NBKRIST

17EE3103

LINEAR CONTROL SYSTEMS

LECTURE NOTES

Prepared by :

E. Vinod Arun Kumar, Sk. Manjoor Ahmad & R.Ramprasad

Department of Electrical & Electronics Engineering

UNIT – I

Introduction to classical control system

A System is a combination or an arrangement of different physical components which act together as an entire unit to achieve certain objective

- **Control System**

A control system is a system of devices or set of devices, that manages, commands, directs or regulates the behaviour of other device(s) or system(s) to achieve desire results.

A control system is a system, which controls other system.

- The main **feature of control system** is, there should be a clear mathematical relation between input and output of the system.
- When the relation between input and output of the system can be represented by a linear proportionality, the system is called linear control system.
- when the relation between input and output cannot be represented by single linear proportionality, rather the input and output are related by some non-linear relation, the system is referred as non-linear control system.

- **Requirement of Good Control System**

  - **Accuracy :** Accuracy is the measurement tolerance of the instrument and defines the limits of the errors made when the instrument is used in normal operating conditions. Accuracy can be improved by using feedback elements. To increase accuracy of any control system error detector should be present in control system.

  - **Sensitivity :** The parameters of control system are always changing with change in surrounding conditions, internal disturbance or any other parameters. This change can be expressed in terms of sensitivity. Any control system should be insensitive to such parameters but sensitive to input signals only.
  - **Noise :** An undesired input signal is known as noise. A good control system should be able to reduce the noise effect for better performance.
  - **Stability :** It is an important characteristic of control system. For the bounded input signal, the output must be bounded and if input is zero

then output must be zero then such a control system is said to be stable system.

- **Bandwidth :** An operating frequency range decides the bandwidth of control system. Bandwidth should be large as possible for frequency response of good control system.
- **Speed :** It is the time taken by control system to achieve its stable output. A good control system possesses high speed. The transient period for such system is very small.
- **Oscillation :** A small numbers of oscillation or constant oscillation of output tend to system to be stable.

Control Systems can be classified as open loop control systems and closed loop control systems based on the **feedback path**.

- **Open Loop Control System**

A control system in which the control action is totally independent of output of the system then it is called **open loop control system**.

In **open loop control systems**, output is not fed-back to the input. So, the control action is independent of the desired output.

Manual control system is also an open loop control system.

The following figure shows the block diagram of the open loop control system.

Here, an input is applied to a controller and it produces an actuating signal or controlling signal. This signal is given as an input to a plant or process which is to be controlled. So, the plant produces an output, which is controlled. The traffic lights control system which we discussed earlier is an example of an open loop control system.

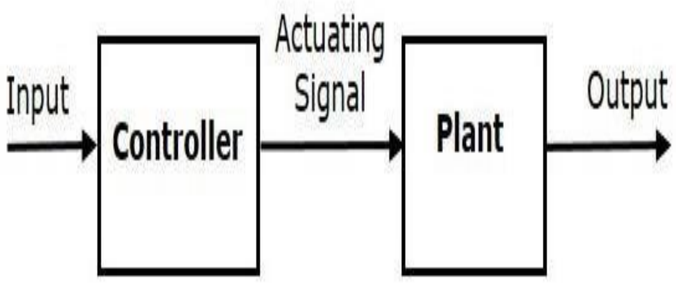


Figure 1.1: Block diagram of open loop control system

❖ Practical Examples of Open Loop Control System

1. **Electric Hand Drier -** Hot air (output) comes out as long as you keep your hand under the machine, irrespective of how much your hand is dried.
2. **Automatic Washing Machine -** This machine runs according to the pre-set time irrespective of washing is completed or not.
3. **Bread Toaster -** This machine runs as per adjusted time irrespective of toasting is completed or not.
4. **Automatic Tea/Coffee Maker -** These machines also function for pre adjusted time only.
5. **Timer Based Clothes Drier -** This machine dries wet clothes for pre-adjusted time, it does not matter how much the clothes are dried.
6. **Light Switch -** Lamps glow whenever light switch is on irrespective of light is required or not.
7. **Volume on Stereo System -** Volume is adjusted manually irrespective of output volume level.

❖ Advantages of Open Loop Control System

1. Simple in construction and design.
2. Economical.
3. Easy to maintain.
4. Generally stable.
5. Convenient to use as output is difficult to measure.

❖ Disadvantages of Open Loop Control System

1. They are inaccurate.
2. They are unreliable.
3. Any change in output cannot be corrected automatically.

➢ Closed Loop Control System

Control system in which the output has an effect on the input quantity in such a manner that the input quantity will adjust itself based on the output generated is called **closed loop control system**.

In **closed loop control systems**, output is fed back to the input. So, the control action is dependent on the desired output.

The following figure shows the block diagram of negative feedback closed loop control system.

The error detector produces an error signal, which is the difference between the input and the feedback signal. This feedback signal is obtained from the block (feedback elements) by considering the output of the overall system as an input to this block. Instead of the direct input, the error signal is applied as an input to a controller.

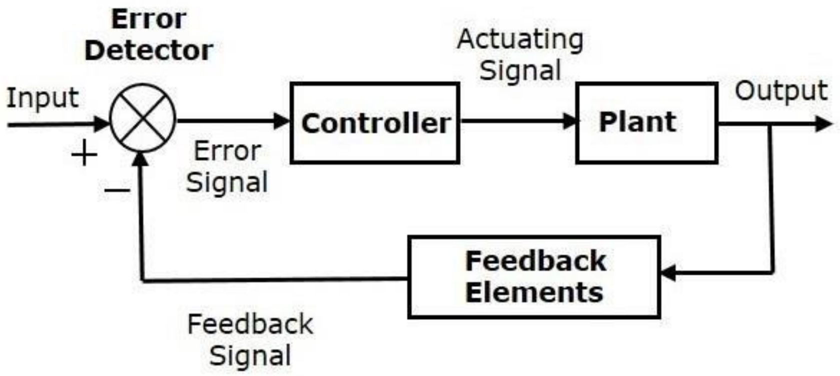


Figure 1.2:Block diagram of closed loop control system

So, the controller produces an actuating signal which controls the plant. In this combination, the output of the control system is adjusted automatically till we get the desired response. Hence, the closed loop control systems are also called the automatic control systems. Traffic lights control system having sensor at the input is an example of a closed loop control system.

- ❖ **Practical Examples of Closed Loop Control System**

1. **Automatic Electric Iron -** Heating elements are controlled by output temperature of the iron.
2. **Servo Voltage Stabilizer -** Voltage controller operates depending upon output voltage of the system.
3. **Water Level Controller -** Input water is controlled by water level of the reservoir.
4. **Missile Launched and Auto Tracked by Radar -** The direction of missile is controlled by comparing the target and position of the missile.

5. **An Air Conditioner -** An air conditioner functions depending upon the temperature of the room.
6. **Cooling System in Car -** It operates depending upon the temperature which it controls.

❖ Advantages of Closed Loop Control System

1. Closed loop control systems are more accurate even in the presence of non-linearity.
2. Highly accurate as any error arising is corrected due to presence of feedback signal.
3. Bandwidth range is large.
4. Facilitates automation.
5. The sensitivity of system may be made small to make system more stable.
6. This system is less affected by noise.

❖ Disadvantages of Closed Loop Control System

1. They are costlier.
2. They are complicated to design.
3. Required more maintenance.
4. Feedback leads to oscillatory response.
5. Overall gain is reduced due to presence of feedback.
6. Stability is the major problem and more care is needed to design a stable closed loop system.

- **Comparison of Closed Loop And Open Loop Control System**

Sr. No.	Open loop control system	Closed loop control system
1	The feedback element is absent.	The feedback element is always present.
2	An error detector is not present.	An error detector is always present.
3	It is stable one.	It may become unstable.
4	Easy to construct.	Complicated construction.
5	It is an economical.	It is costly.
6	Having small bandwidth.	Having large bandwidth.
7	It is inaccurate.	It is accurate.
8	Less maintenance.	More maintenance.
9	It is unreliable.	It is reliable.
10	Examples: Hand drier, tea maker	Examples: Servo voltage stabilizer, perspiration

- **Feedback**

If either the output or some part of the output is returned to the input side and utilized as part of the system input, then it is known as **feedback**. Feedback plays an important role in order to improve the performance of the control systems. In this chapter, let us discuss the types of feedback & effects of feedback.

Types of Feedback

There are two types of feedback −

- Positive feedback
- Negative feedback

Positive Feedback

The positive feedback adds the reference input, R(s) and feedback output. The following figure shows the block diagram of **positive feedback control system**.

The concept of transfer function will be discussed in later chapters. For the time being, consider the transfer function of positive feedback control system is,

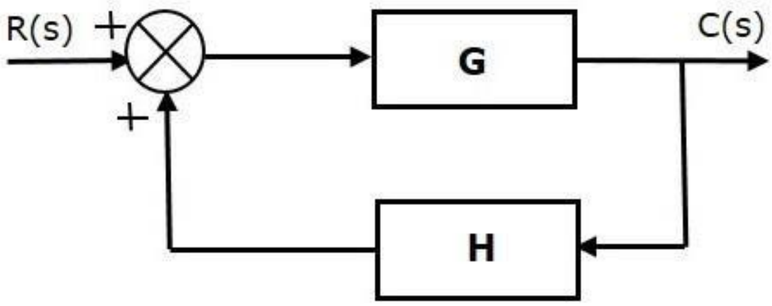


Figure 1.3: Block diagram of positive feedback system

T=G/(1−GH) (Equation 1)
Where,

- **T** is the transfer function or overall gain of positive feedback control system.
- **G** is the open loop gain, which is function of frequency.
- **H** is the gain of feedback path, which is function of frequency.

Negative Feedback

Negative feedback reduces the error between the reference input, R(s)R(s) and system output. The following figure shows the block diagram of the **negative feedback control system**.

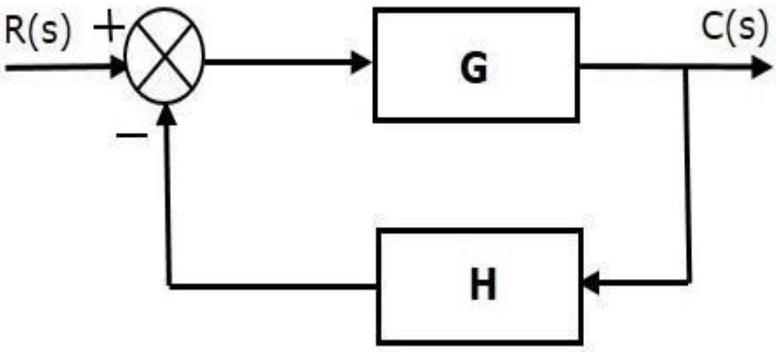


Figure 1.4: Block diagram of negative feedback system

Transfer function of negative feedback control system is,

T=G/(1+GH) (Equation 2)

Effects of Feedback

Let us now understand the effects of feedback.

Effect of Feedback on Overall Gain

- From Equation 2, we can say that the overall gain of negative feedback closed loop control system is the ratio of 'G' and (1+GH). So, the overall gain may increase or decrease depending on the value of (1+GH).
- If the value of (1+GH) is less than 1, then the overall gain increases. In this case, 'GH' value is negative because the gain of the feedback path is negative.
- If the value of (1+GH) is greater than 1, then the overall gain decreases. In this case, 'GH' value is positive because the gain of the feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, the feedback will increase the overall gain of the system in one frequency range and decrease in the other frequency range.

Effect of Feedback on Sensitivity

Sensitivity of the overall gain of negative feedback closed loop control system *(T)* to the variation in open loop gain (**G**) is defined as $S_G^T = \frac{\frac{\delta T}{T}}{\frac{\delta G}{G}}$ (equation 3)

Where, **∂T** is the incremental change in T due to incremental change in G.

We can rewrite Equation 3 as $S_G^T = \frac{\delta T}{\delta G}\frac{G}{T}$ (equation 4)

Do partial differentiation with respect to G on both sides of Equation 2.

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1-G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2} \qquad \text{(Equation 5)}$$

From Equation 2, you will get

$$\frac{G}{T} = 1 + GH \qquad \text{(equation 6)}$$

Substitute Equation 5 and Equation 6 in Equation 4.

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

So, we got the **sensitivity** of the overall gain of closed loop control system as the reciprocal of (1+GH). So, Sensitivity may increase or decrease depending on the value of (1+GH).

- If the value of (1+GH) is less than 1, then sensitivity increases. In this case, 'GH' value is negative because the gain of feedback path is negative.

- If the value of (1+GH) is greater than 1, then sensitivity decreases. In this case, 'GH' value is positive because the gain of feedback path is positive.

In general, 'G' and 'H' are functions of frequency. So, feedback will increase the sensitivity of the system gain in one frequency range and decrease in the other frequency range. Therefore, we have to choose the values of 'GH' in such a way that the system is insensitive or less sensitive to parameter variations.

Effect of Feedback on Stability

- A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable.

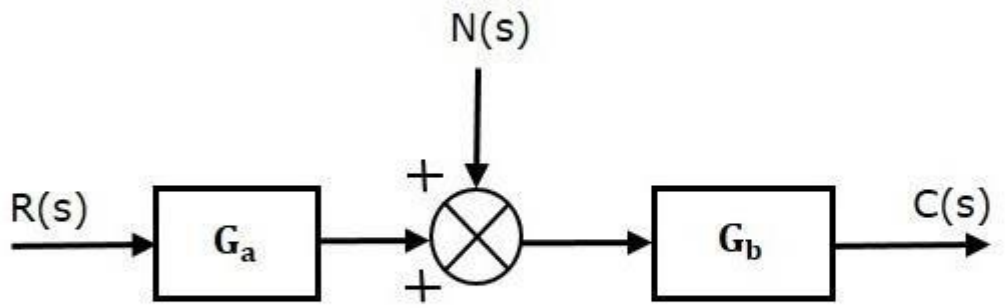
- In Equation 2, if the denominator value is zero (i.e., GH = -1), then the output of the control system will be infinite. So, the control system becomes unstable.

Therefore, we have to properly choose the feedback in order to make the control system stable.

Effect of Feedback on Noise

To know the effect of feedback on noise, let us compare the transfer function relations with and without feedback due to noise signal alone.

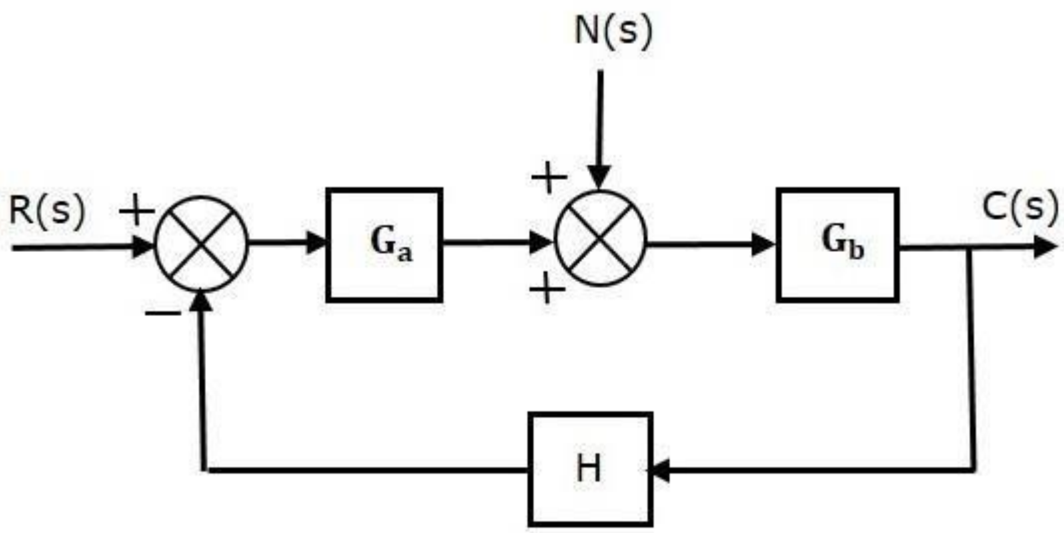
Consider an **open loop control system** with noise signal as shown below.



The **open loop transfer function** due to noise signal alone is

$$\frac{C(s)}{N(s)} = G_b \qquad \text{(Equation 7)}$$

It is obtained by making the other input R(s)R(s) equal to zero.
Consider a **closed loop control system** with noise signal as shown below.



The **closed loop transfer function** due to noise signal alone is

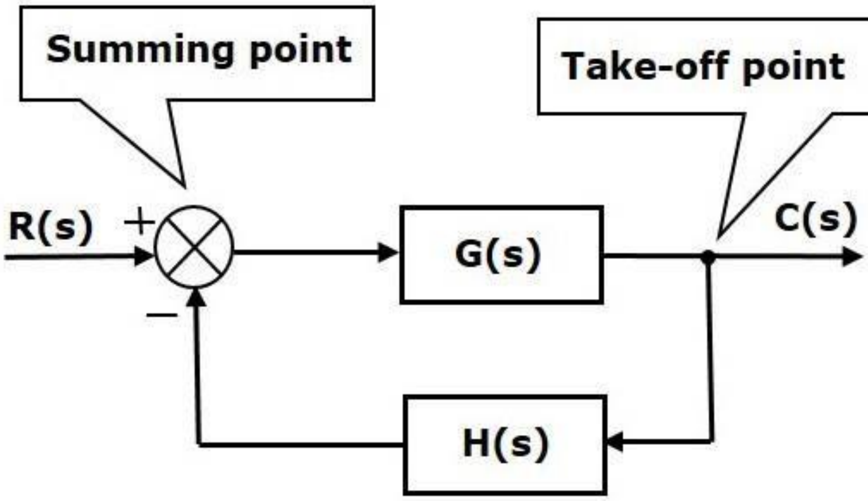
$$\frac{C(s)}{N(s)} = \frac{G_b}{1+G_aG_bH} \qquad \text{(Equation 8)}$$

It is obtained by making the other input R(s) equal to zero.
Compare Equation 7 and Equation 8, In the closed loop control system, the gain due to noise signal is decreased by a factor of $(1+G_aG_bH)$ provided that the term $(1+G_aG_bH)$ is greater than one.

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

Block

The transfer function of a component is represented by a block. Block has single input and single output.

The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

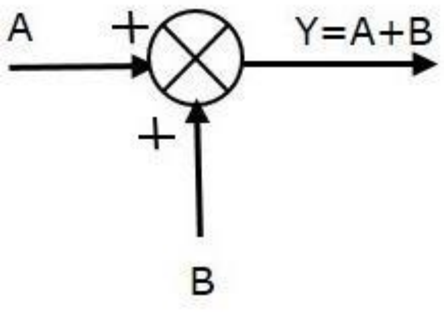
Transfer Function, $G(s) = \frac{Y(s)}{X(s)}$ $\Rightarrow Y(s) = G(s)X(s)$

Output of the block is obtained by multiplying transfer function of the block with input.

Summing Point

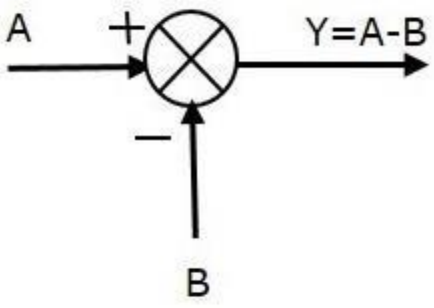
The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one by one.

The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B.**



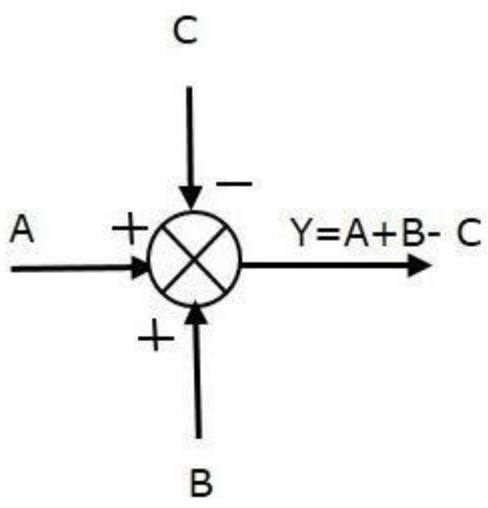
i.e.,Y = A + B.

The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B**.



Y = A + (-B) = A - B.

The following figure shows the summing point with three inputs (A, B, C) and one output (Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

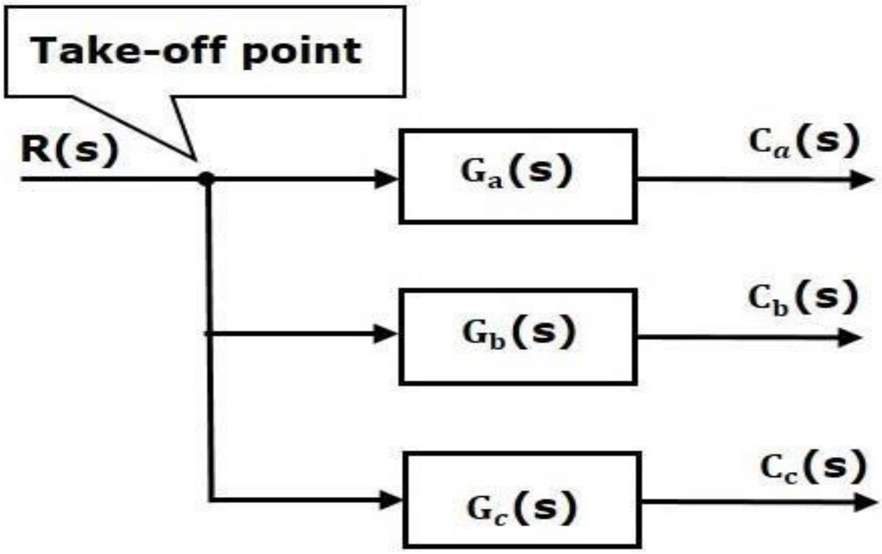


Y = A + B + (−C) = A + B − C.

