UNIT-III

Fourier Transform

Fourier transform is a transformation technique which transforms signals from the continuous time domain to the corresponding frequency domain and vice versa, and which applies for both periodic and aperiodic signals. Fourier transform can be developed by finding the Fourier series of a periodic function and then tending T to infinity.

Definitions of Fourier transform:

The Fourier transform of x(t) is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

Inverse Fourier transform of X(ω) is defined as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty T} x(\omega t)\, e^{j\omega t} d\omega$$

Dirichlet's Conditions: for the Fourier transform to exit for a periodic signal, it must satisfy certain conditions

1. The function x(t) must be a single values function.
2. The function x(t) has only a finite number of maxima and minima in every finite time interval.
3. The function x(t) has a finite number of discontinuities in every finite time interval.
4. The function x(t) is absolutely integral over a period, that is $\int_0^T x(t)\, dt < \infty$.

Properties of Fourier Transform

(1) Linearity:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ and $y(t) \xleftrightarrow{F.T} Y(\omega)$ then

$z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{F.T} a\,X(\omega) + b\,Y(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [a\,x(t) + b\,y(t)]\, e^{-j\omega t} dt$$

$$= a\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt + b\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t} dt$$

$$\therefore \boxed{Z(\omega) = a\,X(\omega) + b\,Y(\omega)}$$

(2) Time Shifting:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $z(t) = x(t - t_0) \xleftrightarrow{F.T} e^{-j\omega t_0} X(\omega)$

$= Z(K)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

Let $t - t_0 = m$

$t = m + t_0$

$dt = dm$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j\omega(m+t_0)}\, dm$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j\omega m} \cdot e^{-j\omega t_0}\, dm$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(m)\, e^{-j\omega m}\, dm$$

replace m by t

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore \boxed{Z(\omega) = e^{-j\omega t_0}\, X(\omega)}$$

Eg: $u(t-3) = e^{-j\omega 5}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$ ∵ F.T of u(t)

(3) Frequency Shift:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then

$z(t) = e^{j\omega_0 t} \cdot x(t) \xleftrightarrow{F.T} X(\omega - \omega_0) = Z(\omega)$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t}\, x(t)\, e^{-j\omega t}\, dt$$

$$Z(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$\therefore \boxed{Z(\omega) = X(\omega - \omega_0)}$$

Eg: $IFT\left[\frac{2}{j(\omega-3)}\right]$

$= sgn(t) \cdot e^{j3t}$

(4) Time Scaling:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then

Eg: $x(-t) = \frac{1}{|-1|} X\left(\frac{\omega}{-1}\right)$

$z(t) = x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

~~Sol~~ Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $at = m \Rightarrow t = m/a$

$a\,dt = dm \Rightarrow dt = dm/a$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j\omega\left(\frac{m}{a}\right)} \frac{dm}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(m)\, e^{-j\omega(m/a)}\, dm$$

replace m by t.

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t)\, e^{-j\left(\frac{\omega}{a}\right)t}\, dt$$

$$\therefore \boxed{Z(\omega) = \frac{1}{a} X\left(\frac{\omega}{a}\right)}$$

(5) Time differentiation / Differentiation in time

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then

$\frac{dx(t)}{dt} \xleftrightarrow{F.T} j\omega\, X(\omega)$

Proof: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

Differentiate w.r.t. 't'

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot j\omega \cdot e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\, j\omega \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[j\omega X(\omega)\right] e^{j\omega t}\, d\omega$$

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{F.T} j\omega X(\omega)}$$

(6) <u>Frequency differentiation:</u>

If $x(t) \overset{F.T}{\longleftrightarrow} X(\omega)$ then

$$-jt\,x(t) \overset{F.T}{\longleftrightarrow} \frac{d}{d\omega}[X(\omega)].$$

<u>proof:</u> $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

Differentiate w.r.t. 'ω'.

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)\; -jt \cdot e^{-j\omega t}\, dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} [-jt\,x(t)]\, e^{-j\omega t}\, dt$$

$$\boxed{\therefore\; -jt\,x(t) \overset{F.T}{\longleftrightarrow} \frac{d}{d\omega}[X(\omega)]}$$

(7) <u>Convolution property:</u>

If $x(t) \overset{F.T}{\longleftrightarrow} X(\omega)$ and $y(t) \overset{F.T}{\longleftrightarrow} Y(\omega)$

then $z(t) = x(t) * y(t) \overset{F.T}{\longleftrightarrow} X(\omega).Y(\omega) = Z(\omega)$

