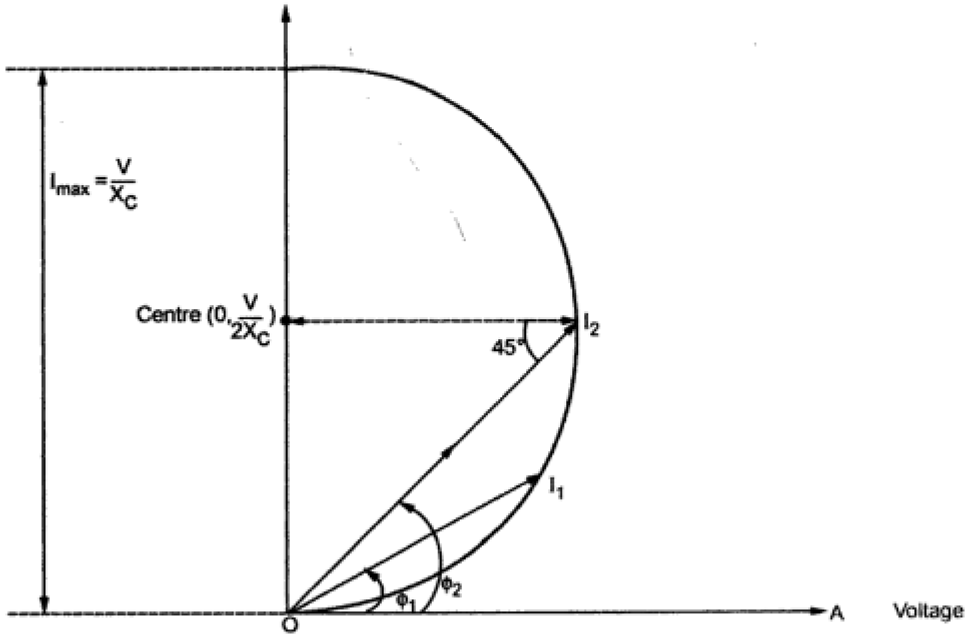
While the centre of the circle is given by,

$$\boxed{\text{Centre } (x_1, y_1) = \left(0, +\frac{V}{2X_C}\right)}$$

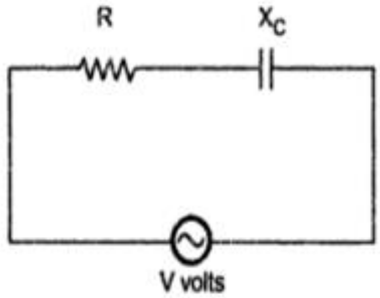
The maximum angle is, $\boxed{\phi = 90°}$

Key Point : *As the centre has positive y-co-ordinate, the semicircle is in the first quadrant.*

The locus diagram is shown in the Fig. 5.13.



Variable X_C and Constant R Circuit



Consider a circuit where X_C acts as a variable element, as shown in the Fig.

The current expression remains same as basically there is no change in the circuit.

The current I leads the voltage.

Now when $X_C = 0$, then the current is maximum and will be in phase with the voltage as circuit becomes purely resistive.

$$\therefore \qquad I_{max} = \frac{V}{R}$$

This is nothing but the diameter of the locus diagram which is a semicircle.

$$\therefore \qquad \boxed{R = \text{Radius} = \frac{V}{2R}}$$

The centre of the semicircle is given by,

$$\boxed{\text{Centre } (x_1, y_1) = \left(\frac{V}{2R}, 0\right)}$$

As centre has positive x-co-ordinate, voltage is x-axis and current leads voltage, hence the locus diagram lies in the first quadrant. The locus diagram is shown in the Fig.