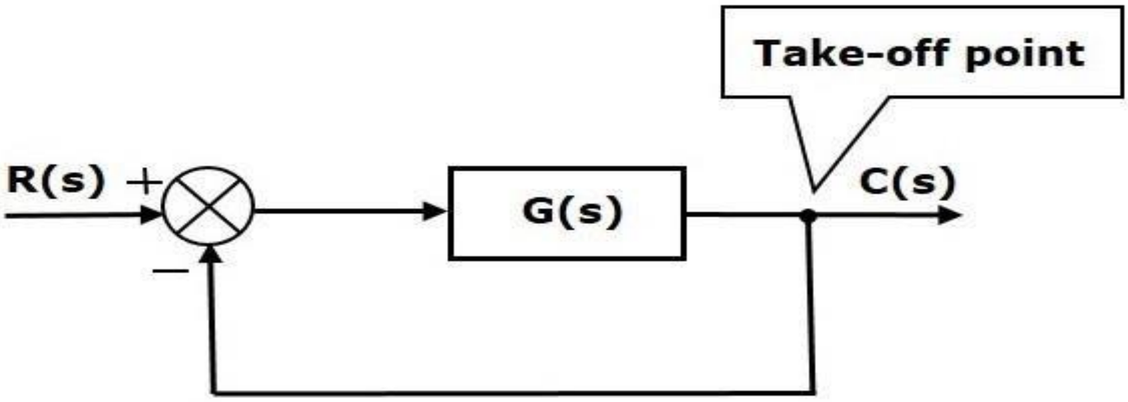
Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points.

In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.



In the following figure, the take-off point is used to connect the output C(s), as one of the inputs to the summing point.



Disadvantages of Block Diagram Representation

- No information about the physical construction
- Source of energy is not shown

Advantages of Block Diagram Representation

- Very simple to construct block diagram for a complicated system
- Function of individual element can be visualized
- Individual & Overall performance can be studied
- Over all transfer function can be calculated easily.

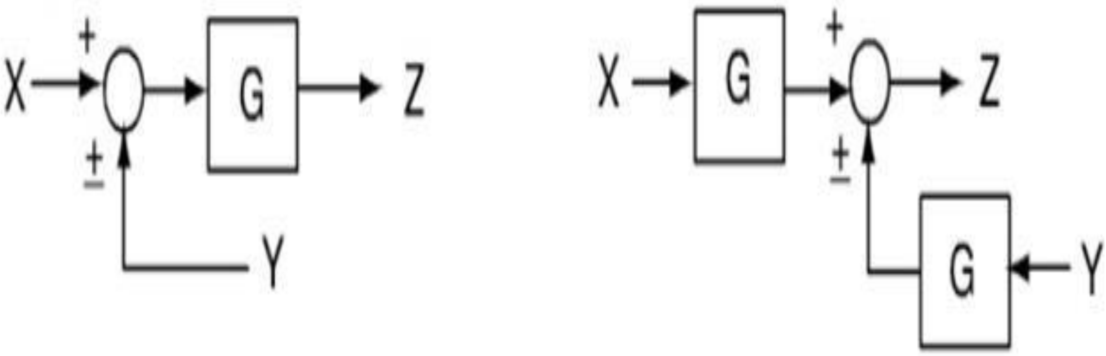
Block diagram reduction technique

Because of their simplicity and versatility, block diagrams are often used by control engineers to describe all types of systems. A block diagram can be used simply to represent the composition and interconnection of a system. Also, it can be used, together with transfer functions, to represent the cause-and-effect relationships throughout the system. Transfer Function is defined as the relationship between an input signal and an output signal to a device.

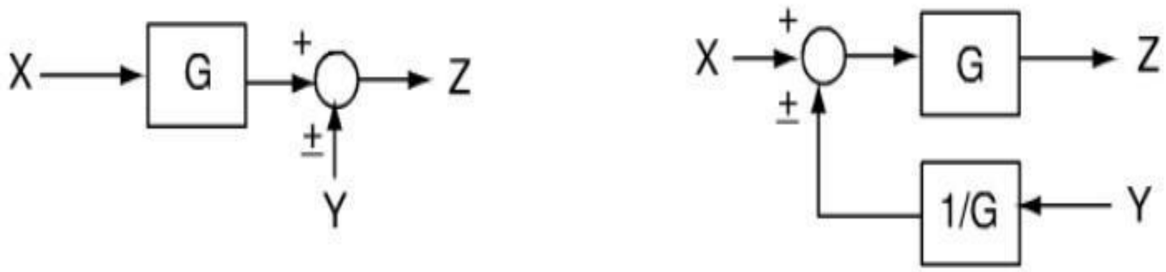
Block diagram rules

Cascaded blocks

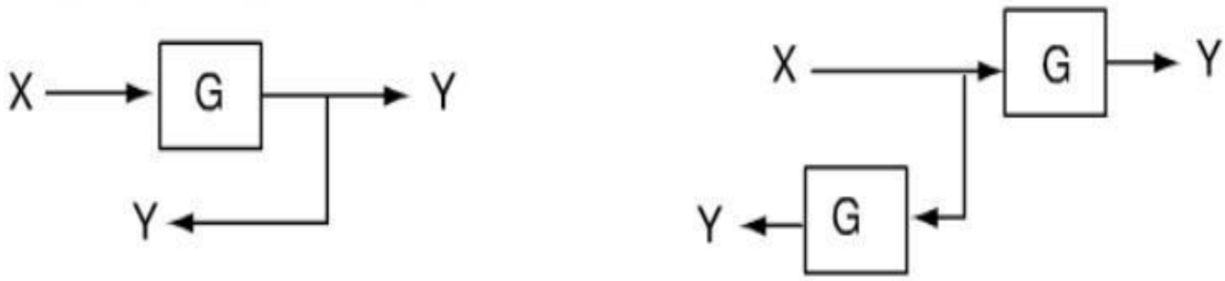
Moving a summer beyond the block



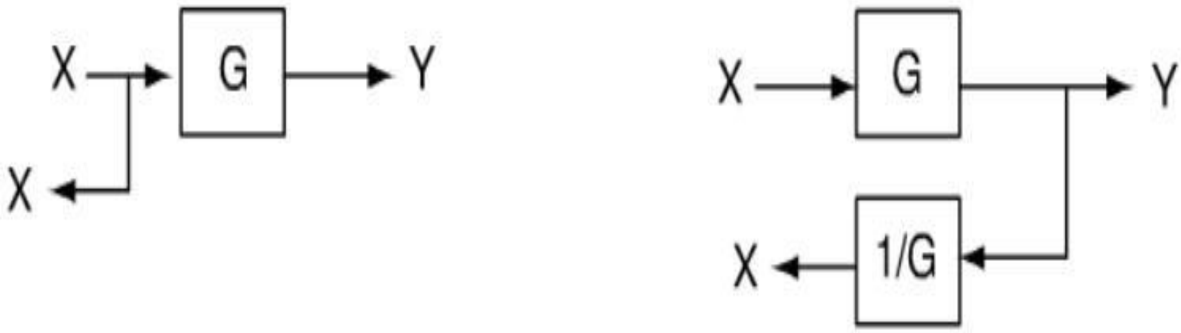
Moving a summer ahead of block



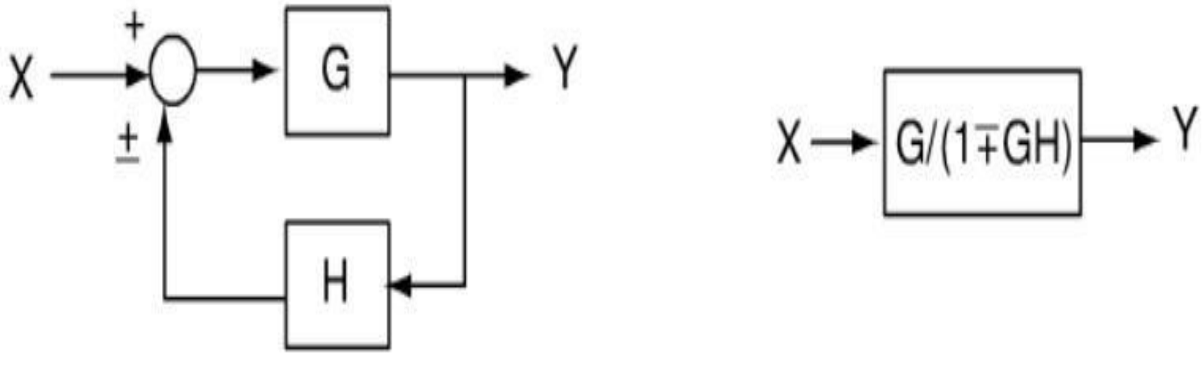
Moving a pick-off ahead of block



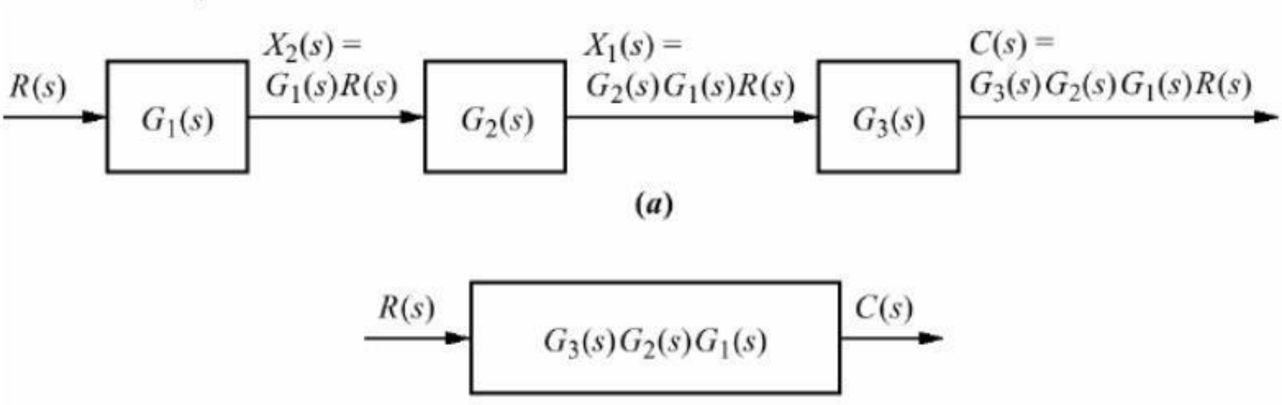
Moving a pick-off behind a block



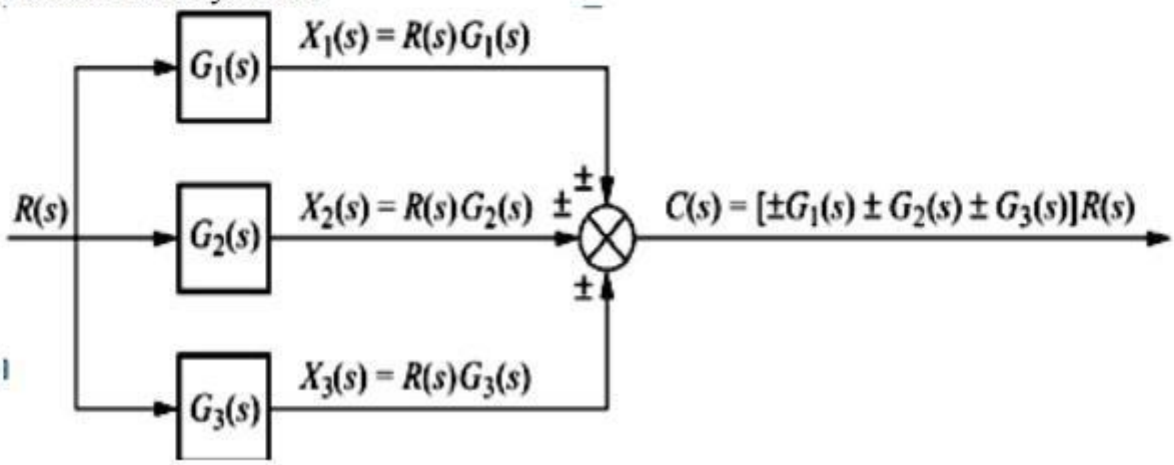
Eliminating a feedback loop



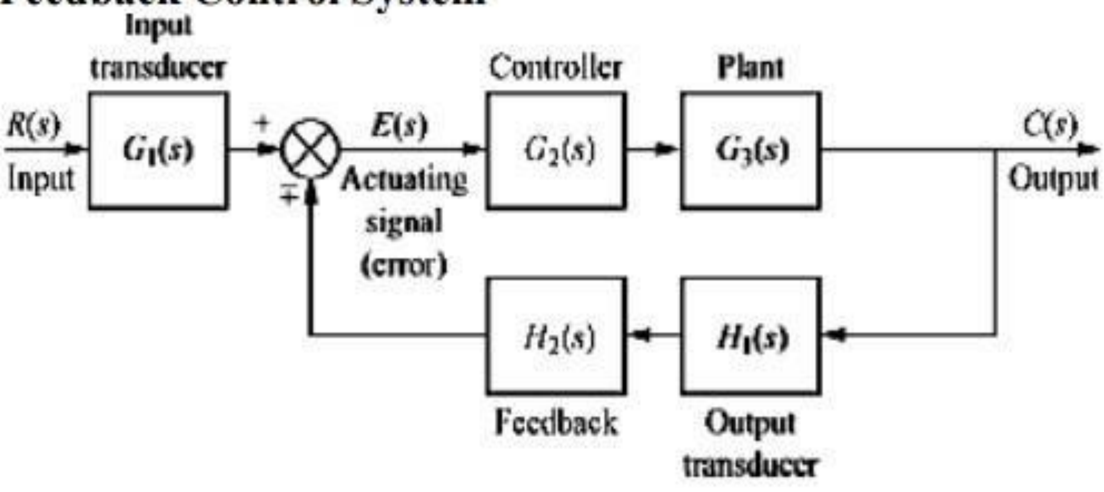
Cascaded Subsystems



Parallel Subsystems



Feedback Control System



Procedure to solve Block Diagram Reduction Problems

Step 1: Reduce the blocks connected in series

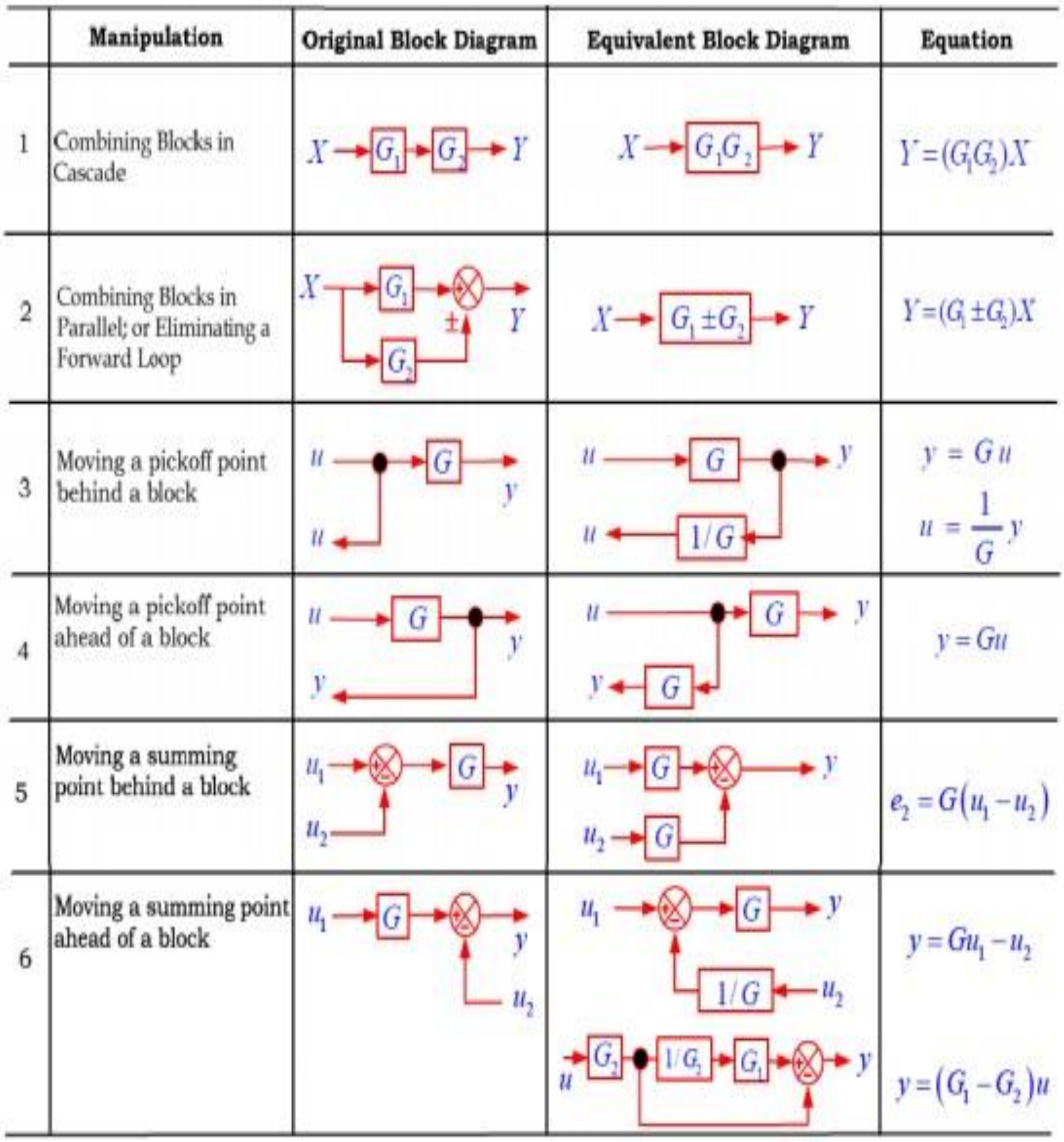
Step2: Reduce the blocks connected in parallel

Step 3: Reduce the minor feedback loops

Step 4: Try to shift take off points towards right and Summing point towards left

Step 5: Repeat steps 1 to 4 till simple form is obtained

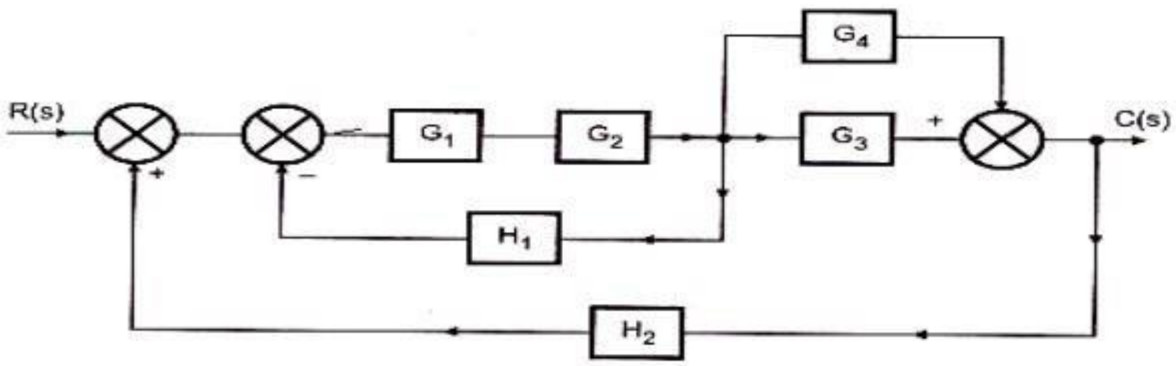
Step 6: Obtain the Transfer Function of Overall System



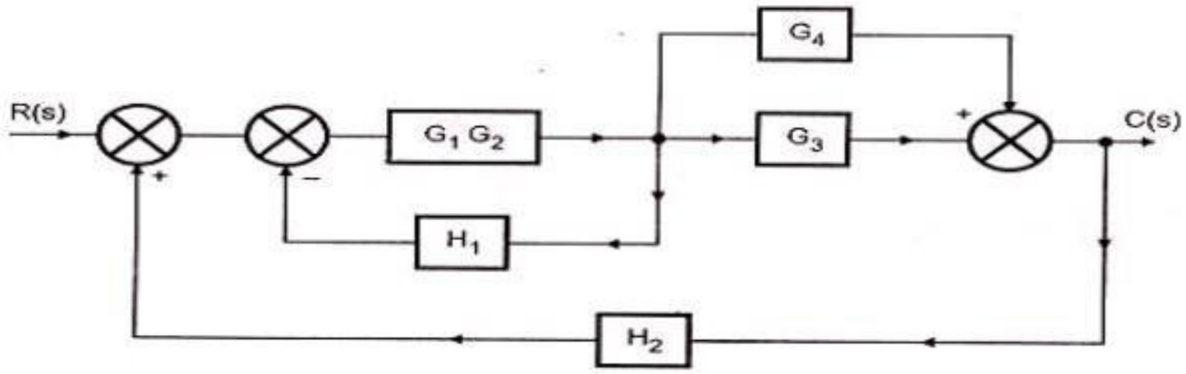
	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining Blocks in Cascade			$Y=(G_1G_2)X$
2	Combining Blocks in Parallel; or Eliminating a Forward Loop			$Y=(G_1 \pm G_2)X$
3	Moving a pickoff point behind a block			$y = Gu$ $u = \frac{1}{G}y$
4	Moving a pickoff point ahead of a block			$y=Gu$
5	Moving a summing point behind a block			$e_2=G(u_1-u_2)$
6	Moving a summing point ahead of a block			$y=Gu_1-u_2$ $y=(G_1-G_2)u$

Problem 1

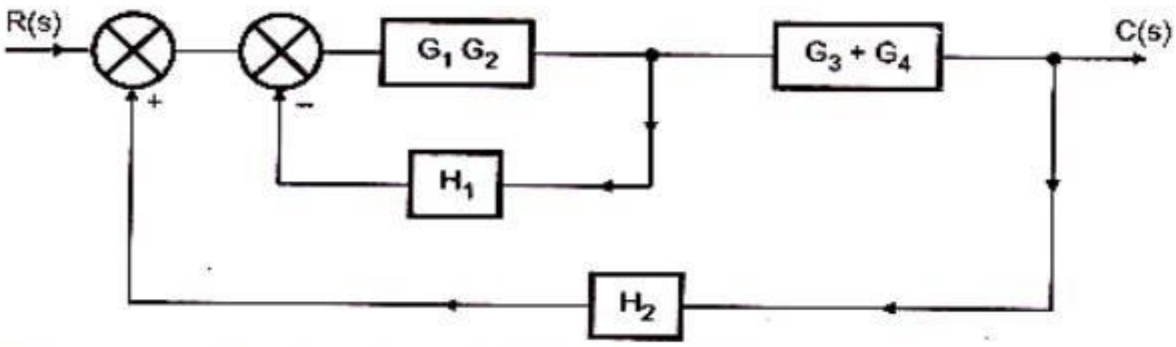
1. Obtain the Transfer function of the given block diagram