<u>proof:</u> $Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$

<u>Eg:</u> $F.T[\operatorname{sgn}(t) * u(t)]$

$= \left(\frac{2}{j\omega}\right) \cdot \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$

$$Z(\omega) = \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$Z(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt.$$

Interchanging the integrations

$$Z(\omega) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt\, d\tau$$

Let $t - \tau = m \Rightarrow t = m + \tau$

$dt = dm$

$$\therefore Z(\omega) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(m)\, e^{-j\omega(m+\tau)}\, dt\, dm$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} y(m)\, e^{-j\omega m}\, dm$$

$$\therefore \boxed{Z(\omega) = X(\omega)\, Y(\omega)}$$

(8) Multiplication / modulation:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ and $y(t) \xleftrightarrow{F.T} Y(\omega)$ then

$$z(t) = x(t) \cdot y(t) \xleftrightarrow{F.T} \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$Z(\omega) = \int_{-\infty}^{\infty} [x(t) \cdot y(t)]\, e^{-j\omega t}\, dt \quad \text{--- (1)}$$

W.K.T, $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, e^{j\tau t}\, d\tau$ --- (2)

Substitute eq(2) in (1)

$$Z(\omega) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau)\, e^{j\tau t}\, d\tau \cdot y(t) \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau) \int_{-\infty}^{\infty} y(t)\, e^{-j(\omega-\tau)t}\, dt\, d\tau.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, Y(\omega-\tau)\, d\tau \quad \left(\because x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau\right)$$

$$= \frac{1}{2\pi}\ \cancel{X(\tau)}\ \left[X(\omega) * Y(\omega)\right]$$

$$\boxed{\therefore Z(\omega) = \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]}$$

(9) Integration property:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{F.T} \frac{1}{j\omega} X(\omega)$

Proof: Let $x(t)$ be expressed as

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

Apply F.T. on both sides

$$F[x(t)] = F\left[\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]\right]$$

By differentiation property, R.H.S of above equation

$$F[x(t)] = j\omega\, F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] \quad (\because \text{from proof 5})$$

$$X(\omega) = j\omega\, F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\frac{1}{j\omega} X(\omega) = F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\therefore \boxed{F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega)}$$

(10) Duality theorem:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$

(10) Duality theorem:

If $x(t) \xleftrightarrow{F.T} X(\omega)$ then $X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$

proof:

I.F.T $\rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$

replace t by ω and ω by t.

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{j\omega t}\, dt$$

Substitute '$-\omega$' in ω.

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t) e^{j(-\omega)t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

$$\boxed{\therefore X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)}$$

11) Parseval's theorem (or) Rayleigh's theorem : (Energy)

If $x(t) \xleftrightarrow{FT} X(\omega)$ then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

Proof:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \int_{-\infty}^{\infty} [x(t) . x^*(t)] dt \quad \text{--- (1)}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega \quad \text{--- (2)}$$

Substitute eq(2) in eq(1)

$$E = \int_{-\infty}^{\infty} x(t) . \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega \right] dt$$

changing the order of integration,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\underline{\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt} \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, \underline{X(\omega)}\, d\omega$$

$$\boxed{\therefore E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

12) Symmetry property:

Let $x(t)$ be real signal and $X(\omega) = X_R(\omega) + jX_I(\omega)$

then $x_e(t) \xleftrightarrow{F.T} X_R(\omega)$

$x_o(t) \xleftrightarrow{F.T} jX_I(\omega)$

Proof: If $x(t)$ is a real signal then

$$x(-t) \xleftrightarrow{F.T} X^*(\omega)$$

$$x(-t) \xleftrightarrow{F.T} X^*(\omega) = X_R(\omega) - jX_I(\omega)$$

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$F[x_e(t)] = \frac{1}{2} F\left((x(t) + x(-t))\right)$$

$$F[x_e(t)] = \frac{1}{2}[X(\omega) + X^*(\omega)]$$

$$= \frac{1}{2}[X_R(\omega) + \cancel{jX_I(\omega)} + X_R(\omega) - \cancel{jX_I(\omega)}]$$

$$= \frac{1}{\cancel{2}} \cancel{2} X_R(\omega)$$

$$\therefore \boxed{F[x_e(t)] = X_R(\omega)} \Rightarrow x_e(t) \xleftrightarrow{F.T} X_R(\omega)$$

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$

$$F[x_0(t)] = \frac{1}{2} F[x(t) - x(-t)]$$

$$= \frac{1}{2}[(\cancel{X_R(\omega)} + jX_I(\omega)) - [\cancel{X_R(\omega)} - jX_I(\omega)]]$$

$$= \frac{1}{\cancel{2}} \cancel{2} j X_I(\omega).$$

$$\therefore \boxed{F[x_0(t)] = jX_I(\omega)} \Rightarrow x_0(t) \xleftrightarrow{F.T} jX_I(\omega).$$