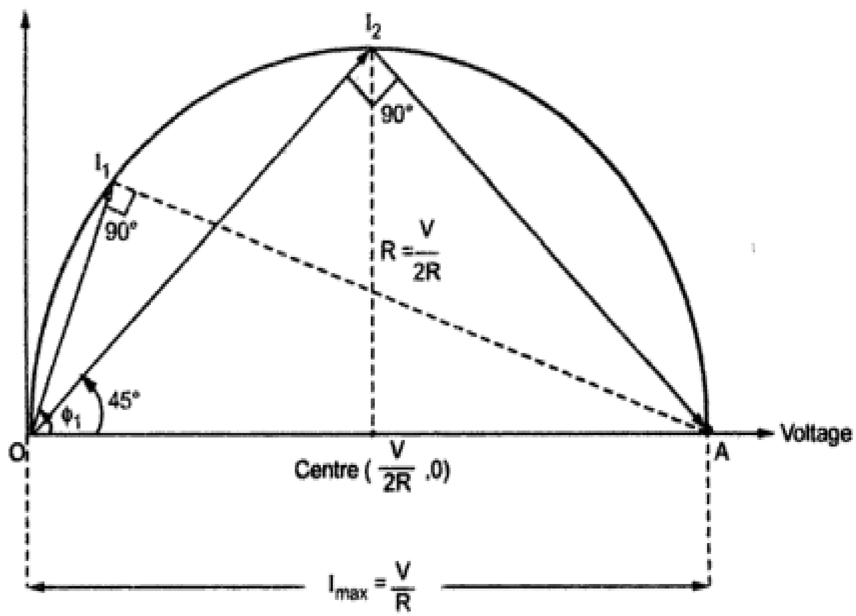
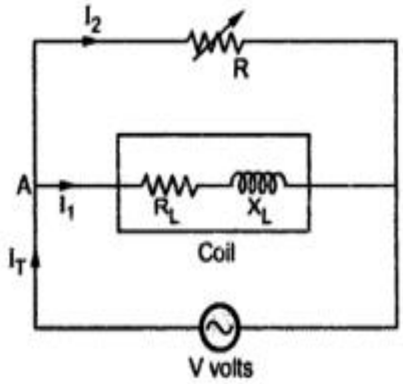


Fig. Locus diagram for variable X_C and fixed R

Locus Diagrams for Parallel Circuits

Let us study the locus diagrams for various parallel combinations of the elements R, L and C.

Practical Coil in Parallel with Variable R



The circuit is shown in the Fig. 5.16.

Let R = Variable resistance

R_L = Resistance of coil

X_L = Inductive reactance of coil = $2\pi f L$

I_1 = Current through coil

I_2 = Current through variable resistance

Applying KCL at node A,

$$\bar{I}_T = \bar{I}_1 + \bar{I}_2$$

Now I_1 is given by, $I_1 = \frac{V}{Z_L}$ where Z_L = Impedance of coil

so $Z_L = R_L + jX_L$

and $|Z_L| = \sqrt{R_L^2 + X_L^2}$ and $\phi = \tan^{-1} \frac{X_L}{R}$

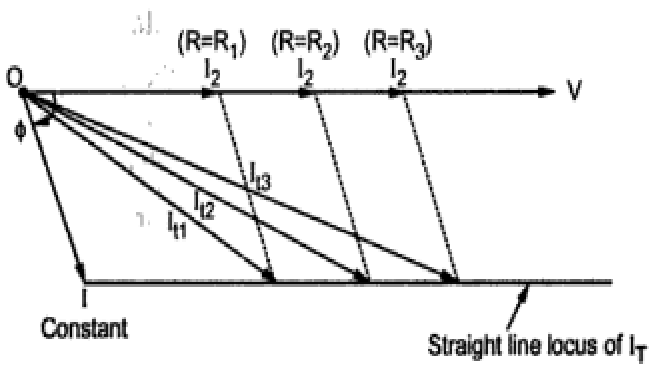
As R_L and X_L are constants, I_1 is constant and all the time lagging voltage V by angle ϕ.

The current I_2 is inversely proportional to resistance R and is always in phase with V.

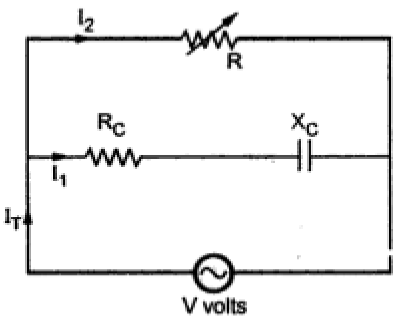
$$I_2 = \frac{V}{R}$$

The phasor sum of I_1 and I_2 is total current I_T.

So the locus of tip of the current phasoor I_T for various values of R is a straight line parallel to voltage axis as shown in the Fig.



R - C in Parallel with Variable R



The circuit is shown in the Fig.