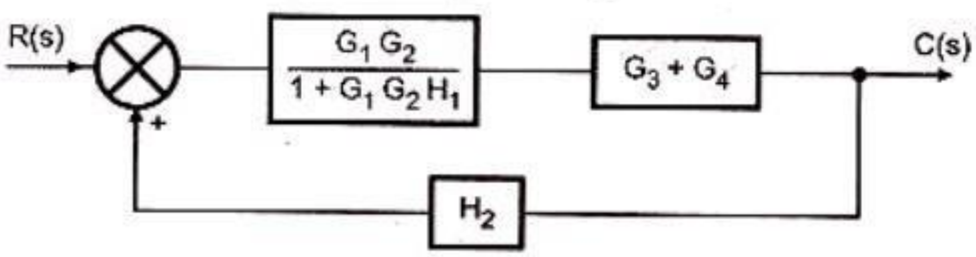
Combine G1, G2 which are in series

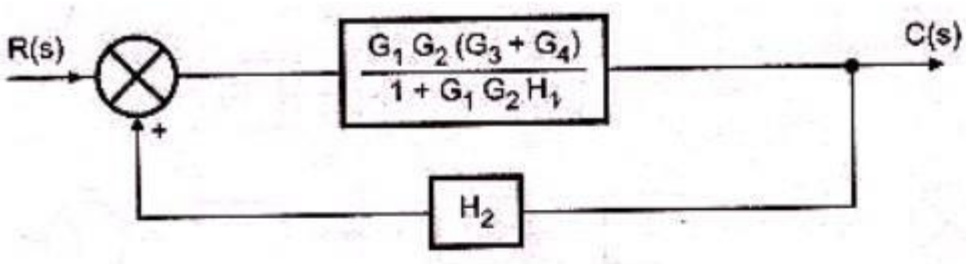


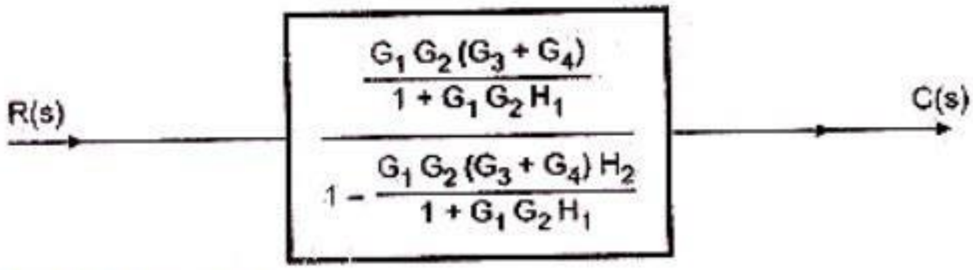
Combine G3, G4 which are in Parallel



Reduce minor feedback loop of G1, G2 and H1



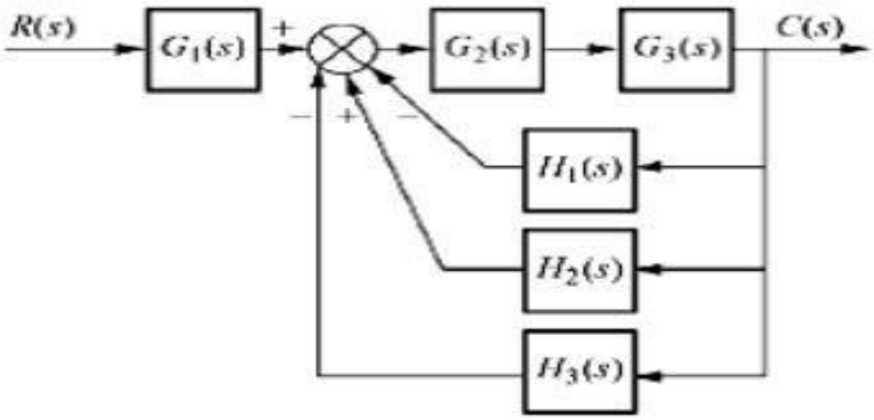




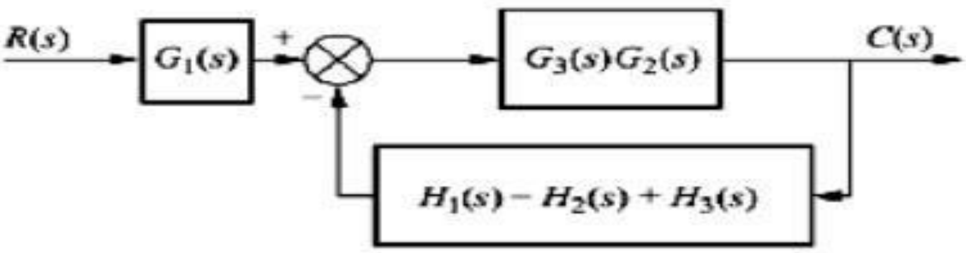
Transfer function

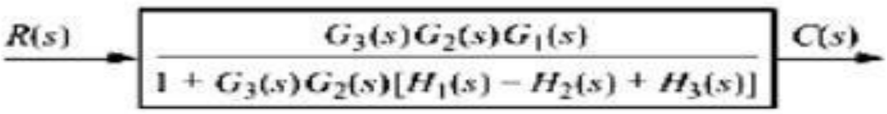
$$\frac{C(s)}{R(s)} = \frac{G_1\ G_2\,(G_3 + G_4)}{1 + G_1\,G_2\,H_1 - G_1\,G_2\,(G_3 + G_4)\,H_2}$$

2. Obtain the transfer function for the system shown in the fig

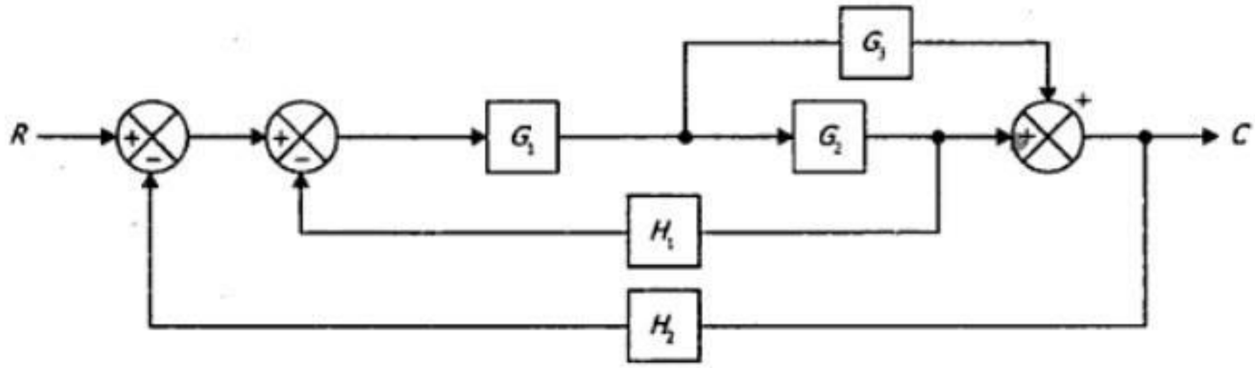


Solution





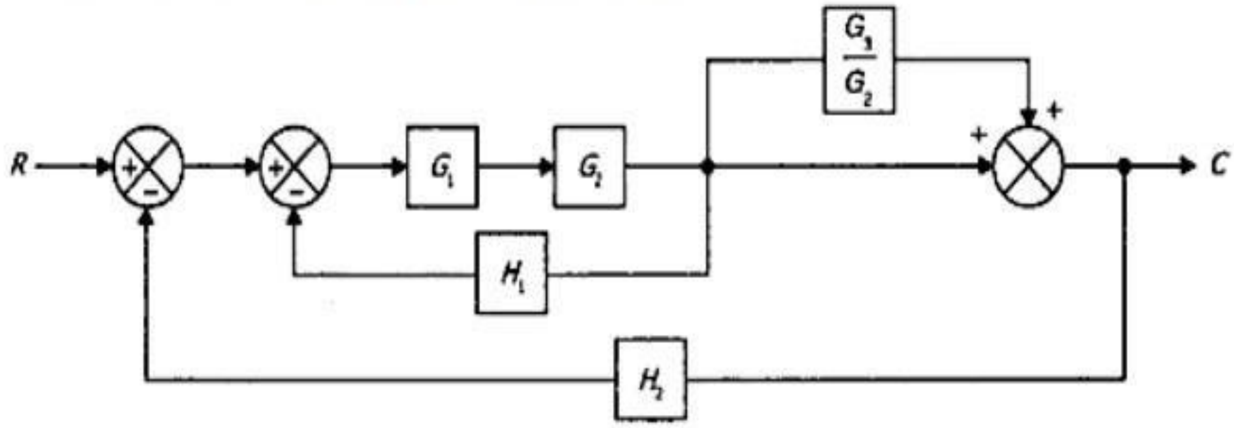
3. Obtain the transfer function C/R for the block diagram shown in the fig



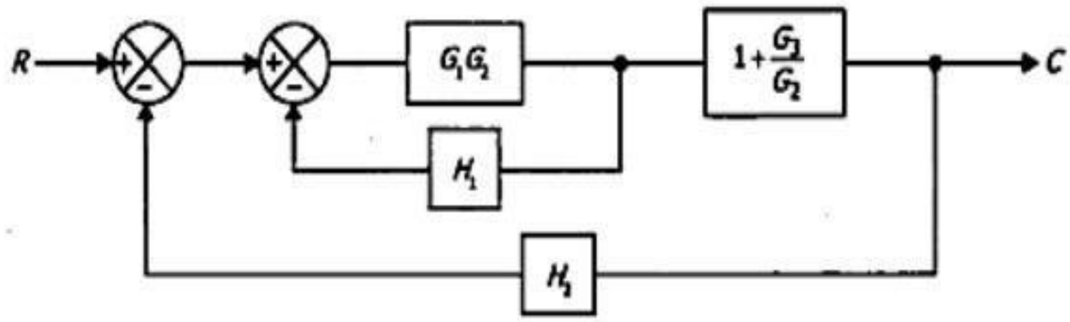
Solution

The take-off point is shifted after the block G2

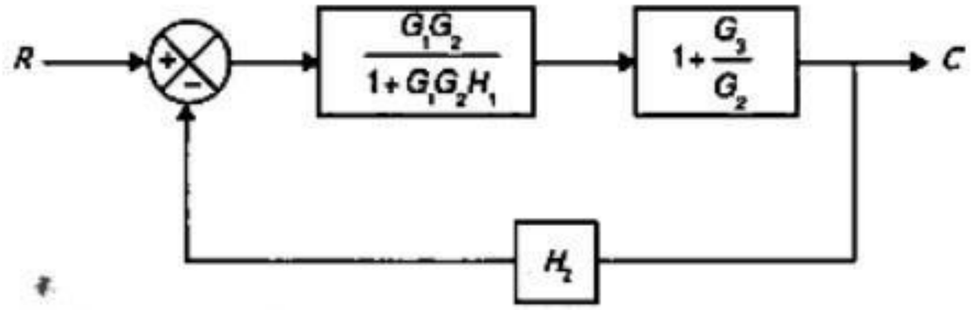
The take-off point is shifted after the block G2



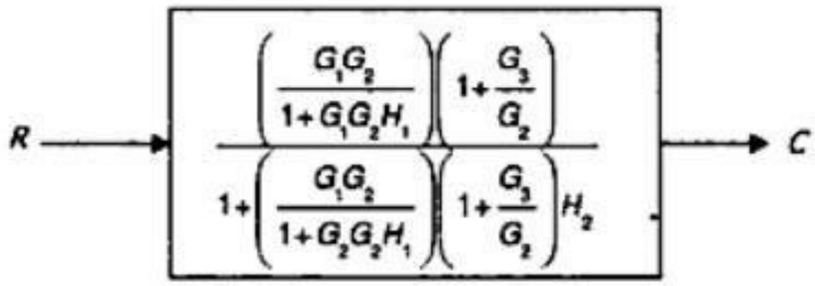
Reducing the cascade block and parallel block



Replacing the internal feedback loop



Equivalent block diagram

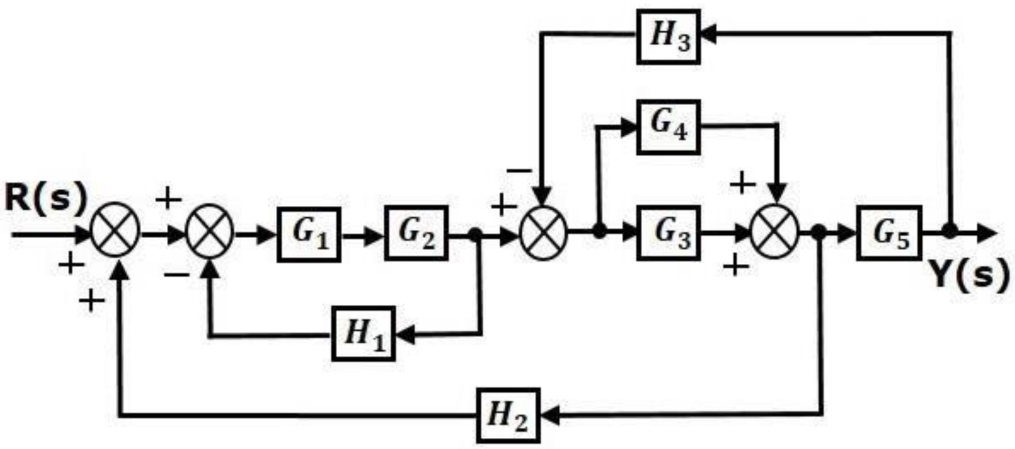


Transfer function

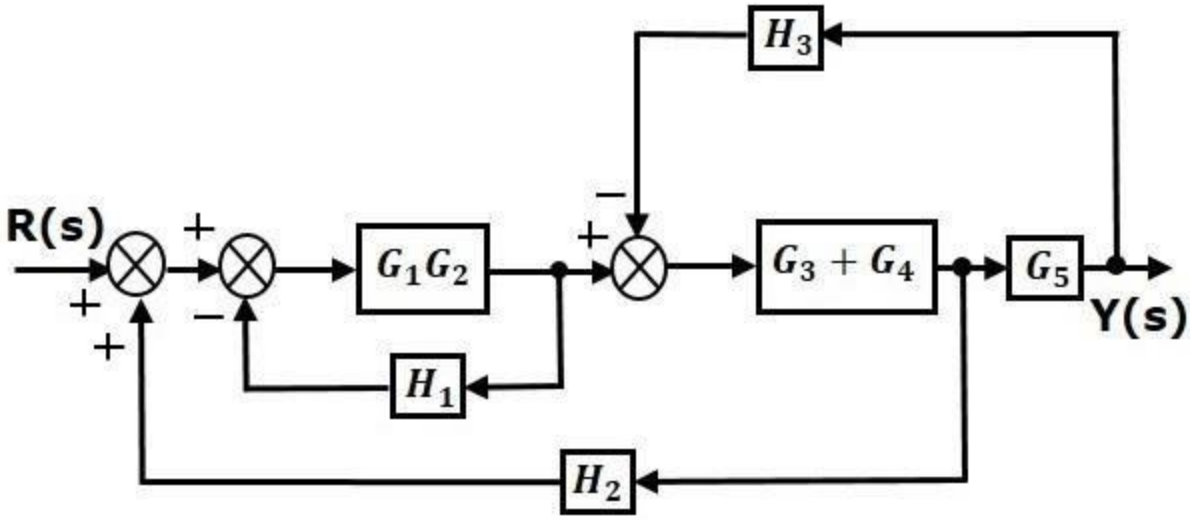
$$\frac{C}{R}=\frac{\dfrac{G_1(G_2+G_3)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2+G_3)H_2}{1+G_1G_2H_1}}$$

$$=\frac{G_1(G_2+G_3)}{1+G_1G_2(H_1+H_2)+G_1G_3H_2}$$

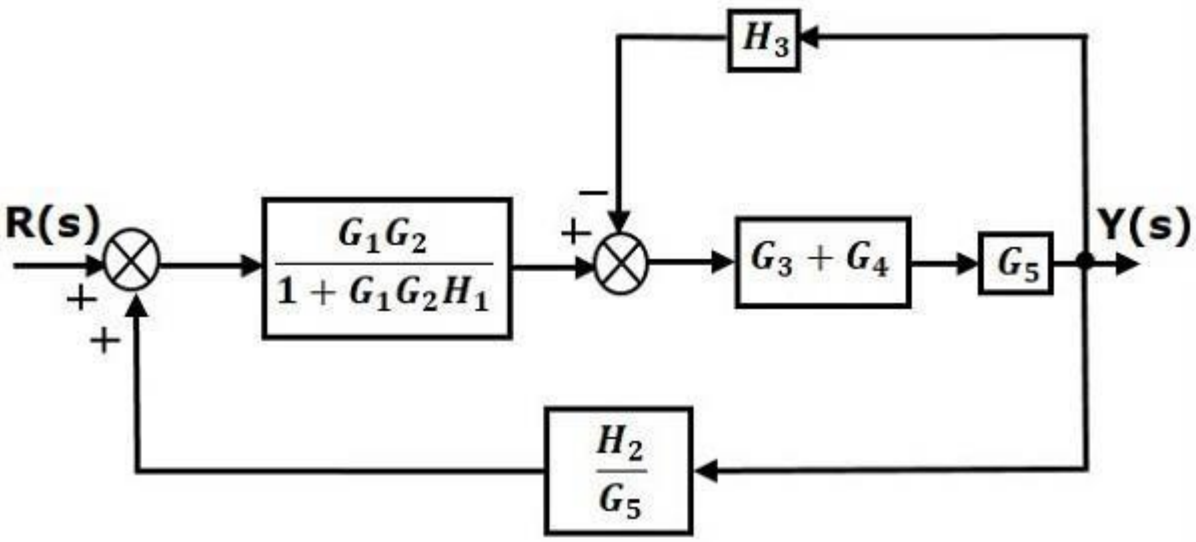
4. Obtain the transfer function C/R for the block diagram shown in the fig using the block diagram reduction rules.



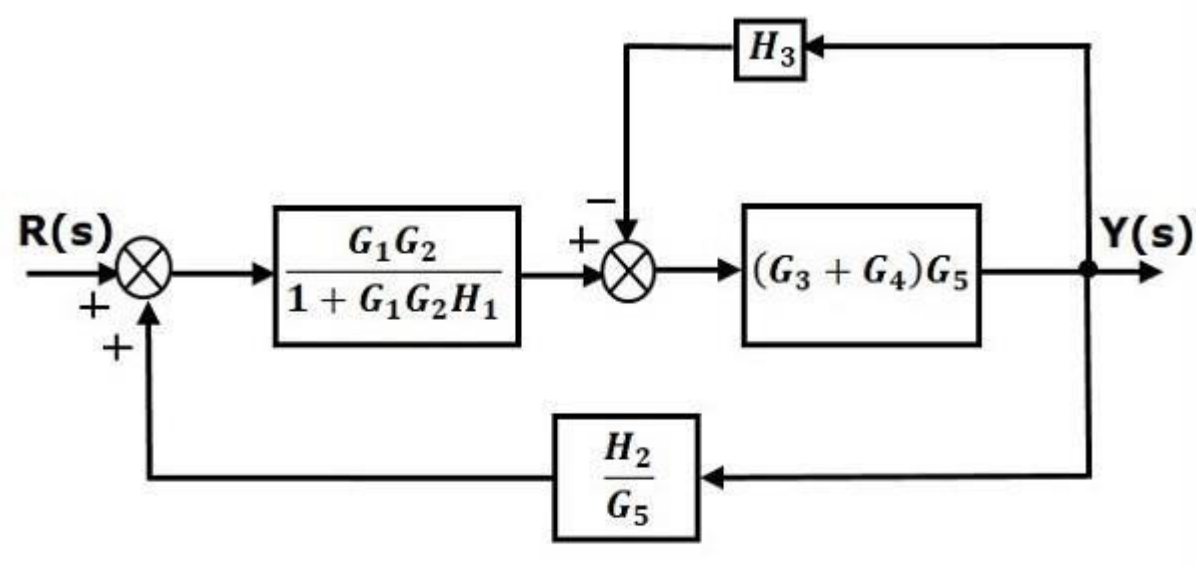
Step 1 – Use Rule 1 for blocks G_1 and G2. Use Rule 2 for blocks G3 and G4. The modified block diagram is shown in the following figure.



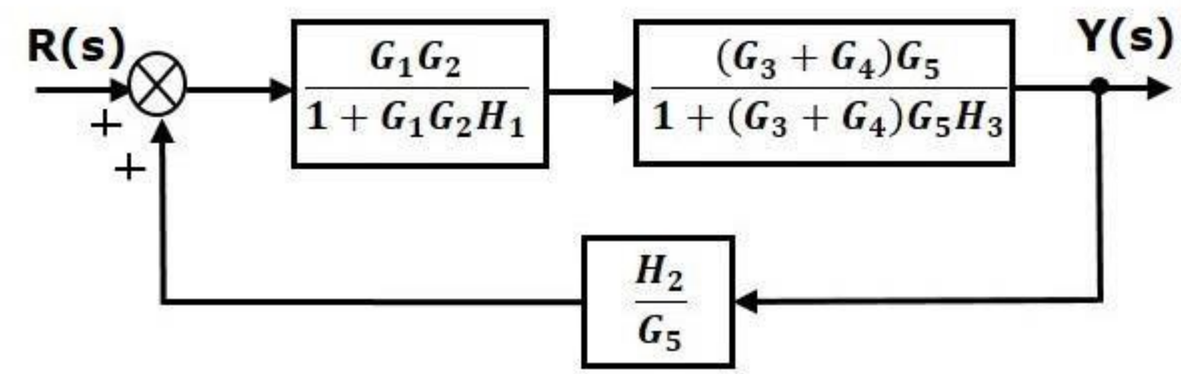
Step 2 – Use Rule 3 for blocks G_1G_2 and H_1. Use Rule 4 for shifting take-off point after the block G_5. The modified block diagram is shown in the following figure.