<u>Fourier transform</u> ~~of~~ <u>for standard Signals:</u>

(1) $e^{-at} u(t)$

Let $x(t) = e^{-at} u(t)$.

$$u(t) = 1, \quad t > 0$$
$$\quad\;\; 0, \quad t < 0$$

$$\therefore x(t) = e^{-at}, \quad t > 0$$
$$\quad\quad\;\; 0, \quad t < 0$$

By F.T,

$$X(\omega) = \int_{-\infty}^{0} e^{-at} u(t) e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

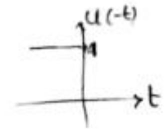
$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^\infty = 0 - \frac{1}{-(a+j\omega)} = \frac{1}{(a+j\omega)}$$

$$\therefore\ F\left[e^{-at}u(t)\right] \xleftrightarrow{F.T.} \frac{1}{(a+j\omega)}$$

2) $e^{at}u(-t)$

<u>Proof</u>: Let $x(t)=e^{at}u(-t)$



$$u(-t)=1\ ,\ t<0$$
$$=0\ ,\ t>0$$

By F.T,

$$X(\omega)=\int_{-\infty}^{\infty} e^{at}u(-t)\,e^{-j\omega t}\,dt.$$

$$=\int_{-\infty}^{0} e^{at}(1)\,e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{at}(0)\,e^{-j\omega t}\,dt.$$

$$=\int_{-\infty}^{0} e^{at-j\omega t}\,dt$$

$$=\int_{-\infty}^{0} e^{(a-j\omega)t}\,dt$$

$$=\left[\frac{e^{a-j\omega}}{a-j\omega}\right]_{-\infty}^{0}$$

$$=\frac{1}{a-j\omega}-0=\frac{1}{a-j\omega}.$$

$$\boxed{\therefore\ e^{at}u(-t) \xleftrightarrow{F.T.} \frac{1}{a-j\omega}}$$

4) $t.e^{-at}u(t)$

Proof: Let $x(t) = te^{-at}u(t)$

By F.T, $X(\omega) = \int_{-\infty}^{\infty} te^{-at}u(t)\, e^{-j\omega t}dt$

$$= \int_{-\infty}^{0} te^{-at}\underbrace{u(t)}_{0} e^{-j\omega t}dt + \int_{0}^{\infty} te^{-at}u(t)\, e^{-j\omega t}dt.$$

$$= \int_{0}^{\infty} te^{-at}\underbrace{u(t)}_{1} e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} te^{-at}e^{-j\omega t}dt.$$

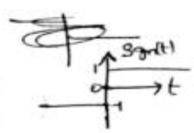
$$= \int_{0}^{\infty} t\, e^{-(a+j\omega)t}dt$$

$$= \left[t\, \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} - \frac{e^{-(a+j\omega)t}}{(-(a+j\omega))^2} \right]_{0}^{\infty}$$

$$= (0) - \left[0 - \frac{1}{(a+j\omega)^2}\right] = \frac{1}{(a+j\omega)^2}$$

$$\boxed{\therefore\ te^{-at}u(t) \xleftrightarrow{F.T} \frac{1}{(a+j\omega)^2}}$$

5) $e^{-a|t|}\,sgn(t)$



Proof: $sgn(t) = 1 \ , t>0$

$\qquad\quad = -1, \ t<0.$

Let $x(t) = e^{-a|t|}sgn(t)$

By F.T, $X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}sgn(t)\, e^{-j\omega t}dt$

$$= \int_{-\infty}^{0} e^{-at}(-1)\, e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}(1)\, e^{-j\omega t}dt$$

$$= -\int_{-\infty}^{0} e^{-(a+j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= -\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= -\left(\frac{1}{-(a+j\omega)} - 0\right) + \left[0 - \left(\frac{-1}{a+j\omega}\right)\right]$$

$$= \frac{1}{a+j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2}{a+j\omega}$$

$$\therefore \boxed{e^{-at}\,\mathrm{Sgn}(t) \xleftrightarrow{F.T} \frac{2}{a+j\omega}}$$

6) Sgn(t)

Proof: Let $x(t) = \mathrm{Sgn}(t)$

$$\mathrm{Sgn}(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \end{cases}$$

By F.T, $X(\omega) = \int_{-\infty}^{\infty} \mathrm{Sgn}(t)\, e^{-j\omega t}\, dt$

