UNIT-II

Fourier series

The representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions is called Fourier series. It is applicable for only periodic signals.

Dirichlet's Conditions: for the Fourier series to exit for a periodic signal, it must satisfy certain conditions

1. The function x(t) must be a single values function.
2. The function x(t) has only a finite number of maxima and minima.
3. The function x(t) has a finite number of discontinuities.
4. The function x(t) is absolutely integral over a period, that is $\int_0^T x(t)\, dt < \infty$.

Classification of Fourier series:

Three important classes of Fourier series methods are available. They are

1. Trigonometric form
2. Exponential form
3. Cosine form

1. Trigonometric form of Fourier series:

A periodic function x(t) of period T can be represented as

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos kwt + \sum_{k=1}^{\infty} b(k)\sin kwt$$

Where a(k) and b(k) are constants and a(0) is DC component.

$$a(0) = \frac{1}{T}\int_0^T x(t)dt$$

$$a(k) = \frac{2}{T}\int_0^T x(t)\cos kwtdt$$

$$b(k) = \frac{2}{T}\int_0^T x(t)\sin kwtdt$$

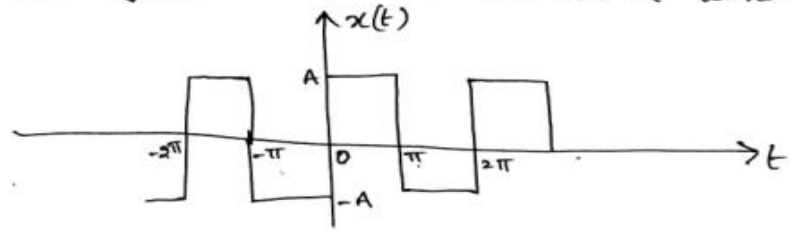
Note:

If x(t) has even symmetry, then b(k)=0 & a(0) and a(k) are to be evaluated.

If x(t) has odd symmetry, then a(k)=0 and a(0)=0 & b(k) is to be evaluated.

If x(t) has half wave symmetry, then a(0)=0 and only Odd harmonics exist. e^{jkwt}

Obtain The trigonometric Fourier Series for the wave form



Solution:-

the given signal $x(t) = \begin{cases} A & 0 \le t \le \pi \\ -A & \pi \le t \le 2\pi \end{cases}$ & $t_0 = 0$, $t_0 + T = 2\pi$

Fundamental period $w_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

the waveform has odd symmetry. So $a(0) = 0$, $a(k) = 0$ and

$$b(k) = \frac{4}{T}\int_0^{T/2} x(t)\sin k\omega t \, dt = \frac{4}{2\pi}\int_0^{\pi} A \sin kt \, dt$$

$$= \frac{2A}{n\pi}\left[1 - (-1)^n\right]$$

$$b(k) = \begin{cases} 4A/n\pi & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

the trigonometric Fourier series is

$$x(t) = a(0) + \sum_{k=1}^{\infty} a(k)\cos k\omega t + \sum_{k=1}^{\infty} b(k)\sin k\omega t$$

$$x(t) = \sum_{k=\text{odd}}^{\infty} \frac{4A}{k\pi}\sin kt = \frac{4A}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{k}\sin kt$$

$$\boxed{x(t) = \frac{4A}{\pi}\sin t + \frac{4A}{3\pi}\sin 3t + \frac{4A}{5\pi}\sin 5t + \cdots}$$