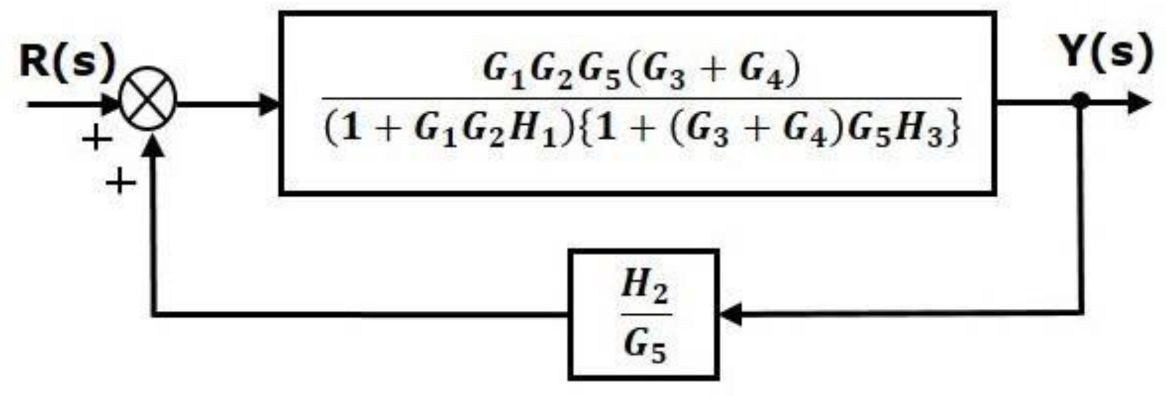
Step 3 – Use Rule 1 for blocks (G_3+G_4) and G_5. The modified block diagram is shown in the following figure.



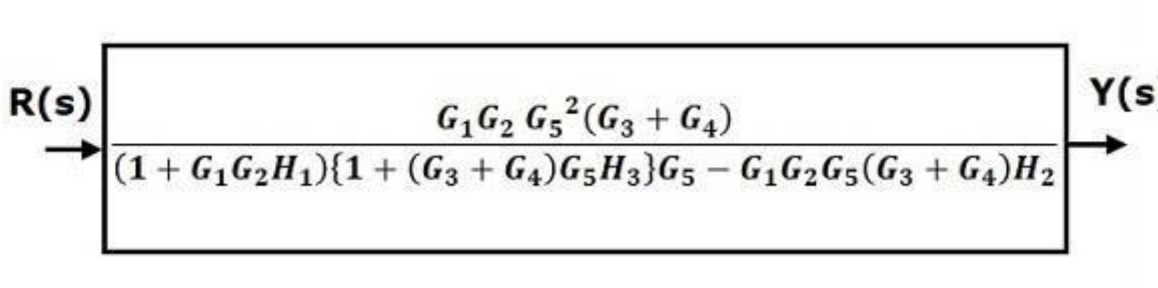
Step 4 – Use Rule 3 for blocks $(G_3+G_4)G_5$ and H_3. The modified block diagram is shown in the following figure.



Step 5 – Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



Step 6 – Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



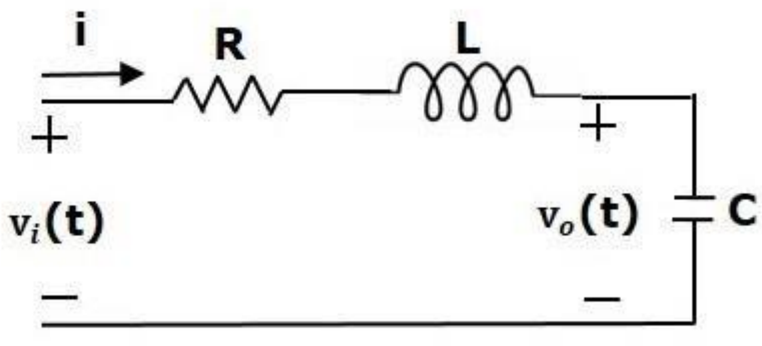
Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)(1+(G_3+G_4)G_5H_3)G_5-G_1G_2G_5(G_3+G_4)H_2}$$

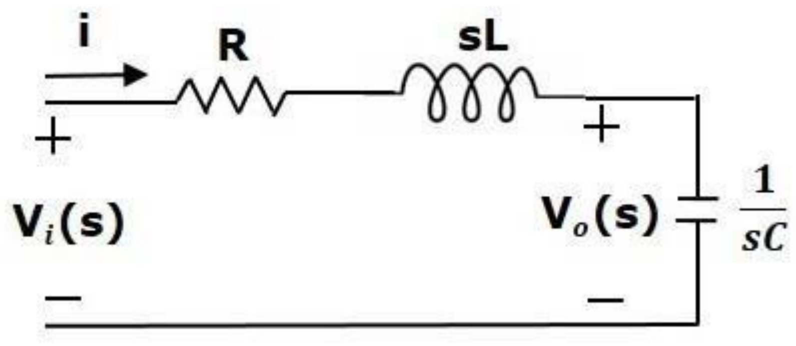
- **Block Diagram Representation of Electrical Systems**

In this section, let us represent an electrical system with a block diagram. Electrical systems contain mainly three basic elements — **resistor, inductor and capacitor**.

Consider a series of RLC circuit as shown in the following figure. Where, $V_i(t)$ and $V_o(t)$ are the input and output voltages. Let i(t) be the current passing through the circuit. This circuit is in time domain.



By applying the Laplace transform to this circuit, will get the circuit in s-domain. The circuit is as shown in the following figure.



From the above circuit, we can write

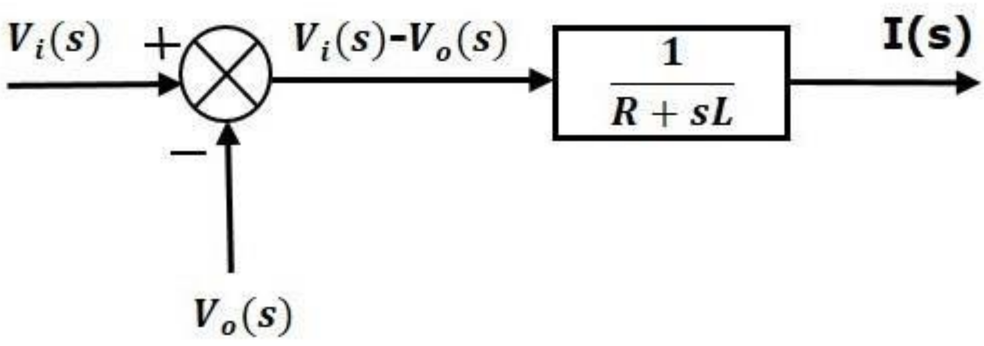
$$I(s) = \frac{V_i(s) - V_o(s)}{R + sL}$$

$$\Rightarrow I(s) = \left\{\frac{1}{R+sL}\right\}\{V_i(s) - V_o(s)\} \qquad \textbf{(Equation 1)}$$

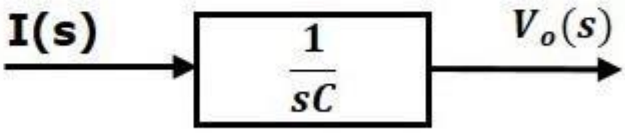
$$V_o(s) = \left(\frac{1}{sC}\right) I(s) \qquad \textbf{(Equation 2)}$$

Let us now draw the block diagrams for these two equations individually. And then combine those block diagrams properly in order to get the overall block diagram of series of RLC Circuit (s-domain).

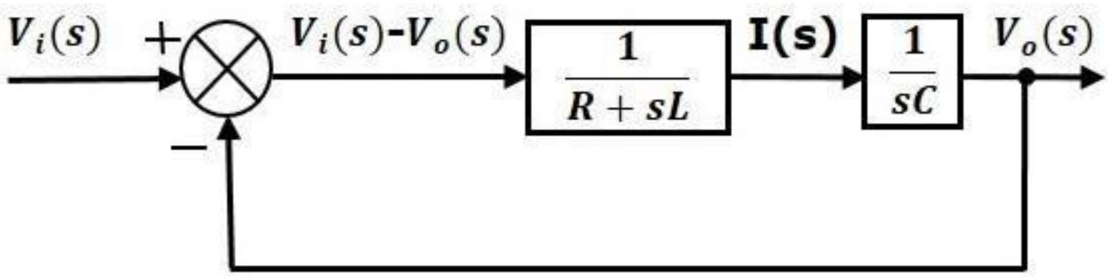
Equation 1 can be implemented with a block having the transfer function, $\frac{1}{R+SL}$ .The input and output of this block are $\{V_i(s)−V_o(s)\}$ and I(s). We require a summing point to get $\{V_i(s)−V_o(s)\}$. The block diagram of Equation 1 is shown in the following figure.



Equation 2 can be implemented with a block having transfer function, 1sC1sC. The input and output of this block are I(s)I(s) and Vo(s)Vo(s). The block diagram of Equation 2 is shown in the following figure.



The overall block diagram of the series of RLC Circuit (s-domain) is shown in the following figure.



Similarly, you can draw the **block diagram** of any electrical circuit or system just by following this simple procedure.

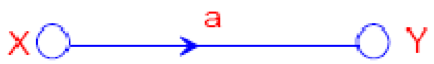
- Convert the time domain electrical circuit into an s-domain electrical circuit by applying Laplace transform.
- Write down the equations for the current passing through all series branch elements and voltage across all shunt branches.
- Draw the block diagrams for all the above equations individually.
- Combine all these block diagrams properly in order to get the overall block diagram of the electrical circuit (s-domain).

The block diagram reduction process takes more time for complicated systems. Because, we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation) i.e., how to represent signal flow graph from a given block diagram and calculation of transfer function just by using a gain formula without doing any reduction process.

- **SIGNAL FLOW GRAPHS**

For complex control systems, the block diagram reduction technique is cumbersome. An alternative method for determining the relationship between system variables has been developed by ***Mason*** and is based on a signal flow graph. A signal flow graph is a diagram that consists of nodes that are connected by branches. A node is assigned to each variable of interest in the system, and branches are used to relate the different variables. The main advantage for using SFG is that a straight forward procedure is available for finding the transfer function in which it is not necessary to move pickoff point around or to redraw the system several times as with block diagram manipulations.

SFG is a diagram that represents a set of simultaneous linear algebraic equations which describe a system. Let us consider an equation, y = a x. It may be represented graphically as,



Definitions:

Node: A node is a point representing a variable or signal.

Branch: A branch is a directed line segment joining two nodes.

Transmittance: It is the gain between two nodes.

Input node: A node that has only outgoing branche(s). It is also, called as source and corresponds to independent variable.

Output node: A node that has only incoming branches. This is also called as sink and corresponds to dependent variable.

Path: A path is a traversal of connected branches in the direction of branch arrow.

Loop: A loop is a closed path.

Self loop: It is a feedback loop consisting of single branch.

Loop gain: The loop gain is the product of branch transmittances of the loop.

Nontouching loops: Loops that do not posses a common node. Forward path: A path from source to sink without traversing an node more than once.

Feedback path: A path which originates and terminates at the same node. Forward path gain: Product of branch transmittances of a forward path.

Properties of Signal Flow Graphs:

1) Signal flow applies only to linear systems.

2) The equations based on which a signal flow graph is drawn must be algebraic equations in the form of effects as a function of causes. Nodes are used to represent variables. Normally the nodes are arranged left to right, following a succession of causes and effects through the system.

3) Signals travel along the branches only in the direction described by the arrows of the branches.

4) The branch directing from node Xk to Xj represents dependence of the variable Xj on Xk but not the reverse.

5) The signal traveling along the branch Xk and Xj is multiplied by branch gain akj and signal akjXk is delivered at node Xj.

Guidelines to Construct the Signal Flow Graphs: The signal flow graph of a system is constructed from its describing equations, or by direct reference to block diagram of the system. Each variable of the block diagram becomes a node and each block becomes a branch. The general procedure is

1) Arrange the input to output nodes from left to right.

2) Connect the nodes by appropriate branches.

3) If the desired output node has outgoing branches, add a dummy node and a unity gain branch.

4) Rearrange the nodes and/or loops in the graph to achieve pictorial clarity.

Algebra Addtion rule The value of the variable designated by a node is equal to the sum of all signals entering the node.

Transmission rule The value of the variable designated by a node is transmitted on every branch leaving the node.

Multiplication rule A cascaded connection of n-1 branches with transmission functions can be replaced by a single branch with new transmission function equal to the product of the old ones.

Masons Gain Formula The relationship between an input variable and an output variable of a signal flow graph is given by the net gain between input and output nodes and is known as overall gain of the system. Masons gain formula is used to obtain the over all gain (transfer function) of signal flow graphs.

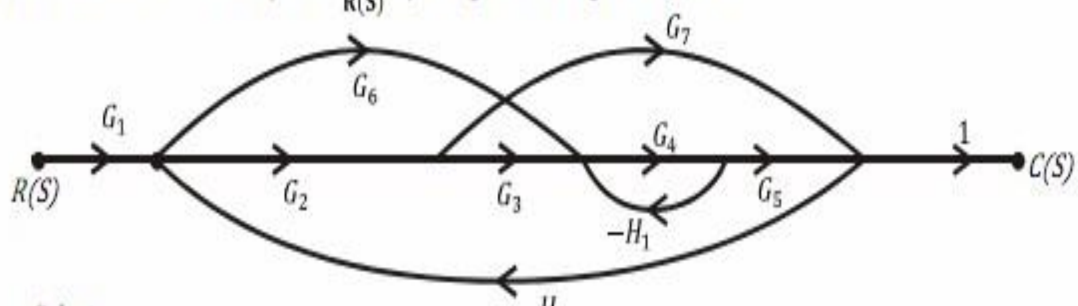
Gain P is given by $P = \frac{1}{\Delta} \sum_k P_k \Delta_k$

Where, P_k is gain of kth forward path, Δ is determinant of graph

Δ=1-(sum of all individual loop gains)+(sum of gain products of all possible combinations of two nontouching loops – sum of gain products of all possible combination of three nontouching loops) + ···

Δ_k is cofactor of kth forward path determinant of graph with loops touching kth forward path. It is obtained from Δ by removing the loops touching the path P_k.

(1) Obtain the closed loop TF, $\frac{C(S)}{R(S)}$ by using Mason's gain formula.



Solution:

Mason's gain formula is, $T = \frac{1}{\Delta} \sum P_K \Delta_K$

No. of forward paths:

$P_1 = G_1 G_2 G_3 G_4 G_5 \qquad \Delta_1 = 1$

$P_2 = G_1 G_6 G_4 G_5 \qquad \Delta_2 = 1$

$P_3 = G_1 G_2 G_7 \qquad \Delta_3 = 1 - \{-H_1 G_4\} = 1 + G_4 H_1$

No. of individual loops:

$L_{11} = -G_4 H_1$

$L_{21} = -G_2 G_3 G_4 G_5 H_2$

$L_{31} = -G_2 G_7 H_2$

$L_{41} = -G_6 G_4 G_5 H_2$

Gain Products of all possible combinations of two non-touching loops:

$L_{12} = (L_{11})(L_{31})$

$= H_1 H_2 G_2 G_4 G_7$

No. of three non-touching loops = 0.

$\Rightarrow \quad \Delta = 1 - \{L_{11} + L_{21} + L_{31} + L_{41}\} + L_{12}$

$$T = \frac{1}{1 - \{L_{11} + L_{21} + L_{31} + L_{41}\} + L_{12}} \times \{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3\}$$

$$\therefore \quad \boxed{T = \frac{G_1 G_2 G_3 G_4 G_5 + G_1 G_6 G_4 G_5 + G_1 G_2 G_7[1 + G_4 H_1]}{1 + G_4 H_1 + G_2 G_3 G_4 G_5 H_2 + G_2 G_7 H_2 + G_6 G_4 G_5 H_2 + H_1 H_2 G_2 G_4 G_7}}$$

UNIT-II

MATHEMATICAL MODELING OF PHYSICAL SYSTEMS

In this chapter, let us discuss the **differential equation modeling** of mechanical systems. There are two types of mechanical systems based on the type of motion.

- Translational mechanical systems
- Rotational mechanical systems

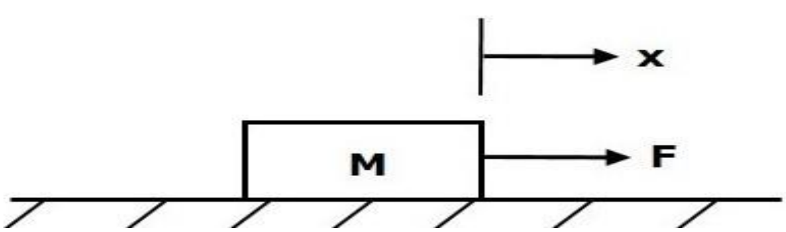
Modeling of Translational Mechanical Systems

Translational mechanical systems move along a **straight line**. These systems mainly consist of three basic elements. Those are mass, spring and dashpot or damper.

If a force is applied to a translational mechanical system, then it is opposed by opposing forces due to mass, elasticity and friction of the system. Since the applied force and the opposing forces are in opposite directions, the algebraic sum of the forces acting on the system is zero. Let us now see the force opposed by these three elements individually.

MASS:

Mass is the property of a body, which stores kinetic energy. If a force is applied on a body having mass M, then it is opposed by an opposing force due to mass. This opposing force is proportional to the acceleration of the body. Assume elasticity and friction are negligible.



$$F_m \propto a$$
$$\Rightarrow F_m = Ma = M\,\mathrm{d}^2x/\mathrm{d}t^2$$
$$F = F_m = M\,\mathrm{d}^2x/\mathrm{d}t^2$$

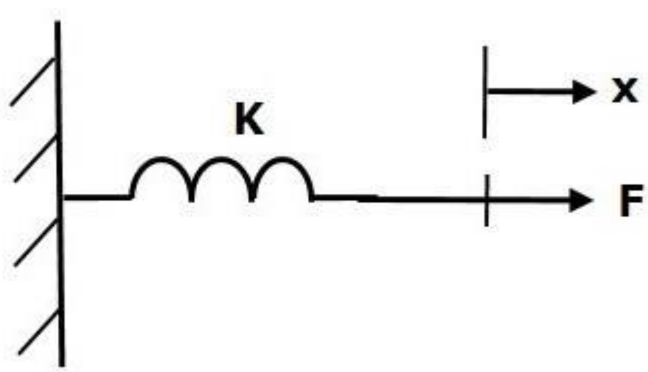
Where,

- F is the applied force
- Fm is the opposing force due to mass

- M is mass
- a is acceleration
- x is displacement

SPRING

Spring is an element, which stores potential energy. If a force is applied on spring K, then it is opposed by an opposing force due to elasticity of spring. This opposing force is proportional to the displacement of the spring. Assume mass and friction are negligible.



$$F \propto x$$

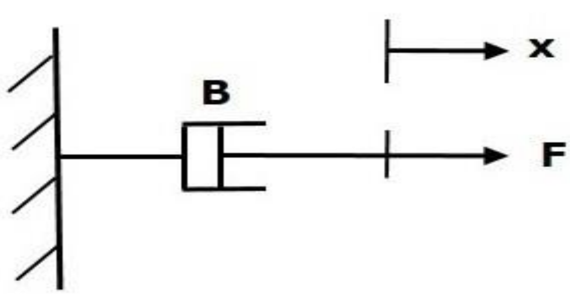
$$\Rightarrow F_k = Kx \quad F = F_k = Kx$$

Where,

- F is the applied force
- F_k is the opposing force due to elasticity of spring
- K is spring constant
- x is displacement

DASHPOT

If a force is applied on dashpot B, then it is opposed by an opposing force due to friction of the dashpot. This opposing force is proportional to the velocity of the body. Assume mass and elasticity are negligible



$$F_b \propto v$$

$$\Rightarrow F_b = Bv = B\,dx/dt$$
$$F = F_b = B\,dx/dt$$

Where,

- Fb is the opposing force due to friction of dashpot
- B is the frictional coefficient
- v is velocity
- x is displacement

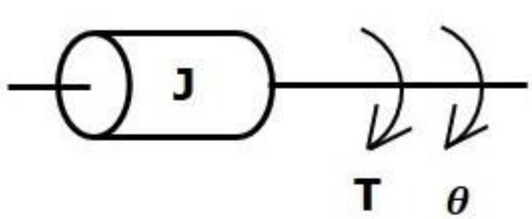
Modeling of Rotational Mechanical Systems

- Rotational mechanical systems move about a fixed axis. These systems mainly consist of three basic elements. Those are moment of inertia, torsional spring and dashpot.
- If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero. Let us now see the torque opposed by these three elements individually.

Moment of Inertia

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores kinetic energy.

If a torque is applied on a body having moment of inertia J, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible.



$$T_j \propto \alpha$$
$$\Rightarrow T_j = J\alpha = J\frac{d^2\theta}{dt^2}$$
$$T = T_j = J\frac{d^2\theta}{dt^2}$$

Where,

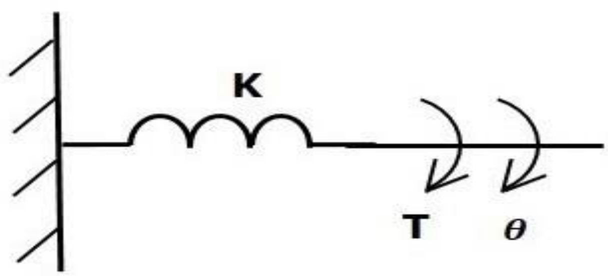
- T is the applied torque

- T_j is the opposing torque due to moment of inertia
- J is moment of inertia
- α is angular acceleration
- θ is angular displacement

TORSIONAL SPRING

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores potential energy.