It can be seen that,

$$Z_C = R_C - jX_C$$

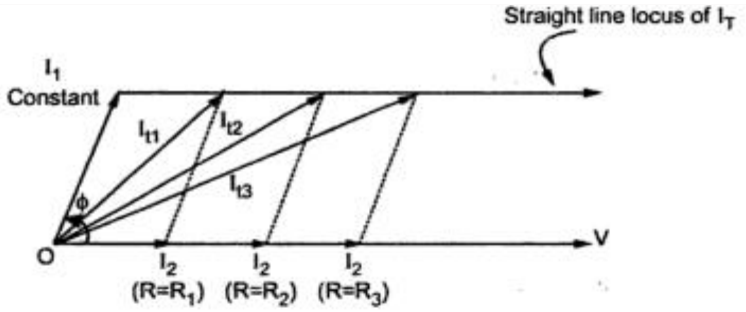
and $|Z_C| = \sqrt{R_C^2 + X_C^2}$

and $\phi = \tan^{-1} \frac{X_C}{R}$ leading

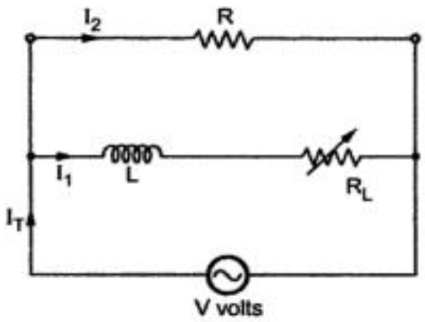
Applying KCL at node A,

$$\bar{I}_T = \bar{I}_1 + \bar{I}_2 \quad \text{phasor sum}$$

The analysis is exactly same as the previous case. The only change is that current I_1 is constant and leads the voltage V by angle ϕ. The current I_2 is variable and is always in phase with the voltage. So locus of I_T is again a straight line but parallel to the voltage axis in the first quadrant. This is shown in the Fig.



R_L- L Series in Parallel with R with R_L Variable



Consider the parallel combination of R_L-L in series, parallel with the resistance R but now R_L is variable and L , R are constants.

Now the current I_2 is fixed and is given by,

$$I_2 = \frac{V}{R}$$

Applying KCL,

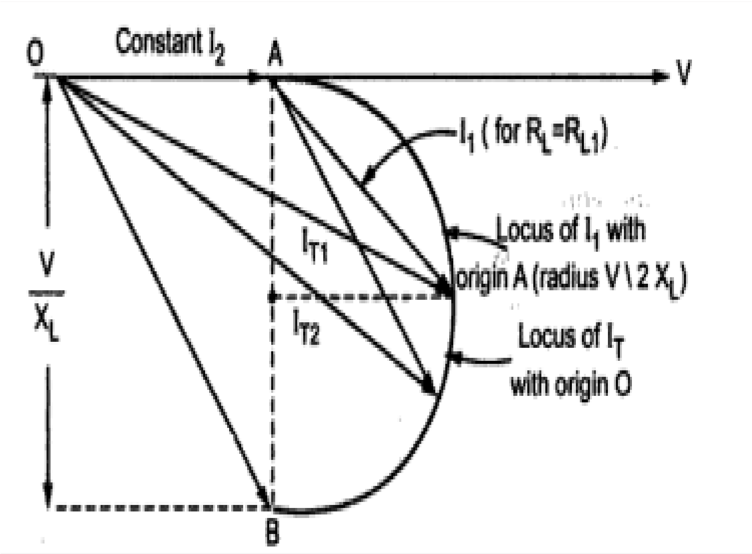
$$\bar{I}_T = \bar{I}_1 + \bar{I}_2$$

But now I_1 is current through series R_L-L circuit consisting of variable R_L. As seen earlier, the locus of I_1 is a semicircle whose diameter is perpendicular to voltage axis, lagging with respect to voltage. The radius of this semicircle is,

$$\boxed{R = \frac{V}{2X_L} = \text{Radius of semicircle}}$$

And the centre is given by,

$$\boxed{\text{Centre} = \left(0, -\frac{V}{2X_L}\right)}$$


O
Constant I_2
A
V
I_1 (for R_L=R_{L1})
Locus of I_1 with
origin A (radius V \ 2 X_L)
I_{T1}
I_{T2}
Locus of I_T
with origin O
$\frac{V}{X_L}$
B