2. Exponential form of Fourier series:

A periodic function x(t) of period T can be represented as

$$x(t) = \sum_{k=-\infty}^{\infty} X(K)\, e^{jkwt}$$

Where

$$X(K) = \frac{1}{T}\int_0^{T} x(t)\, e^{-jkwt} dt$$

3. Cosine form of Fourier series:

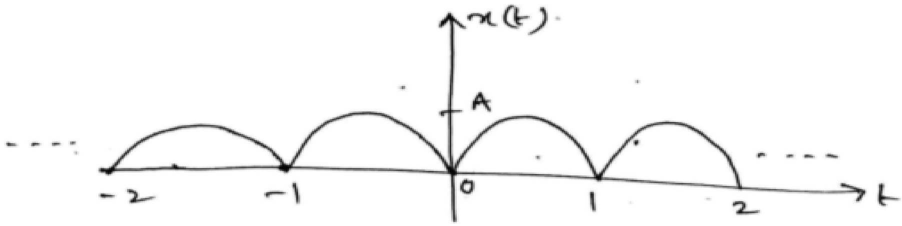
A periodic function x(t) of period T can be represented as

$$x(t) = \sum_{k=1}^{\infty} C(k)\cos(kwt + Q(k))$$

Where $Q(k) = \tan^{-1}[b(k)/a(k)]$

$$C(k) = \sqrt{a^2(k) + b^2(k)}$$

Find The Exponential Fourier Series for the Rectified Sine wave



the given signal $x(t) = A \sin \omega t \quad 0 \le t \le 1 \quad \therefore \omega = \frac{2\pi}{2} = \pi, \; \& \; T = 1$

$$X(0) = \frac{1}{T} \int_0^T x(t)\, dt = \frac{1}{1} \int_0^1 A \sin \pi t \, dt = \frac{2A}{\pi}$$

$$X(k) = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega t} dt$$

$$= \frac{1}{1} \int_0^1 A \sin \pi t \; e^{-j2\pi k t} dt$$

$$= \frac{A}{2j} \int_0^1 \left[e^{j\pi t} e^{-j2\pi k t} - e^{-j\pi t} e^{-2jk\pi t} \right]$$

$$X(k) = \frac{2A}{\pi(1-4k^2)}$$

the Exponential Fourier Series is

$$x(t) = X(0) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} X(k)\, e^{jk\omega t}$$

$$\boxed{x(t) = \frac{2A}{\pi} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{2A}{\pi(1-4k^2)} e^{j2k\pi t}}$$

Properties of Fourier series

1) Linearity:

If $x(t) \xleftrightarrow{F.S} X(k)$ and $y(t) \xleftrightarrow{F.S} Y(k)$

then $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{F.S} a\,X(k) + b\,Y(k)$

Proof:

$$Z(k) = \frac{1}{T} \int_{\langle T \rangle} z(t)\, e^{-jk\omega t} dt$$

$$Z(k) = \frac{1}{T} \int_{\langle T \rangle} [a\,x(t) + b\,y(t)]\, e^{-jk\omega t} dt$$

$$= \frac{1}{T} \int_{\langle T \rangle} (a\,x(t)\, e^{-jk\omega t} + b\,y(t).\, e^{-jk\omega t}) dt$$

$$= a\,\frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega t} dt + b\,\frac{1}{T} \int_{\langle T \rangle} y(t)\, e^{-jk\omega t} dt$$

$$\therefore Z(k) = a\,X(k) + b\,Y(k)$$

Properties of fourier Series:

1) Linearity:

If $x(t) \xleftrightarrow{F.S} X(K)$ and $y(t) \xleftrightarrow{F.S} Y(K)$

then $z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{F.S} aX(K) + bY(K)$

Proof:

$$Z(K) = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jK\omega t}\, dt$$

$$Z(K) = \frac{1}{T}\int_{\langle T\rangle} [a\,x(t) + b\,y(t)]\, e^{-jK\omega t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} \left(a\,x(t)\, e^{-jK\omega t} + b\,y(t).\, e^{-jK\omega t}\right) dt$$

$$= a\,\frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-jK\omega t}\, dt + b\,\frac{1}{T}\int_{\langle T\rangle} y(t)\, e^{-jK\omega t}\, dt$$

$$\therefore Z(K) = a\,X(K) + b\,Y(K)$$

2) Time shifting:

If $x(t) \xleftrightarrow{F.S} X(K)$

$z(t) = x(t - t_0) \xleftrightarrow{F.S} X(K)\, e^{-jK\omega t_0} = Z(K)$

Proof: $Z(K) = \frac{1}{T}\int_{\langle T\rangle} x(t - t_0)\, e^{-jK\omega t}\, dt$

Let $t - t_0 = m$

$dt = dm$.

$$Z(K) = \frac{1}{T}\int_{\langle T\rangle} x(m)\, e^{-jK\omega(m + t_0)}\, dm$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(m)\, e^{-jK\omega m} . e^{-jK\omega t_0}\, dm$$

$$= e^{-jK\omega t_0}\, \frac{1}{T}\int_{\langle T\rangle} x(m)\, e^{-jK\omega m}\, dm$$

replace m by t.

$$\therefore Z(K) = \left(e^{-jK\omega t_0}\right) \frac{1}{T}\int_{\langle T\rangle} x(t).\, e^{-jK\omega t}\, dt$$

$$= e^{-jK\omega t}\quad X(K)$$

$$\boxed{\therefore Z(K) = X(K)\, e^{-jK\omega t_0}}$$

(3) Frequency Shift:

If $X(k)$ is a fourier coefficient of $x(t)$ then

$$x(t)\, e^{+j\omega k_0 t} \xleftrightarrow{F.S.} X(K-k_0)$$

proof: Let $z(t) = x(t)\, e^{jK_0\omega t}$

$$Z(K) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{j\omega k_0 t}\, e^{-j\omega K t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-j(K-k_0)\omega t}\, dt$$

$$\boxed{Z(K) = X(K-K_0)} \qquad \left(X(K) = \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-j\omega t}\, dt \right)$$

(4) Time differentiation (or) Differentiation in time:

If $X(k)$ is a fourier Series of $x(t)$ then

$$\frac{d\,x(t)}{dt} \xleftrightarrow{F.S} jK\omega .\, X(k)$$

Proof: $$x(t) = \sum_{K=-\infty}^{\infty} X(k).\, e^{j\omega k t}$$

$$\frac{d\,x(t)}{dt} = \frac{d}{dt}\left(\sum_{K=-\infty}^{\infty} X(k).\, e^{j\omega t K}\right)$$

$$\frac{d\,x(t)}{dt} = \sum_{K=-\infty}^{\infty} \left[X(k).\, jK\omega\right] e^{j\omega K t}$$

$$\boxed{\frac{d x(t)}{dt} \xleftrightarrow{F.S} X(K).\, jK\omega}$$

(5) Convolution in time:

If $x(t) \xleftrightarrow{F.S} X(K)$, $y(t) \xleftrightarrow{F.S} Y(K)$ then

$$z(t) = x(t) * y(t) \xleftrightarrow{F.S} T.\, X(K).\, Y(K)$$

Proof: $$Z(K) = \frac{1}{T}\int_{\langle T\rangle} x(t) * y(t)\, e^{-j\omega K t}\, dt$$

In the above eq., $x(t) * y(t) = \int_{\langle T\rangle} x(\tau).\, y(t-\tau)\, d\tau$

$$Z(K) = \frac{1}{T}\int_{\langle T\rangle}\int_{\langle T\rangle} x(\tau)\, y(t-\tau)\, e^{-jK\omega t}\, dt\, d\tau$$

Interchanging the order of integration

$$Z(k)=\frac{1}{T}\int_{\langle T\rangle}x(\tau)\int_{\langle T\rangle}y(t-\tau)\,e^{-jk\omega t}\,d\tau\,dt$$