If a torque is applied on torsional spring K, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.



$$T_k \propto \theta$$

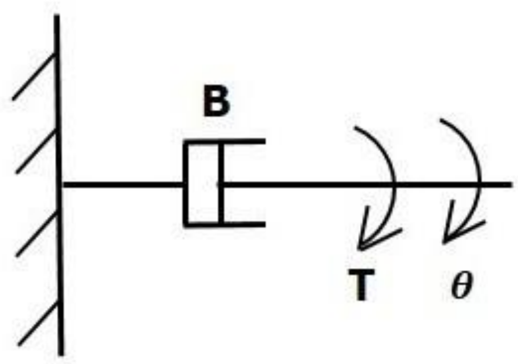
$$\Rightarrow T_k = K\theta \quad T = T_k = K\theta$$

Where,

- T is the applied torque
- T_k is the opposing torque due to elasticity of torsional spring
- K is the torsional spring constant
- θ is angular displacement

Dashpot

If a torque is applied on dashpot B, then it is opposed by an opposing torque due to the rotational friction of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.



$T_b \propto \omega$

$$\Rightarrow T_b = B\omega = B\frac{d\theta}{dt}$$

$$T = T_b = B\frac{d\theta}{dt}$$

Where,

- Tb is the opposing torque due to the rotational friction of the dashpot
- B is the rotational friction coefficient
- ω is the angular velocity
- θ is the angular displacement

Two systems are said to be analogous to each other if the following two conditions are satisfied.

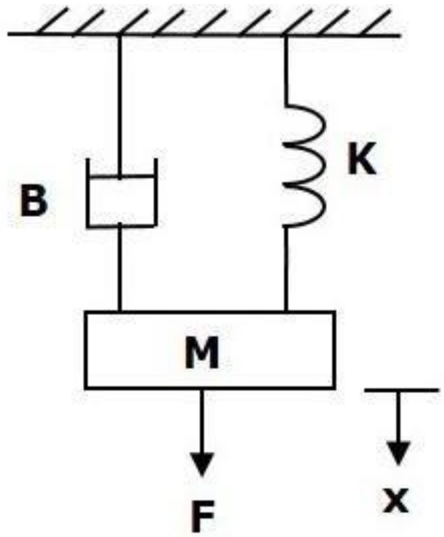
- The two systems are physically different
- Differential equation modelling of these two systems are same

Electrical systems and mechanical systems are two physically different systems. There are two types of electrical analogies of translational mechanical systems. Those are force voltage analogy and force current analogy.

Force Voltage Analogy

In force voltage analogy, the mathematical equations of translational mechanical system are compared with mesh equations of the electrical system.

Consider the following translational mechanical system as shown in the following figure.

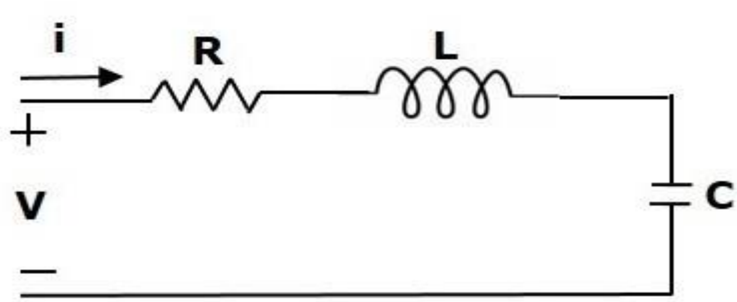


The **force balanced equation** for this system is

$$F=F_m+F_b+F_k$$

$$\Rightarrow F=M(d^2x/dt^2)+B(dx/dt)+K*x \qquad \textbf{(Equation 1)}$$

Consider the following electrical system as shown in the following figure. This circuit consists of a resistor, an inductor and a capacitor. All these electrical elements are connected in a series. The input voltage applied to this circuit is V volts and the current flowing through the circuit is i Amps.



Mesh equation for this circuit is

$$V=Ri+L\ didt+1_c\int idt \qquad \textbf{(Equation 2)}$$

Substitute, i=dq/dt in Equation 2.

$$V=R\ (dq/dt)+L(d^2q/dt^2)+qC$$

$$\Rightarrow V = L(d^2q/dt^2) + R(dq/dt) + (1_c)q \quad \textbf{(Equation 3)}$$

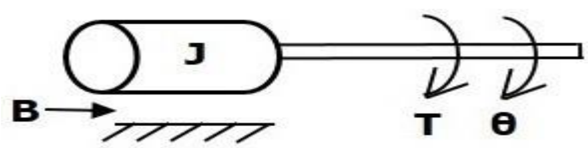
By comparing Equation 1 and Equation 3, we will get the analogous quantities of the translational mechanical system and electrical system. The following table shows these analogous quantities.

Translational Mechanical System	**Electrical System**
Force(F)	Voltage(V)
Mass(M)	Inductance(L)
Frictional Coefficient(B)	Resistance(R)
Spring Constant(K)	Reciprocal of Capacitance $(1/c)$
Displacement(x)	Charge(q)
Velocity(v)	Current(i)

Similarly, there is torque voltage analogy for rotational mechanical systems. Let us now discuss about this analogy.

Torque Voltage Analogy

In this analogy, the mathematical equations of rotational mechanical system are compared with mesh equations of the electrical system.
Rotational mechanical system is shown in the following figure.



The torque balanced equation is

$$T = Tj + Tb + Tk$$

$$\Rightarrow T = Jd^2\theta/dt^2 + Bd\theta/dt + k\theta \quad \textbf{(Equation 4)}$$

By comparing Equation 4 and Equation 3, we will get the analogous quantities of rotational mechanical system and electrical system. The following table shows these analogous quantities.

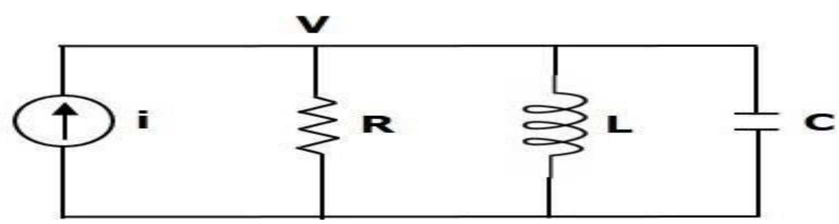
Rotational Mechanical System	**Electrical System**
Torque(T)	Voltage(V)
Moment of Inertia(J)	Inductance(L)
Rotational friction coefficient(B)	Resistance(R)

Torsional spring constant(K)	Reciprocal of Capacitance $(\frac{1}{c})$
Angular Displacement(θ)	Charge(q)
Angular Velocity(ω)	Current(i)

Force Current Analogy

In force current analogy, the mathematical equations of the translational mechanical system are compared with the nodal equations of the electrical system.

Consider the following electrical system as shown in the following figure. This circuit consists of current source, resistor, inductor and capacitor. All these electrical elements are connected in parallel.



The nodal equation is $i=V*R+(1/L)\int Vdt$+C*dV/dt

(Equation 5)

Substitute, V=dΨ/dt in Equation 5.

$$i=(1/R)(d\Psi/dt)+(1/L)\Psi+C(d^2\Psi/dt^2)$$

$\Rightarrow i=C(d^2\Psi/dt^2)+(1/R)(d\Psi/dt)+(1/L)\Psi$ **(Equation 6)**

By comparing Equation 1 and Equation 6, we will get the analogous quantities of the translational mechanical system and electrical system. The following table shows these analogous quantities.

Translational Mechanical System	**Electrical System**
Force(F)	Current(i)
Mass(M)	Capacitance(C)
Frictional coefficient(B)	Reciprocal of Resistance$(\frac{1}{R})$
Spring constant(K)	Reciprocal of Inductance$(\frac{1}{L})$
Displacement(x)	Magnetic Flux(ψ)
Velocity(v)	Voltage(V)

Similarly, there is a torque current analogy for rotational mechanical systems. Let us now discuss this analogy.

Torque Current Analogy

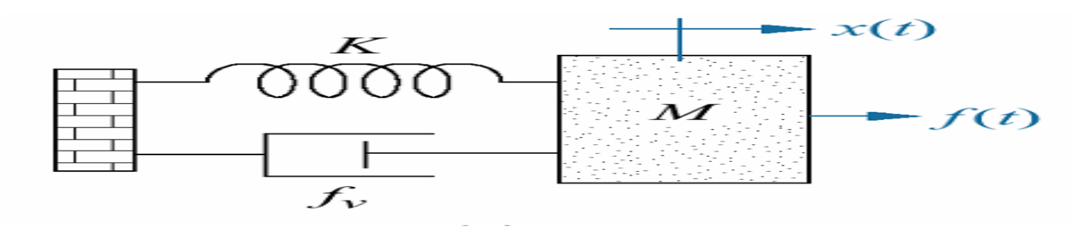
In this analogy, the mathematical equations of the rotational mechanical system are compared with the nodal mesh equations of the electrical system.

By comparing Equation 4 and Equation 6, we will get the analogous quantities of rotational mechanical system and electrical system. The following table shows these analogous quantities.

Rotational Mechanical System	**Electrical System**
Torque(T)	Current(i)
Moment of inertia(J)	Capacitance(C)
Rotational friction coefficient(B)	Reciprocal of Resistance$(1R)$
Torsional spring constant(K)	Reciprocal of Inductance$(1L)$
Angular displacement(θ)	Magnetic flux(ψ)
Angular velocity(ω)	Voltage(V)

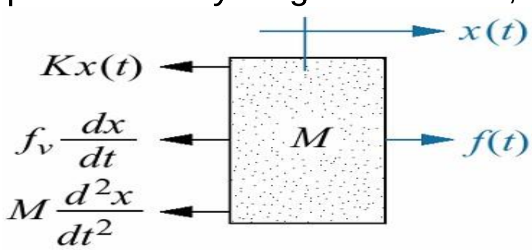
Example:

Find the transfer function X(s) / F(s) for the system given below

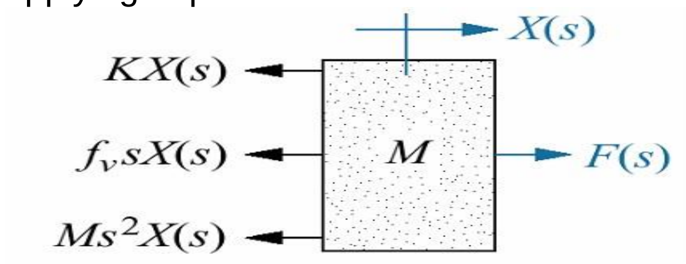


Solution:

Step1: Free-body diagram of mass, spring, and damper system



Applying Laplace transform



We now write the differential equation of motion using Newton's law

$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t)$$

Taking the Laplace transform, assuming zero initial conditions

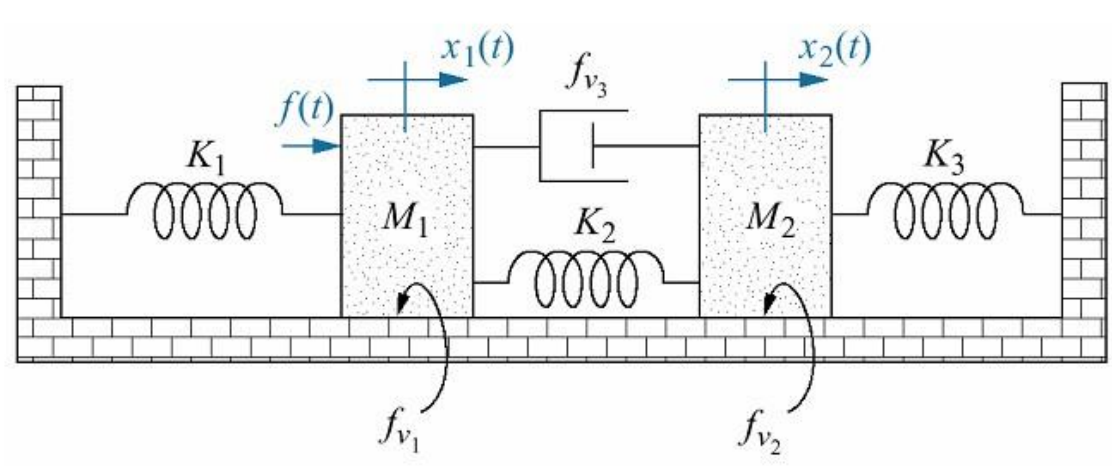
$$Ms^2X(s)+f_vsX(s)+KX(s)=F(s)$$

$$(Ms^2+f_vs+K)X(s)=F(s)$$

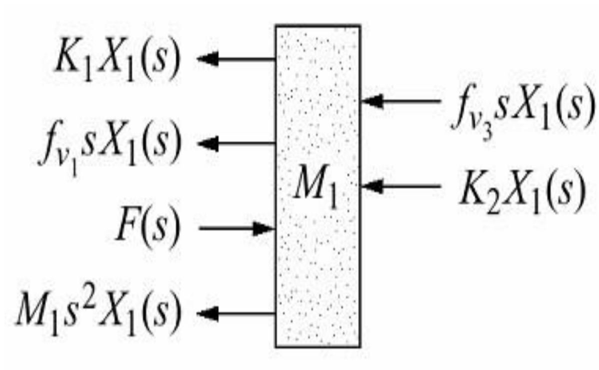
Solving for transfer function yields

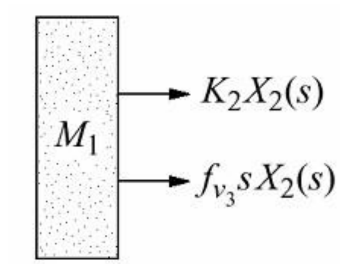
$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+f_v+K}$$

2. Find the transfer function X(s)/F(s)

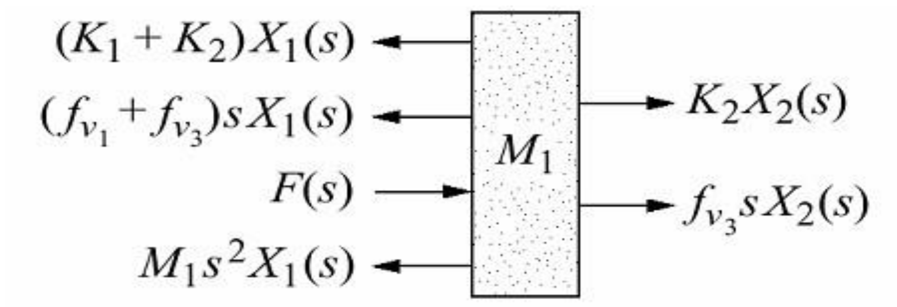


Forces on M1 due only to motion of M





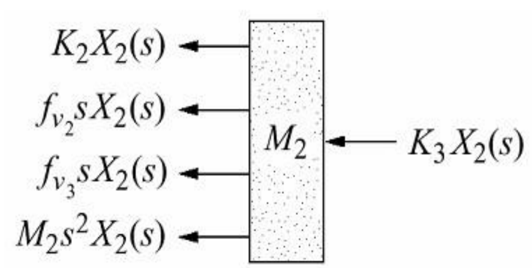
Final force on mass1

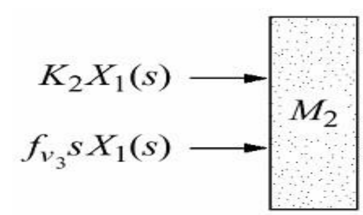


We now write the differential equation of motion using Newton's law

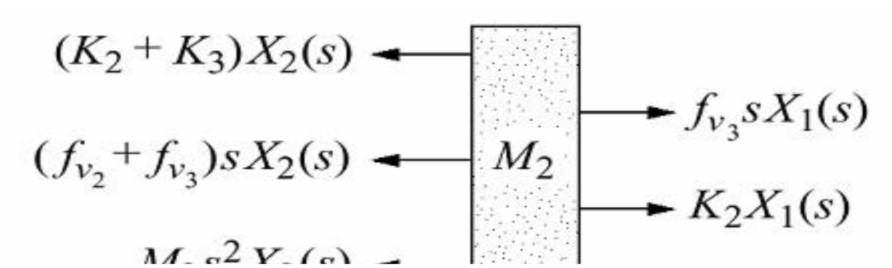
$$[M_1s^2 + (f_{v1}+f_{v3})s + (K_1+K_2)]\, X_1(s) - (f_{v3}s+K_2)X_2(s) = F(s)$$

Forces on M2





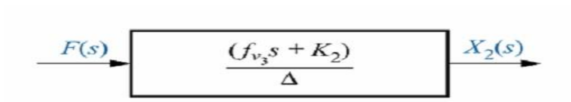
Total force on M2



$$-(f_{v3}s + K_2)X_1(s) + [M_2s^2 + (f_{v2}+f_{v3})s + (K_2 + K_3)]X_2(s) = 0$$

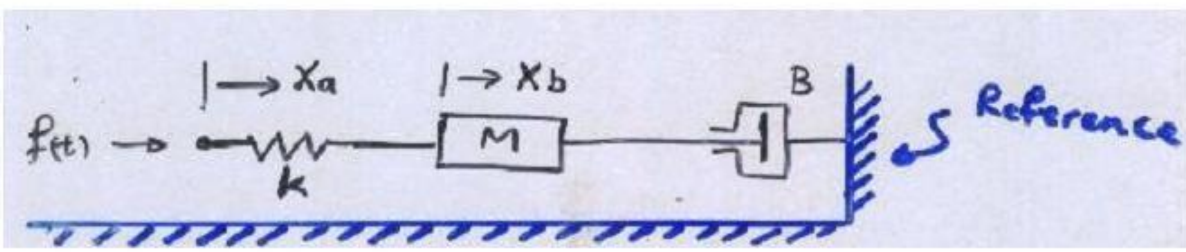
By solving above 2 equations

The transfer function $X_2(s)/F(s)$ is

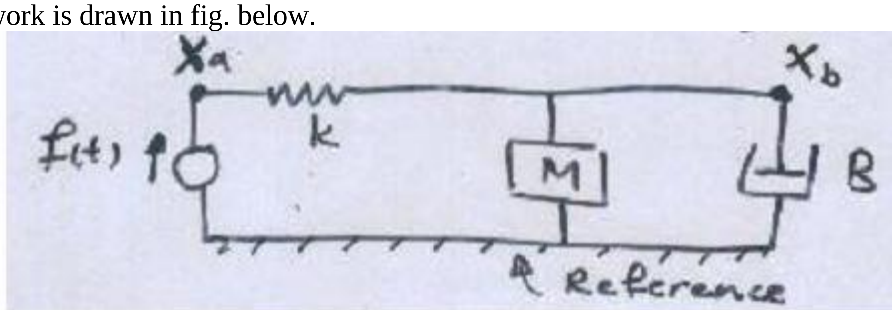


$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v1}+f_{v3})s + (K_1+K_2)] & -(f_{v3}s+K_2) \\ -(f_{v3}s+K_2) & [M_2 s^2 + (f_{v2}+f_{v3})s + (K_2+K_3)] \end{vmatrix}$$

3. Find the T.F. of simple mass-spring-damper mechanical system



To draw the mechanical network, the points xa, xband the reference are located. The complete mechanical network is drawn in fig. below.



For nodes a & b

$$f = f_k = k(x_a - x_b) \qquad (1)$$

$$f_k = f_m + f_B = MD^2 x_b + BDx_b \qquad (2) \text{ where } D=d/dt$$

Automobile suspension some system of one wheel

M_1= mass of the automobile

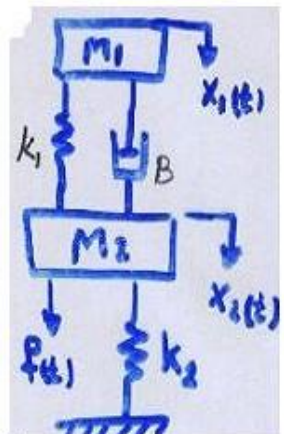
B = the shock absorber

k_1 =the spring

k_2= elestance of the tire

M_2= mass of the wheel

Two independent displacement exist, so we must write Two equations



$$M_1 \frac{d^2x_1}{dt^2} = -B(\frac{dx_1}{dt} - \frac{dx_2}{dt}) - k_1(x_1 - x_2)$$

$$M_1 \frac{d^2x_2}{dt^2} = f(t) - B(\frac{dx_2}{dt} - \frac{dx_1}{dt}) - k_1(x_2 - x_1) - k_2x_2$$

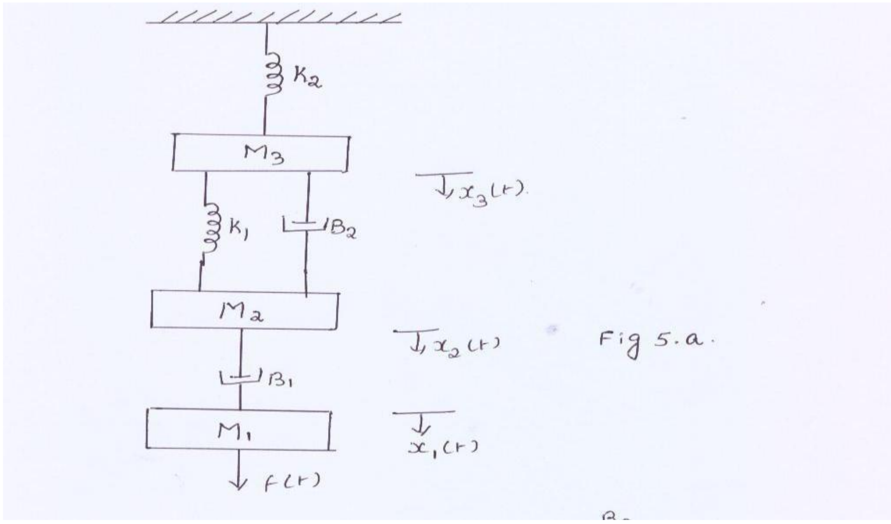
$M_1s^2X_1(s) + B(sX_1(s)-sX_2(s))+k_1(X_1(s)-X_2(s)) =0$

$M_2s^2X_2(s) + B(sX_2(s)-sX_1(s))+k_1(X_2(s)-X_1(s))+k_2X_2(s) =F(s)$

$$T(s) = \frac{X_1(s)}{F(s)} = \frac{B_s + k_1}{M_1M_2s^4 + B(M_1+M_2)s^3 + (K_1M_2 + k_2M_1)s^2 + k_2Bs + k_1k_2}$$

$f(s) \quad Ms^2 + Bs$

5.Obtain the analogous electrical network for the system shown in fig.5.(AU:Nov./Dec.-2007)



The Mass M1 is under the displacement x1(t).