Let $x(t) = \mathrm{Sgn}(t)$

$$= 2u(t) - 1$$

$$\frac{dx(t)}{dt} = 2\frac{d}{dt}[u(t)].$$

$$\frac{dx(t)}{dt} = 2\,\delta(t) \qquad \left(\because \frac{d}{dt}[\underbrace{u(t)}_{\text{Step function}}] = \underbrace{\delta(t)}_{\text{Impulse func.}}\right)$$

$$F\left[\frac{d}{dt}x(t)\right] = 2\,F(\delta(t))$$

$$j\omega X(\omega) = 2[F(\delta t)]$$

$$\boxed{X(\omega) = \frac{2}{j\omega}\,F(\delta t)}$$

$$X(\omega) = \frac{2}{j\omega}(1) \qquad F(\delta t) = 1.$$

$$= \frac{2}{j\omega}$$

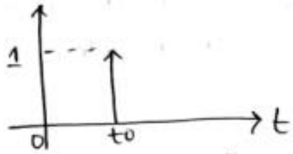
$$\boxed{\mathrm{Sgn}(t) \xleftrightarrow{F.T} \frac{2}{j\omega}}$$

Impulse function($\delta(t)$):

$$\delta(t) = 1, \; t = 0$$

$$= 0, \; t \neq 0$$

$$\delta(t-t_0) = 1 \;, \; t = t_0$$
$$= 0 \;, \; \text{otherwise}$$



The area under the pulse curve is always unity

i.e. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

The shifting or sampling may also be done at any instant $t = t_0$, if you define the impulse function mathematically.

$$\int_{-\infty}^{\infty} x(t) . \delta(t-t_0) = x(t_0)$$

7) $\delta(t)$

Proof: Let $x(t) = \delta(t)$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= \left[e^{-j\omega t} \right]_{t=0}$$

$\star$ $\boxed{\therefore F[\delta(t)] = 1}$

8) Gate function:

$$\Pi(t/T) = A \quad |t| < T/2$$
$$= 0 \quad \text{otherwise.}$$

$$F[\Pi(t/T)] = \int_{-T/2}^{T/2} A\, e^{-j\omega t} dt$$

$$= A \left(\frac{e^{-j\omega t}}{-j\omega} \right]_{-T/2}^{T/2}$$

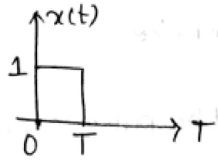
$$= \frac{-A}{j\omega} \left[e^{\frac{-j\omega T}{2}} - e^{\frac{j\omega T}{2}} \right]$$

$$= \frac{-2A}{j\omega} \left[\frac{e^{-j\omega T/2} - e^{j\omega T/2}}{2j} \right] \quad \left(\because \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta \right)$$

$$\left(\because \text{sinc} = \frac{\sin t}{t} \right)$$

$$= \frac{-2A}{\omega} \left(-\sin \omega T/2 \right) = \frac{2A}{\omega} \sin \omega T/2 \;// = AT . \text{sinc}(\omega T/2)$$

Find the fourier transform of rectangular pulse given below.



Sol: $x(t) = 1 \quad 0 < t < T$

$= 0,$ otherwise

$$\therefore X(\omega) = \int_0^T 1 \cdot e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T$$

$$= \frac{-1}{j\omega}\left(e^{-j\omega T} - 1\right)$$

$$= \frac{1}{j\omega}\left(1 - e^{-j\omega T}\right)$$

$$= \frac{1}{j\omega}\left(e^{-j\omega T/2} \cdot e^{j\omega T/2} - e^{-j\omega T/2} e^{-j\omega T/2}\right)$$

$$= \frac{2}{\omega} e^{-j\omega T/2} \left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right)$$

$$= \frac{2}{\omega} e^{-j\omega T/2} \sin\omega T/2 \times \frac{T/2}{T/2}$$

$$= e^{-j\omega T/2} \frac{\sin\omega T/2}{\omega T/2} \cdot \cancel{2} \times \frac{T}{\cancel{2}}$$

$$= e^{-j\omega T/2} \operatorname{Sinc}(\omega T/2) T \quad \left(\because \operatorname{Sinc} t = \frac{\sin t}{t}\right)$$

$$= T \cdot e^{-j\omega T/2} \operatorname{Sinc}(\omega T/2)$$

Fourier transform of periodic signal :-

Let the signal $x(t)$ be periodic with period T_0 such signal can be expressed by exponential fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} x(k) e^{jk\omega_0 t}$$

Apply fourier transform on b. s

$$F[x(t)] = \sum_{k=-\infty}^{\infty} x(k) \; F[e^{jk\omega_0 t}]$$

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} x(k) \; 2\pi \, \delta(\omega - k\omega_0)}$$