Let $t-\tau=m$

$t=\tau+m$

$dt=dm$

$$Z(k)=\frac{1}{T}\int_{\langle T\rangle}x(\tau)\int_{\langle T\rangle}y(m)\,e^{-jk\omega(\tau+m)}\,d\tau\,dm$$

$$=\frac{1}{T}\left[\int_{\langle T\rangle}x(\tau)e^{-jk\omega\tau}\,d\tau\int_{\langle T\rangle}y(m)\,e^{-jk\omega m}\,dm\right]$$

$$=\frac{1}{T}\left[T.X(k).\ T\ Y(k)\right]$$

$$\left(\because X(k)=\frac{1}{T}\int_{\langle T\rangle}x(t)e^{-jk\omega t}dt \quad TX(k)=\int_{\langle T\rangle}x(t)e^{-jk\omega t}dt\right)$$

$$\boxed{\therefore Z(k)=T.X(k).Y(k).}$$

(6) Time Scaling:

If $x(t)\xleftrightarrow{F.S}X(k)$ then

$Z(t)=x(at)\xleftrightarrow{F.S}Z(k)=X(k)$

Proof: $X(k)=\frac{1}{T}\int_{\langle T\rangle}x(t)e^{-j\omega kt}dt$

$x(t)$ is a periodic then $z(t)=x(at)$ is also Periodic.

If 'T' is the period of $x(t)$ then period of $z(t)$ will be $\frac{T}{a}$

Similarly, the frequency of $x(t)$ is ω then the frequency of $Z(t)$ will be $a\omega$.

$$Z(k)=\frac{1}{T/a}\int_{\langle T/a\rangle}x(at)\,e^{-jk(a\omega)t}\,dt$$

$$= \frac{a}{T} \int_{\langle T/a \rangle} x(at)\, e^{-jk(a\omega)t}\, dt$$

Let $at = m \Rightarrow t = m/a$

$a\,dt = dm \Rightarrow dt = \frac{dm}{a}$

$$= \frac{a}{T} \int_{\langle T/a \rangle} x(m)\, e^{-jk(a\omega)\frac{m}{a}} \frac{dm}{a}$$

$$= \frac{a}{T} \times \frac{1}{a} \int_{\langle T/a \rangle} x(m)\, e^{-jk\omega m}\, dm$$

$$= \frac{1}{T} \int_{\langle T/a \rangle} x(m)\, e^{-jk\omega m}\, dm$$

replace m by t

$$= \frac{1}{T} \int_{\langle T/a \rangle} x(t)\, e^{-jk\omega t}\, dt$$

$$\boxed{\therefore Z(K) = X(K)}$$

(7) Parseval's theorem: (power)

If $x(t)$ is the periodic power signal with fourier coefficient $X(K)$ then average power in the signal is given by:
$$P = \sum_{K=-\infty}^{\infty} |X(K)|^2$$

Proof: The power in the periodic signal is given as

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} [x(t)\; x^*(t)]\, dt \quad \text{—— (1)}$$

$x(t)$ can be expressed as

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{jk\omega t}$$

$$x^*(t) = \sum_{K=-\infty}^{\infty} \left[X(K)\, e^{jk\omega t}\right]^*$$

$$\Rightarrow x^*(t) = \sum_{K=-\infty}^{\infty} X^*(K)\, e^{-jk\omega t} \quad \text{—— (2)}$$

Substitute eq. (2) in eq. (1)

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot \left[\sum_{K=-\infty}^{\infty} X^*(K) e^{-j\omega Kt} \right] dt$$

Interchange the order of integration and summation

$$P = \frac{1}{T} \left[\sum_{K=-\infty}^{\infty} X^*(K) \int_{-T/2}^{T/2} x(t) e^{-j\omega Kt} dt \right]$$

$$= \frac{1}{T} \left[\sum_{K=-\infty}^{\infty} X^*(K) \, T \cdot X(K) \right]$$

$$= \sum_{K=-\infty}^{\infty} \left[X^*(K) \cdot X(K) \right]$$

$$\boxed{P = \sum_{K=-\infty}^{\infty} |X(K)|^2}$$

(8) Symmetry properties:

If $x(t)$ is real then $X^*(K) = X(-K)$

If $x(t)$ is imaginary then $X^*(K) = -X(-K)$

If $x(t)$ is real and even then imaginary of $X(K) = 0$

If $x(t)$ is real and odd then real of $X(K) = 0$