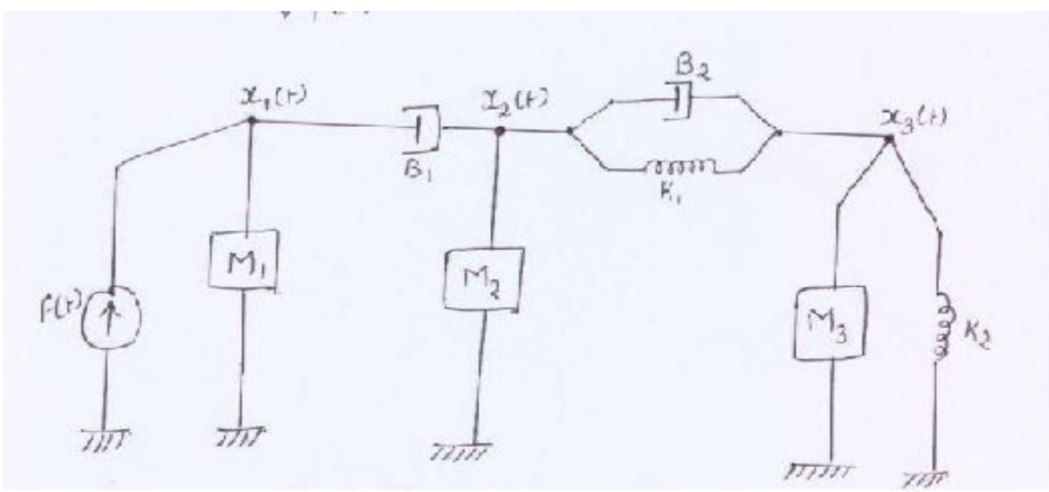
The friction B1 is responsible to change the displacement from x1(t) to

x2(t) The Mass M2 is under the displacement x2(t).

The friction B2 and spring K1 are responsible to change the displacement from x2(t) to x3(t)

The Mass M3 and spring K2 are under the influence of displacement

x3(t). The equivalent Mechanical system is shown in fig.5.a.



The equilibrium equations are

F(t) = M1(d2x1(t)/ dt2) + B1d(x1(t)- x2(t))/dt -- (1)

$0 = B1d(x2(t) - x1(t))/dt + M2(d2x2(t)/dt2) + K1(x2(t) - x3(t)) + B2d(x2(t) - x3(t))/dt$ -------- (2)

$0 = K1(x3(t) - x2(t)) + B2d(x3(t) - x2(t))/dt + M3(d2x3(t)/dt2) + K3x3(t)$ --------------------- (3)

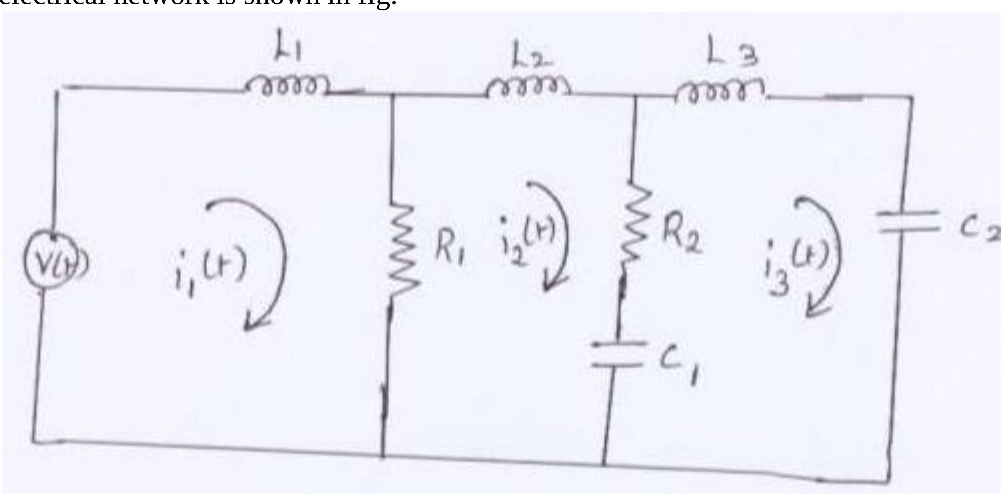
Using force- voltage analogy ,

Mass is replaced by inductance, friction or dashpot is replaced by resistance, spring is replaced by reciprocal of capacitance, displacement is replaced by charge. Rate of change of displacement is replaced by current, force is replaced by voltage.

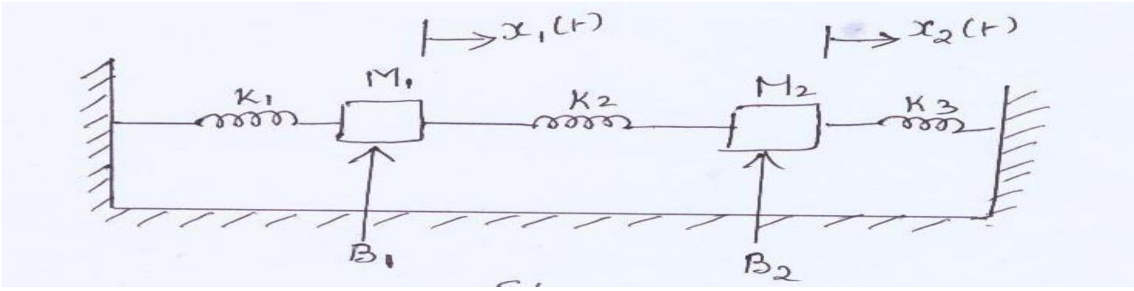
$V(t) = L_1 di_1(t)/dt + R_1 (i_1(t) - i_2(t))$ ------------------ (4)

$0 = R_1 (i_2(t) - i_1(t)) + L_2 di_2(t)/dt + 1/C_1 \int (i_2(t) - i_3(t))dt + R_2 (i_2(t) - i_3(t))$------- (5)

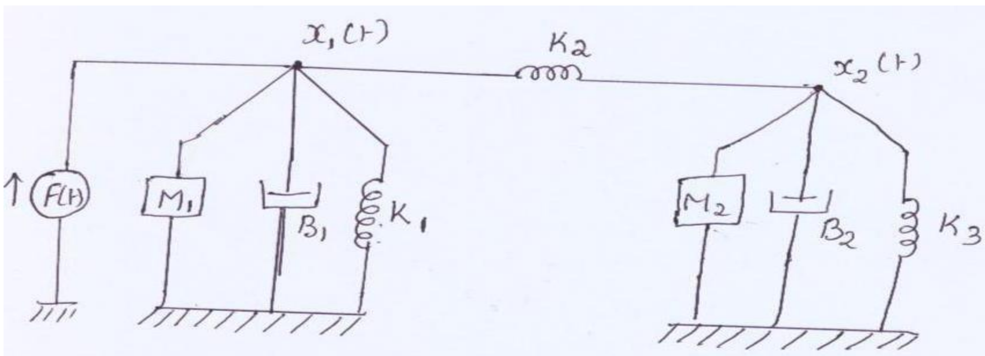
$0 = 1/C_1 \int (i_3(t) - i_2(t))\, dt + R_2 (i_3(t) - i_2 +(t)) + L_3 di_3(t)/dt + 1/C_2 \int (i_3(t)dt.$------- (6) The analogous electrical network is shown in fig.



6.Draw the equivalent mechanical system of the system shown in fig.. write the set of equilibrium equations for it and obtain electrical analogous circuits using i) F-V analogy ii)F-I analogy.



As shown in fig.6. M_1,K_1,and B_1 are under the displacement x_1 as K_1 and B_1 are with respect to rigid support. K_2 is between x_1 and x_2 as it is responsible for the change in displacement. While M_2, K_3 and B_2 are under the displacement x_2. Hence the equivalent mechanical system is as shown in fig



The equilibrium equations are

$$F(t) = M_1(d^2 x_1(t)/ dt^2) + B_1dx_1(t)/dt + K_1x_1(t) + K_2(x_1(t)- x_2(t)) \text{ ------(1)}$$

$$0 = M_2(d^2 x_2(t)/ dt^2) + B_2dx_2(t)/ dt + K_2(x_2(t)- x_1(t)) + K_3x_2(t) \text{ ------(2)}$$

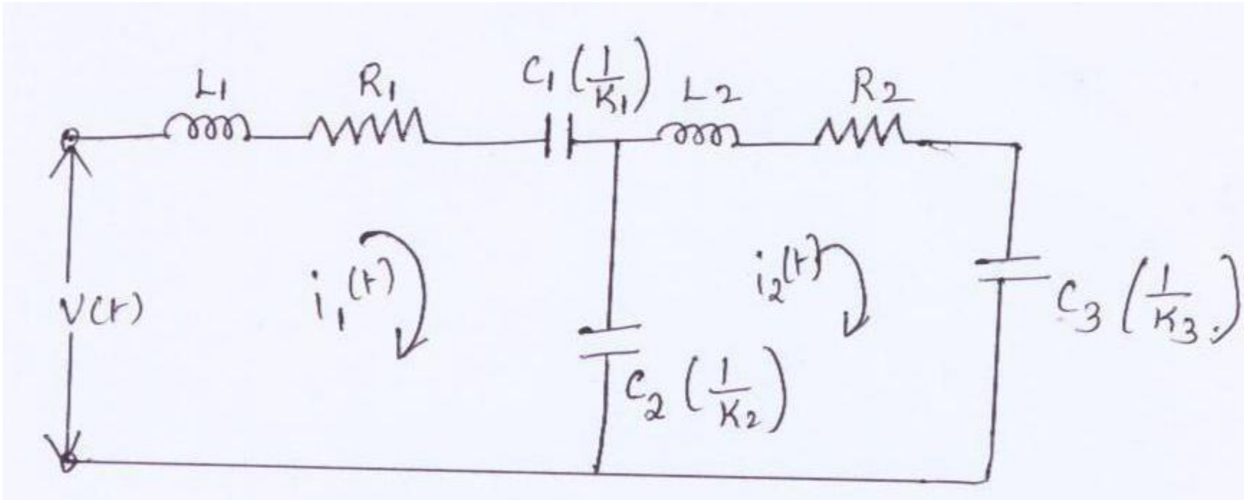
Using force- voltage analogy ,

Mass is replaced by inductance, friction or dashpot is replaced by resistance, spring is replaced by reciprocal of capacitance, displacement is replaced by charge. Rate of change of displacement is replaced by current, force is replaced by voltage.

$$V(t) = L_1di_1(t)/dt + R_1 i_1(t) + 1/C_1\int(i_1(t) dt + 1/C_2\int(i_1(t) – i_2(t)) dt \text{ ----------(3)}$$

$$0 = L_2di_2(t)/dt + R_1 i_2(t) +1/C_2\int(i_2(t) – i_1(t))dt + 1/C_3\int(i_2(t) dt\text{---4)}$$

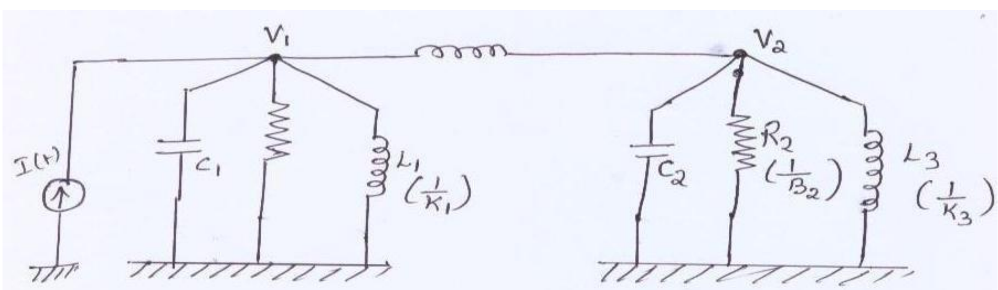
The analogous system for force voltage analogy is shown in fig.6.c.



Using force- current analogy ,

Mass is replaced by capacitance, friction or dashpot is replaced by reciprocal of resistance, spring is replaced by reciprocal of inductance, displacement is replaced by flux. Rate of change of displacement is replaced by voltage, force is replaced by current.

The analogous system for force current analogy is shown in fig

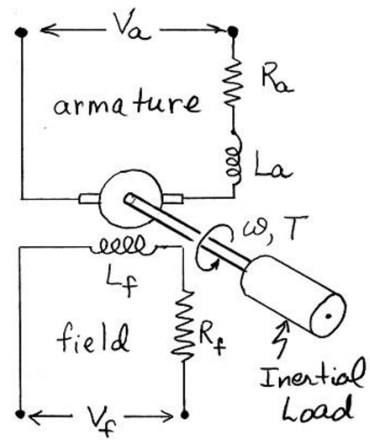


$I(t) = C_1 dV_1(t)/dt + 1/R_1\, V_1(t) + 1/L_1 \int (V_1(t)\, dt + 1/L_2 \int (V_1(t) - V_2(t))\, dt$.. (5)

$0 = C_2 dV_2(t)/dt + 1/R_2\, V_2(t) + 1/L_2 \int (V_2(t) - V_1(t))dt + 1/L_3 \int (V_2(t)\, dt$------6)

Armature Controlled DC Motor Transfer Functions

In a armature-current controlled DC motor, the field current i_f is held constant, and the armature current is controlled through the armature voltage V_a. In this case, the ***motor torque increases linearly with the armature current***.



DC Motor

$$\boxed{T_m = K_{ma}\, i_a} \tag{1}$$

K_{ma} is a ***constant*** that depends on the chosen motor. The ***transfer function*** from the input armature current to the resulting motor torque is

$$\boxed{\frac{T_m}{I_a}(s) = K_{ma}} \tag{2}$$

The ***voltage/current relationship*** for the armature side of the motor is

$$\boxed{V_a = V_R + V_L + V_b = R_a i_a + L_a\left(di_a/dt\right) + V_b} \tag{3}$$

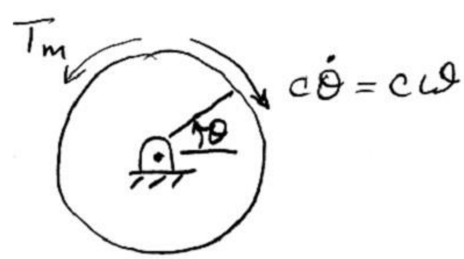
V_b represents the "***back EMF***" ***induced by the rotation of the armature windings in a magnetic field***. V_b is proportional to the rotational speed ω, i.e. $\boxed{V_b(s) = K_b\omega(s)}$.

Taking Laplace transforms of Eq. (3) gives

$$\boxed{V_a(s) - V_b(s) = \left(R_a + L_a s\right) I_a(s)} \quad \text{or} \quad \boxed{V_a(s) - K_b\omega(s) = \left(R_a + L_a s\right) I_a(s)} \tag{4}$$

Applying ***Newton's Law*** (by summing moments) for the rotational motion of the motor gives

$$\boxed{\sum M = T_m - c\omega = J\dot{\omega}} \quad \text{(CCW positive)}$$



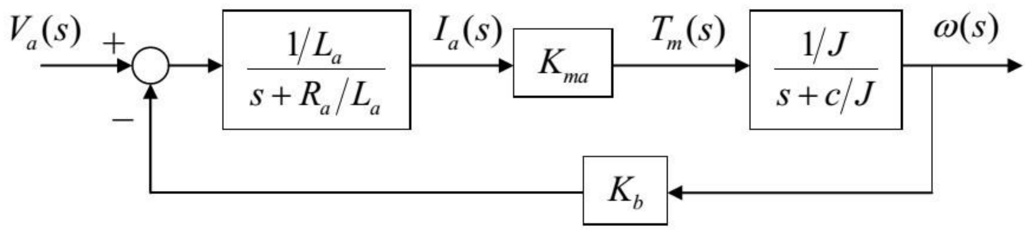
Free Body Diagram of the Inertial Load

$$\boxed{J\dot{\omega} + c\omega = T_m} \tag{5}$$

Thus, the ***transfer function*** from the input motor torque to rotational speed changes is

$$\boxed{\frac{\omega}{T_m}(s)=\frac{(1/J)}{s+(c/J)}} \qquad (1^{st} \text{ order system}) \qquad (6)$$

Together, Eqs. (2), (4) and (6) can be represented by the ***closed loop block diagram***:



Block diagram reduction gives the transfer function from the input armature voltage to the resulting speed change.

$$\boxed{\frac{\omega}{V_a}(s)=\frac{(K_{ma}/L_aJ)}{(s+R_a/L_a)(s+c/J)+(K_bK_{ma}/L_aJ)}} \qquad (2^{nd} \text{ order system}) \qquad (7)$$

If we assume the ***time constant of the electrical circuit is small*** compared to the time constant of the load dynamics, the transfer function of Eq. (7) may be reduced to a first order transfer function

$$\boxed{\frac{\omega}{V_a}(s)=\frac{K_{ma}/R_aJ}{s+(cR_a+K_bK_{ma})/R_aJ}} \qquad (1^{st} \text{ order system}) \qquad (8)$$

The transfer function from the input armature voltage to the resulting ***angular position*** change is found by multiplying Eqs. (7) and (8) by $1/s$.

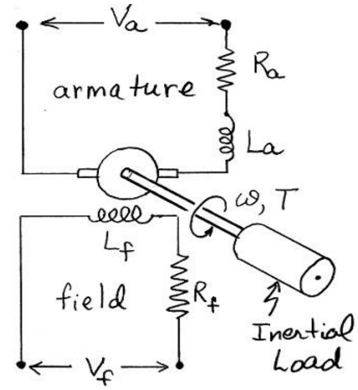
$$\boxed{\frac{\theta}{V_a}(s)=\frac{K_{ma}/R_aJ}{s(s+(cR_a+K_bK_{ma})/R_aJ)}} \qquad (2^{nd} \text{ order system}) \qquad (9)$$

Note that this transfer function also represents a second order differential equation with inertia and damping, but no stiffness (same form as for a hydraulic cylinder!).

Field Controlled DC Motor

The figure at the right represents a DC motor attached to an inertial load. The voltages applied to the field and armature sides of the motor are represented by V_f and V_a. The resistances and inductances of the field and armature sides of the motor are represented by R_f, L_f, R_a, and L_a. The torque generated by the motor is proportional to i_f and i_a the currents in the field and armature sides of the motor.

$$\boxed{T_m = K\, i_f\, i_a} \tag{1.1}$$



DC Motor

Field-Current Controlled:

In a field-current controlled motor, the armature current i_a is held constant, and the field current is controlled through the field voltage V_f. In this case, the motor torque increases linearly with the field current. We write

$$T_m = K_{mf}\, i_f$$

By taking Laplace transforms of both sides of this equation gives the transfer function from the input current to the resulting torque.

$$\boxed{\frac{T_m(s)}{I_f(s)} = K_{mf}} \tag{1.2}$$

For the field side of the motor the voltage/current relationship is

$$\begin{aligned} V_f &= V_R + V_L \\ &= R_f i_f + L_f\left(di_f/dt\right) \end{aligned}$$

The transfer function from the input voltage to the resulting current is found by taking Laplace transforms of both sides of this equation.

$$\boxed{\frac{I_f(s)}{V_f(s)} = \frac{\left(1/L_f\right)}{s + \left(R_f/L_f\right)}} \qquad \text{(1}^{\text{st}}\text{ order system)} \qquad (1.3)$$

The transfer function from the input voltage to the resulting motor torque is found by combining equations (1.2) and (1.3).

$$\boxed{\frac{T_m(s)}{V_f(s)} = \frac{T_m(s)}{I_f(s)}\frac{I_f(s)}{V_f(s)} = \frac{\left(K_{mf}/L_f\right)}{s + \left(R_f/L_f\right)}} \qquad \text{(1}^{\text{st}}\text{ order system)} \qquad (1.4)$$

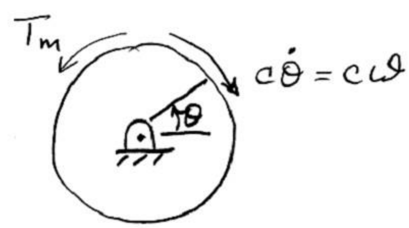
So, a step input in field voltage results in an exponential rise in the motor torque.

An equation that describes the rotational motion of the inertial load is found by summing moments

$$\sum M = T_m - c\omega = J\dot{\omega} \qquad \text{(counterclockwise positive)}$$

or

$$\boxed{J\dot{\omega} + c\omega = T_m}$$



Free Body Diagram of the Inertial Load

Thus, the transfer function from the input motor torque to rotational speed changes is

$$\boxed{\frac{\omega(s)}{T_m(s)} = \frac{\left(1/J\right)}{s + \left(c/J\right)}} \qquad \text{(1}^{\text{st}}\text{ order system)} \qquad (1.5)$$

Combining equations (1.4) and (1.5) gives the transfer function from the input field voltage to the resulting speed change

$$\boxed{\frac{\omega(s)}{V_f(s)} = \frac{\omega(s)}{T_m(s)}\frac{T_m(s)}{V_f(s)} = \frac{\left(K_{mf}/L_f J\right)}{\left(s + c/J\right)\left(s + R_f/L_f\right)}} \qquad \text{(2}^{\text{nd}}\text{ order system)} \qquad (1.6)$$

Finally, since $\omega = d\theta/dt$, the transfer function from input field voltage to the resulting rotational position change is

$$\boxed{\frac{\theta(s)}{V_f(s)} = \frac{\theta(s)}{\omega(s)}\frac{\omega(s)}{V_f(s)} = \frac{\left(K_{mf}/L_f J\right)}{s\left(s + c/J\right)\left(s + R_f/L_f\right)}} \qquad \text{(3}^{\text{rd}}\text{ order system)} \qquad (1.7)$$

Servo Motor

A servo motor is one of the widely used variable speed drives in industrial production and process automation and building technology worldwide. A servo motor is a linear or rotary actuator that provides fast precision position control for closed-loop position control applications. Unlike large industrial motors, a servo motor is not used for continuous energy conversion.

Types of Servo Motors

asically, servo motors are classified into AC and DC servo motors depending upon the nature of supply used for its operation. Brushed permanent magnet DC servo motors are used for simple applications owing to their cost, efficiency and simplicity.

These are best suited for smaller applications. With the advancement of microprocessor and power transistor, AC servo motors are used more often due to their high accuracy control

DC Servo Motors

A DC servo motor is an assembly of four major components, namely a DC motor, a position sensing device, a gear assembly, and a control circuit. The below figure shows the parts that consisting in RC servo motors in which small DC motor is employed for driving the loads at precise speed and position. A DC reference voltage is set to the value corresponding to the desired output. This voltage can be applied by using another potentiometer, control pulse width to voltage converter, or through timers depending on the control circuitry.

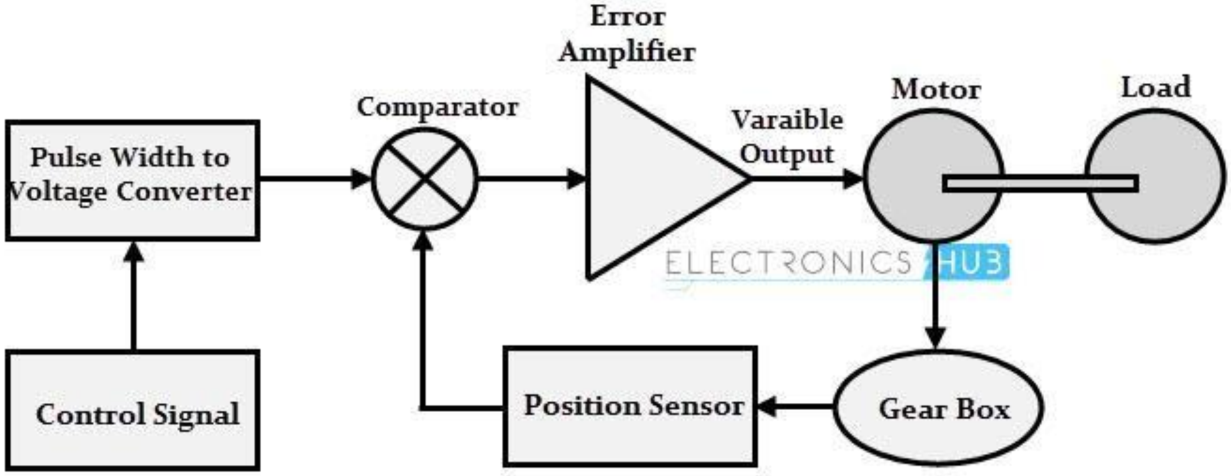
The dial on the potentiometer produces a corresponding voltage which is then applied as one of the inputs to error amplifier.

In some circuits, a control pulse is used to produce DC reference voltage corresponding to desired position or speed of the motor and it is applied to a pulse width to voltage converter.

In this converter, the capacitor starts charging at a constant rate when the pulse high. Then the charge on the capacitor is fed to the buffer amplifier when the pulse is low and this charge is further applied to the error amplifier.

So the length of the pulse decides the voltage applied at the error amplifier as a desired voltage to produce the desired speed or position.

In digital control, microprocessor or microcontroller are used for generating the PWM pluses in terms of duty cycles to produce more accurate control signals.



The feedback signal corresponding to the present position of the load is obtained by using a position sensor. This sensor is normally a potentiometer that produces the voltage corresponding to the absolute angle of the motor shaft through gear mechanism. Then the feedback voltage value is applied at the input of error amplifier (comparator).

The error amplifier is a negative feedback amplifier and it reduces the difference between its inputs. It compares the voltage related to current position of the motor (obtained by potentiometer) with desired voltage related to desired position of the motor (obtained by pulse width to voltage converter), and produces the error either a positive or negative voltage.

This error voltage is applied to the armature of the motor. If the error is more, the more output is applied to the motor armature.

As long as error exists, the amplifier amplifies the error voltage and correspondingly powers the armature. The motor rotates till the error becomes zero. If the error is negative, the armature voltage reverses and hence the armature rotates in the opposite direction.

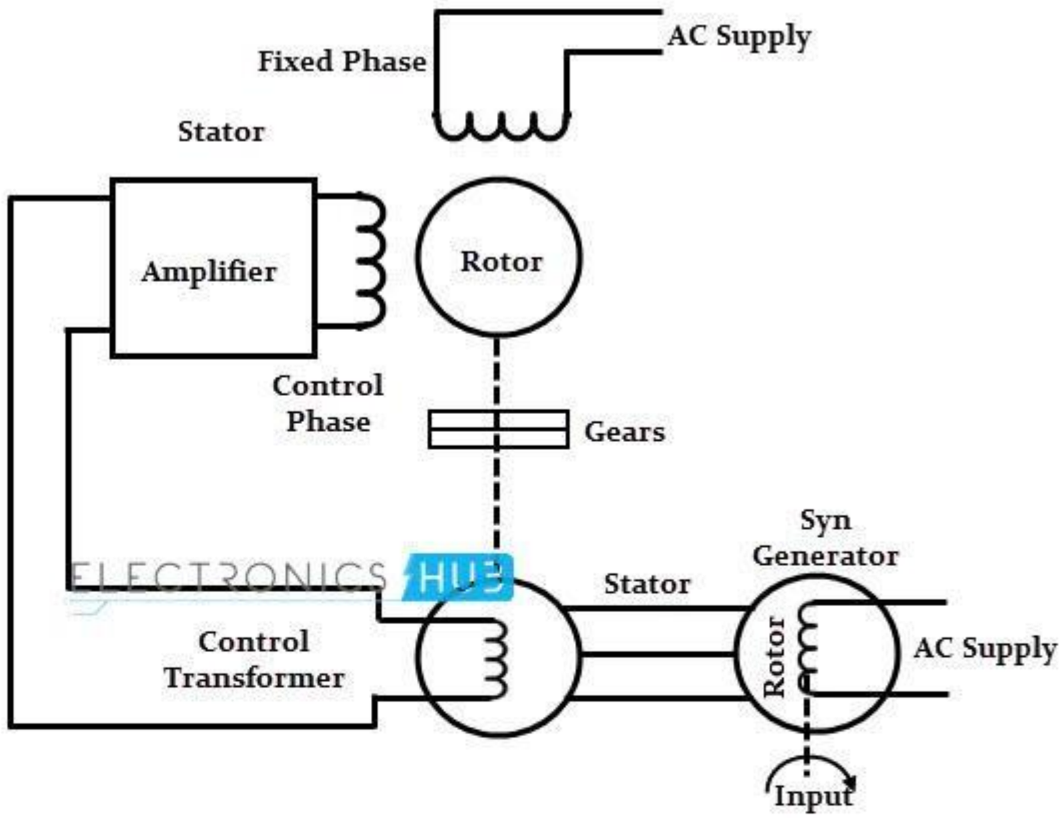
AC Servo Motors

AC servo motors are basically two-phase squirrel cage induction motors and are used for low power applications. Nowadays, three phase squirrel cage induction motors have been modified such that they can be used in high power servo systems.

The main difference between a standard split-phase induction motor and AC motor is that the squirrel cage rotor of a servo motor has made with thinner conducting bars, so that the motor resistance is higher.

Working Principle of AC Servo Motor

The schematic diagram of servo system for AC two-phase induction motor is shown in the figure below. In this, the reference input at which the motor shaft has to maintain at a certain position is given to the rotor of synchro generator as mechanical input theta. This rotor is connected to the electrical input at rated voltage at a fixed frequency.



he three stator terminals of a synchro generator are connected correspondingly to the terminals of control transformer. The angular position of the two-phase motor is transmitted to the rotor of control transformer through gear train arrangement and it represents the control condition alpha.

Initially, there exist a difference between the synchro generator shaft position and control transformer shaft position. This error is reflected as the voltage across the control transformer. This error voltage is applied to the servo amplifier and then to the control phase of the motor.

With the control voltage, the rotor of the motor rotates in required direction till the error becomes zero. This is how the desired shaft position is ensured in AC servo motors.

Alternatively, modern AC servo drives are embedded controllers like PLCs, microprocessors and microcontrollers to achieve variable frequency and variable voltage in order to drive the motor.

SYNCHRO :-

INTRODUCTION

The term synchro is a generic name for a family of inductive devices which works on the principle of a rotating transformer (Induction motor). The trade names for synchronous are Selsyn, Autosyn and Telesyn. Basically they are electro mechanical devices or electromagnetic transducer which produces an output voltage depending upon angular position of the rotor.

A Synchro system is formed by interconnection of the devices called the Synchro Transmitter and the **synchro control transformer**. They are also called as synchro pair. The synchro pair measures and compares two angular displacements and its output voltage is approximately linear with angular difference of the axis of both the shafts. They can be used in the following two ways.

i. To control the angular position of load from a remote place / long distance.

ii. For automatic correction of changes due to disturbance in the angular position of the load.

SYNCHRO TRANSMITTER

The constructional features, electrical circuit and a schematic symbol of **Synchro Transmitter** are shown in figure-2. The two major parts of **Synchro Transmitters** are stator and rotor.The stators identical to the stator of three phase alternator. It is made of laminated silicon steel and slotted on the inner periphery to accommodate a balance three phase winding. The stator winding is concentric type with the axis of the three coil 120° apart. The stator winding is star connected(Y - connection).

The rotor is of dumb bell construction with a single winding. The ends of the rotor winding are terminated on two slip rings. A single phase AC excitation voltage is applied to the rotor through the slip rings.

Working Principles

When the rotor is excited by AC voltage, the rotor current flows, and a magnetic field is produced. The rotor magnetic field induces an emf in the stator coil by transformer action. The effective voltage induced in any stator coil depends upon the angular position of the coils axis with respect to rotor axis.

Constructional Features of **Synchro Transmitter**

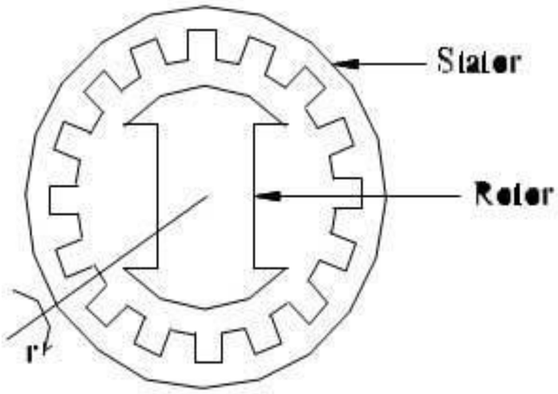


Fig: Constructional Features of Synchro Transmitter

Let er = Instantaneous value of AC voltage applied to rotor.

e ,e ,es1s2 s3 = Instantaneous value of emf induced in stator coils S ,S , S with respect to12 3 neutral respectively.

Er = Maximum value of rotor excitation voltage.

T = Angular frequency of rotor excitation voltage.

K_t = Turns ratio of stator and rotor winding.
K_c = Coupling coefficient.

2 = Angular displacement of rotor with respect to reference.

The instantaneous value of excitation voltage, e = Er sinrTt ---- (1)

Let the rotor rotates in antic lock wise direction. When the rotor rotates by an angle, 2 emfs are induced in stator coils. The frequency of induced emfs is same as that of rotor frequency. The magnitude of induced emfs is proportional to the turn's ratio and coupling coefficient. The turns ratio , K is a constant, but a coupling coefficient, K is a function of rotor angular position.$_t$c

Induced emf in stator coil = K K E since rTt------ (2)

SYNCHRO TRANSMITTER / RECEIVER

Let e be reference vector. With reference to figure 2, when 2 = 0, the flux linkage of coil s i Zero. Hence the flux linkage of coil S is function of cos22 (K = K) Cos c1 2 for coil S). The flux2 linkage of coil S will be maximum after a rotation of 120°in anti-clock wise direction and that3 of S after a rotation of 240°.1

Coupling coefficient, K for coil – S1

Coupling coefficient, K for coil – S2

Coupling coefficient, K for coil – S3

$$e_{S1S2} = e_{S1} - e_{S2} = \sqrt{3}\ KE_r \sin(\theta + 240°) \sin\omega t$$

$$e_{S2S3} = e_{S2} - e_{S3} = \sqrt{3}\ KE_r \sin(\theta + 120°) \sin\omega t$$

$$e_{S3S1} = e_{S3} - e_{S1} = \sqrt{3}\ KE_r \sin\theta \sin\omega t$$

$$e_{S1S2} = e_{S1} - e_{S2} = KE_r \cos(\theta - 240°) \sin\omega t - K E_r \cos\theta \sin\omega t$$

$$= KE_r \left[\cos\theta \cos 240° + \sin\theta \sin 240° - \cos\theta \right] \sin\omega t$$

$$= KE_r \left[\cos\theta (-0.5) + \sin\theta \left(-\frac{\sqrt{3}}{2}\right) - \cos\theta \right] \sin\omega t$$

$$= \sqrt{3}\ KE_r \left[\sin\theta \left(-\frac{1}{2}\right) + \cos\theta \left(-\frac{\sqrt{3}}{2}\right) \right] \sin\omega t$$

$$= \sqrt{3}\ KE_r \left[\sin\theta \cos 240° + \cos\theta \sin 240° \right] \sin\omega t$$

$$= \sqrt{3}\ KE_r \sin(\theta + 240°) \sin\omega t$$

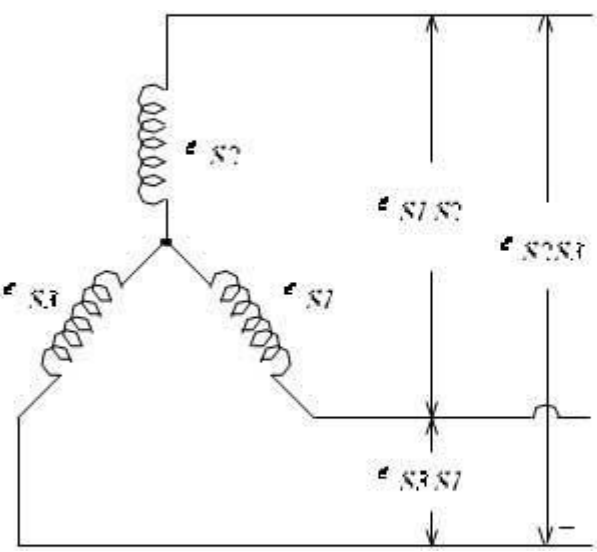


Fig:Induced emf in stator coils

When 2 = 0, from equation 3 we can say that maximum emf is induced in coil S. But from2 equation 8, it is observed that the coil - to coil voltage ES3S1 is zero. This position of the rotor is defined as the electrical zero of the transmitter

$$e_{S2S3} = e_{S2} - e_{S3} = K\,E_r \cos\theta \sin\omega t - KE_r \cos(\theta - 120^\circ) \sin\omega t$$

$$e_{S3S1} = e_{S2} - e_{S3} = KE_r \cos(\theta - 120^\circ) \sin\omega t - K\,E_r \cos(\theta - 240^\circ) \sin\omega t$$

$$= KE_r \left[\cos\theta - \cos\theta \cos 120^\circ - Sin\,\theta\, Sin\, 120^\circ \right] \sin\omega t$$

$$= \sqrt{3}\, KE_r \left[\sin\theta \cos 120^\circ + \cos\theta\, Sin\, 120^\circ \right] \sin\omega t$$

$$= \sqrt{3}\, KE_r \left[\sin\theta \left(-\frac{1}{2}\right) + \cos\theta \left(\frac{\sqrt{3}}{2}\right) \right] \sin\omega t$$

$$= \sqrt{3}\, KE_r \sin(\theta + 120^\circ) \sin\omega t$$

$$= KE_r \left[\cos\theta \cos 120^\circ + Sin\,\theta\, Sin 120^\circ - \cos\theta \cos 240^\circ - Sin\,\theta\, Sin 240^\circ \right] \sin\omega t$$

$$= KE_r \left[\cos\theta\,(-0.5) + \sin\theta \left(\frac{\sqrt{3}}{2}\right) - \cos\theta\,(-0.5) - \sin\theta \left(-\frac{\sqrt{3}}{2}\right) \right] \sin\omega t$$

$$= KE_r \left[\cos\theta - \cos\theta\,(-0.5) - \sin\theta \left(\frac{\sqrt{3}}{2}\right) \right] \sin\omega t$$

$$= \sqrt{3}\, KE_r \sin\theta \sin\omega t$$

angular position of its rotor shaft and the output is a set of three stator coil-to-coil voltages. By measuring and identifying the set of voltages at the stator terminals, it is possible to identify the angular position of the rotor. [A device called synchro / digital converter is available to measure the stator voltages and to calculate the angular measure and then display the direction and angle of rotation of the rotor].

SYNCHRO CONTROL TRANSFORMER

Construction

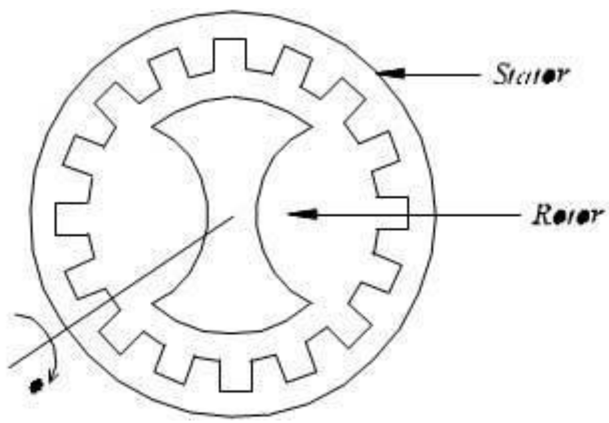


Fig:Constructional Features

The constructional features of **synchro control transformer** are similar to that of **Synchro Transmitter**, except the shape of rotor. The rotor of the control transformer is made cylindrical so that the air gap is practically uniform. This feature of the control transformer minimizes the changes in the rotor impedance with the rotation of the shaft. The constructional features, electrical circuit and a schematic symbol of control transformer are shown in figure 4.

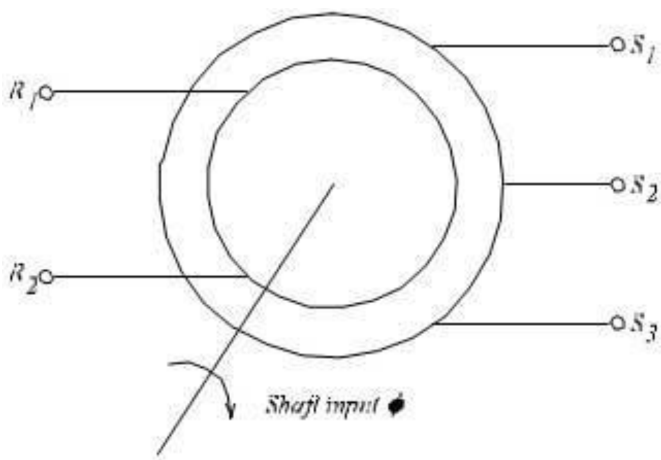


Fig:Schematic Symbol of **synchro control transformer**

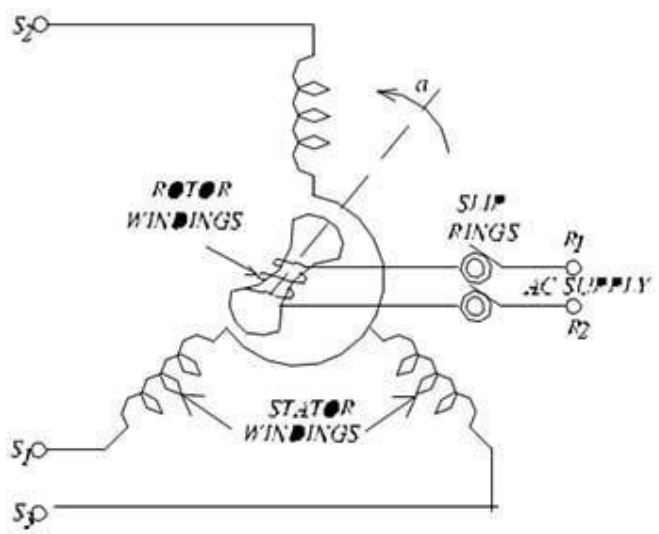


Fig: Electrical Circuit of **synchro control transformer**

Working

The generated emf of the **Synchro Transmitter** is applied as input to the stator coils of control transformer. The rotor shaft is connected to the load whose position has to be maintained at the desired value. Depending on the current position of the rotor and the applied emf on the stator, an emf is induced on the rotor winding. This emf can be measured and used to drive a motor so that the position of the load is corrected.

UNIT-III

3.1 Time-domain Analysis of Control Systems

In time-domain analysis the response of a dynamic system to an input is expressed as a function of time. It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

Usually, the input signals to control systems are not known fully ahead of time. In a radar tracking system, the position and the speed of the target to be tracked may vary in a random fashion. It is therefore difficult to express the actual input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration. The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration. Another standard signal of great importance is a sinusoidal signal.

The time response of any system has two components: transient response and the steady-state response. Transient response is dependent upon the system poles only and not on the type of input. It is therefore sufficient to analyze the transient response using a step input. The steady-state response depends on system dynamics and the input quantity. It is then examined using different test signals by final value theorem.

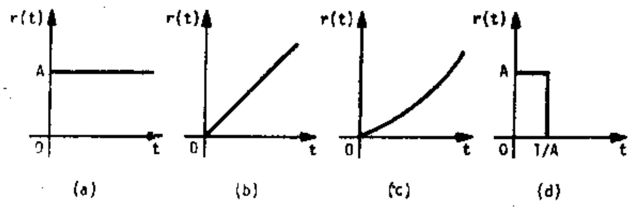
Standard test signals

a) Step signal: $r(t) = Au(t)$.

b) Ramp signal: $r(t) = At; \ \ t > 0$.

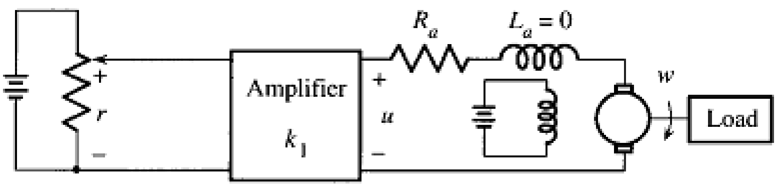
c) Parabolic signal: $r(t) = At^2/2; \ \ t > 0$.

d) Impulse signal: $r(t) = \delta(t)$.



Time-response of first-order systems

Let us consider the armature-controlled dc motor driving a load, such as a video tape. The objective is to drive the tape at constant speed. Note that it is an open-loop system.



$G(s) = \dfrac{W(s)}{R(s)} = \dfrac{k_1 k_m}{\tau_m s + 1}$; If $r(t) = au(t)$, $W(s) = \dfrac{k_1 k_m}{\tau_m s + 1} \cdot \dfrac{a}{s} = \dfrac{a k_1 k_m}{s} - \dfrac{a k_1 k_m}{s + 1/\tau_m}$

$$\Rightarrow w(t) = ak_1k_m - ak_1k_m e^{-t/\tau_m} \ ; \quad \Rightarrow w_{ss}(t) = \lim_{t\to\infty} w(t) = ak_1k_m$$

$w_{ss}(t)$ is the steady-state final speed. If the desired speed is w_r, choosing $a = \dfrac{w_r}{k_1 k_m}$ the motor will eventually reach the desired speed.

We are interested not only in final speed, but also in the speed of response. Here, τ_m is the time constant of motor which is responsible for the speed of response.

The time response is plotted in the Figure in next page. A plot of e^{-t/τ_m} is shown, from where it is seen that, for $t \geq 5\tau_m$ the value of e^{-t/τ_m} is less than 1% of its original value. Therefore, the speed of the motor will reach and stay within 1% of its final speed at 5 time constants.

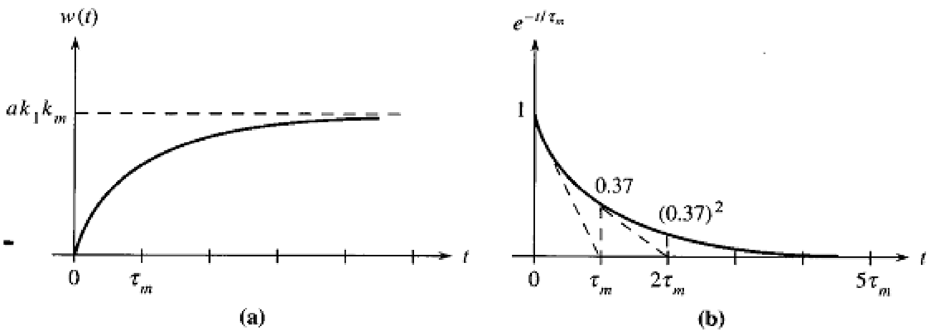
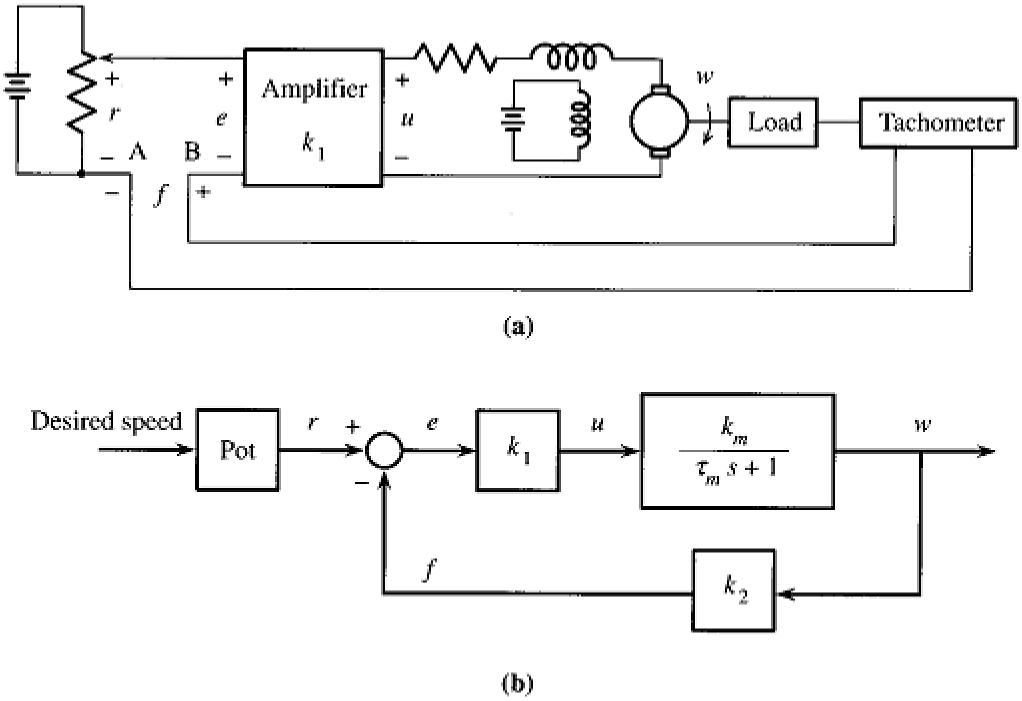


Figure: Time responses

Let us now consider the closed-loop system shown below.



Here, $T(s)=\dfrac{W(s)}{R(s)}=\dfrac{k_1k_m/(\tau_m s+1)}{1+k_1k_2k_m/(\tau_m s+1)}=\dfrac{k_1k_m}{\tau_m s+(1+k_1k_2k_m)}=\dfrac{k_1k_o}{\tau_o s+1}$

where, $k_o=\dfrac{k_m}{1+k_1k_2k_m}$ and $\tau_o=\dfrac{\tau_m}{1+k_1k_2k_m}$.

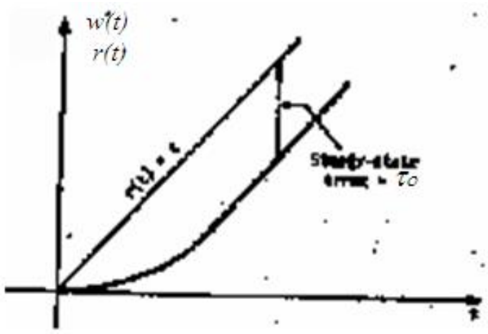
If $r(t)=a$, the response would be, $w(t)=ak_1k_o-ak_1k_oe^{-t/\tau_o}$.

If a is properly chosen, the tape can reach a desired speed. It will reach the desired speed in 5 τ_o seconds. Here, $\tau_o \ll \tau_m$. Thus, we can control the speed of response in feedback system.

Although the time-constant is reduced by a factor $(1+k_1k_2k_m)$, in the feedback system, the motor gain constant is also reduced by the same factor. In order to compensate for this loss of gain, the applied reference voltage must be increased by the same factor.

Ramp response of first-order system

Let, $k_1 k_0 = 1$ for simplicity. Then, $T(s) = \dfrac{1}{\tau_o s + 1} = \dfrac{W(s)}{R(s)}$. Also, let, $r(t) = tu(t)$.



Then, $W(s) = \dfrac{1}{s^2(\tau_o s + 1)} = \dfrac{1}{s^2} - \dfrac{\tau_o}{s} + \dfrac{\tau_o^2}{\tau_o s + 1}$;

$$\Rightarrow w(t) = tu(t) - \tau_o (1 - e^{-t/\tau_o}) u(t)$$

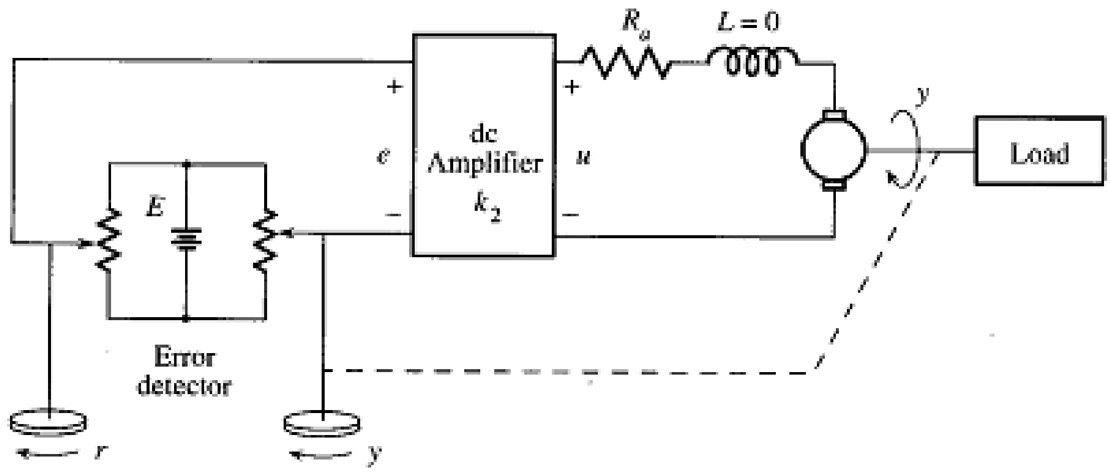
The error signal is, $e(t) = r(t) - w(t)$

Or, $e(t) = \tau_o (1 - e^{-t/\tau_o}) u(t)$

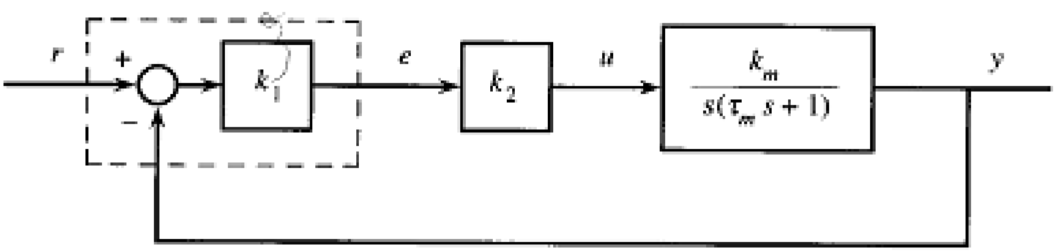
$\Rightarrow e_{ss}(t) = \tau_o$

Thus, the first-order system will track the unit ramp input with a steady-state error τ_o, which is equal to the time-constant of the system.

Time-response of second-order systems



(a)



(b)

Consider the antenna position control system. Its transfer function from *r* to *y* is,

$$T(s) = \frac{Y(s)}{R(s)} = \frac{k_1 k_2 k_m}{\tau_m s^2 + s + k_1 k_2 k_m} = \frac{k_1 k_2 k_m / \tau_m}{s^2 + \dfrac{1}{\tau_m} s + k_1 k_2 k_m / \tau_m} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

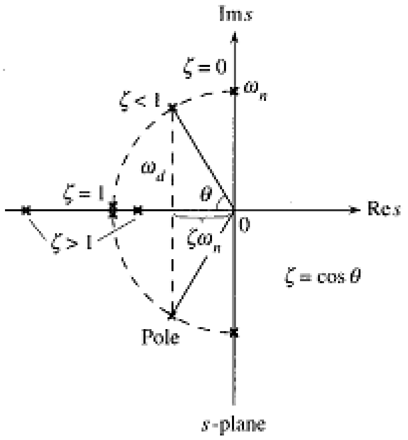
where, we define, $\omega_n^2 = k_1 k_2 k_m / \tau_m$ and $2\zeta\omega_n = \dfrac{1}{\tau_m}$. The constant ζ is called the *damping ratio* and ω_n is called the *natural frequency*. The system above is in fact a standard second order system.

The transfer function $T(s)$ has two poles and no zero. Its poles are,

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d .$$

Here, σ is called the *damping factor*, ω_d is called *damped or actual frequency*.

The location of poles for different ζ are plotted in Figure below. For $\zeta = 0$, the two poles $\pm j\omega_n$ are purely imaginary. If $0 < \zeta < 1$, the two poles are complex conjugate. All possible cases are described in a table shown below.



Damping Ratio	Poles	Remark
$\zeta = 0$	Pure imaginary	Undamped
$0 < \zeta < 1$	Complex conjugate	Underdamped
$\zeta = 1$	Repeated real poles	Critically damped
$\zeta > 1$	Two distinct real poles	Overdamped

Unit step response of second-order systems

Natural frequency, ω_n

The natural frequency of a second order system is the frequency of oscillation of the system without damping.

Damping ratio, ζ

The damping ratio is defined as the ratio of the damping factor, σ to the natural frequency ω_n.

Suppose, $T(s) = \dfrac{b}{s^2 + as + b}$.

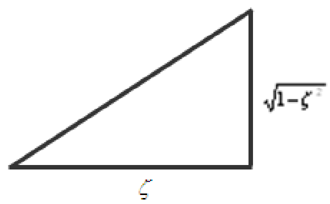
Comparing with standard equation, $a = 2\zeta\omega_n$ and $\omega_n^2 = b$.

Suppose, $r(t) = u(t), \Rightarrow R(s) = \dfrac{1}{s}$; $Y(s) = \dfrac{1}{s}\cdot\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \dfrac{1}{s} - \dfrac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Or,

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{1}{s} - \frac{s + \zeta\omega_n + \zeta\omega_n}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

Performing inverse Laplace transform,



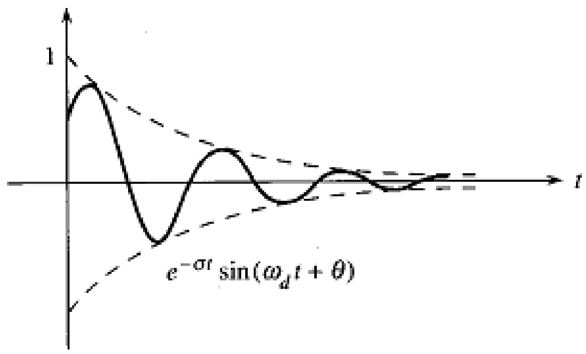
$$y(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2})t - e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2})t \cdot \frac{\zeta}{\sqrt{1-\zeta^2}}$$

or, $y(t) = 1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2})t + \zeta\sin(\omega_n\sqrt{1-\zeta^2})t\right]$

or, $y(t) = 1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$,where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\theta = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right) = \cos^{-1}\zeta$

or, $y(t) = 1 - \dfrac{\omega_n}{\omega_d} e^{-\sigma t}\sin(\omega_d t + \theta)$ ……………………………………………………….(01)

The plot of $e^{-\sigma t}\sin(\omega_d t + \theta)$ is shown in Figure.

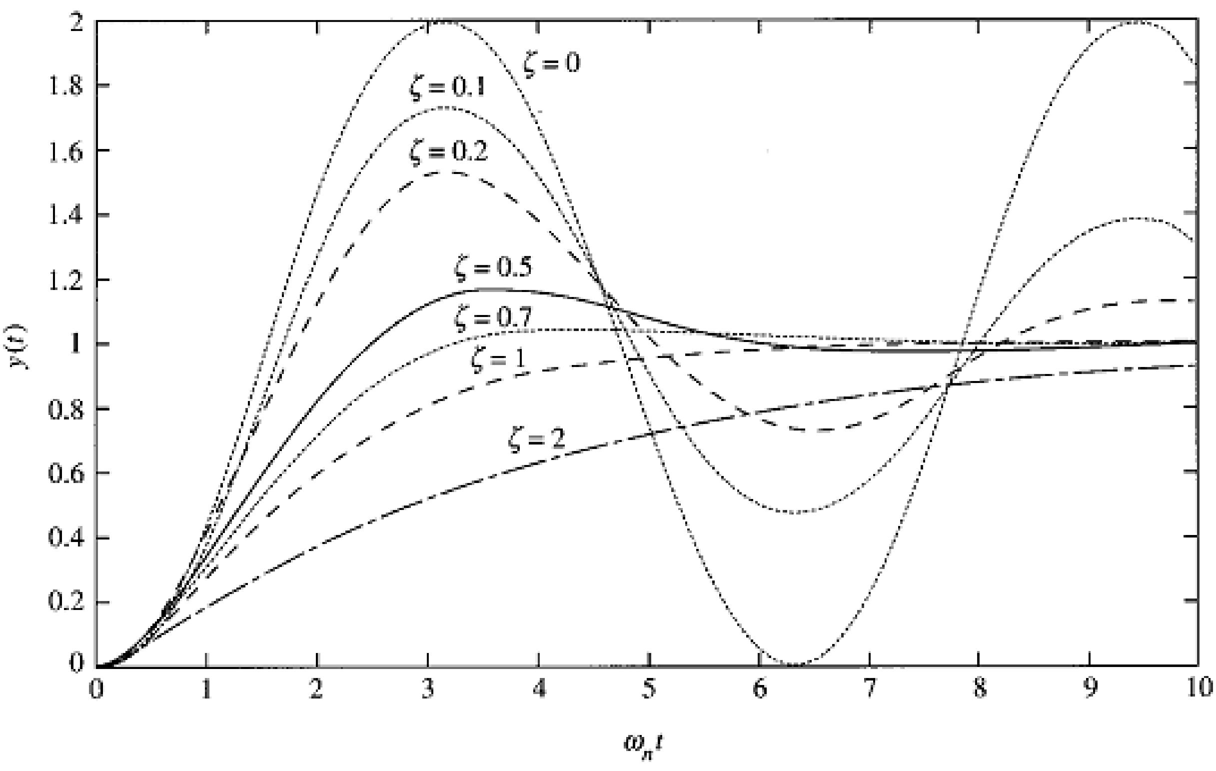


The steady-state response is,

$y_{ss}(t) = \lim_{t\to\infty} y(t) = 1$

Thus, the system has zero steady-state error.
The pole of $T(s)$ dictates the response,
$e^{-\sigma t}\sin(\omega_d t + \theta)$.

The response $y(t)$ for different ζ is shown in Figure below.

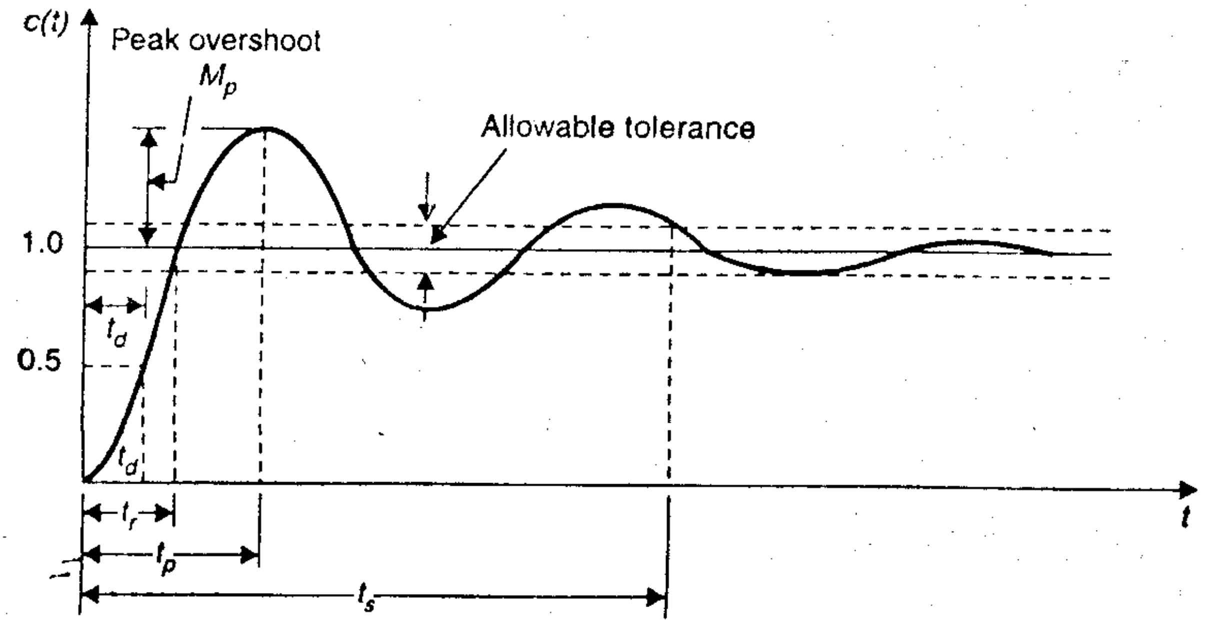


3.2 Time response specifications

Control systems are generally designed with damping less than one, i.e., oscillatory step response. Higher order control systems usually have a pair of complex conjugate poles with damping less than unity that dominate over the other poles. Therefore the time response of second- and higher-order control systems to a step input is generally of damped oscillatory nature as shown in Figure next (next page).

In specifying the transient-response characteristics of a control system to a unit step input, we usually specify the following:

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Peak overshoot, M_p
5. Settling time, t_s
6. Steady-state error, e_{ss}



1. ***Delay time,*** t_d ***:*** It is the time required for the response to reach 50% of the final value in first attempt.
2. ***Rise time,*** t_r ***:*** It is the time required for the response to rise from 0 to 100% of the final value for the underdamped system.
3. ***Peak time,*** t_p ***:*** It is the time required for the response to reach the peak of time response or the peak overshoot.
4. ***Settling time,*** t_s ***:*** It is the time required for the response to reach and stay within a specified tolerance band (2% or 5%) of its final value.
5. ***Peak overshoot,*** M_p ***:*** It is the normalized difference between the time response peak and the steady output and is defined as,

$$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

6. ***Steady-state error,*** e_{ss} ***:*** It indicates the error between the actual output and desired output as 't' tends to infinity.

$$e_{ss} = \lim_{t \to \infty}[r(t) - c(t)].$$

Let us now obtain the expressions for the rise time, peak time, peak overshoot, and settling time for the second order system.

1. ***Rise time,*** t_r ***:*** Put $y(t) = 1$ at $t = t_r$, $\Rightarrow \sin(\omega_d t_r + \theta) = 0 = \sin \pi$, $\Rightarrow t_r = \dfrac{\pi - \theta}{\omega_d}; \theta = \cos^{-1} \zeta$.

2. ***Peak time,*** t_p ***:*** Put $\dfrac{dy}{dt} = 0$ and solve for $t = t_p$; $0 = \dfrac{\sigma\omega_n}{\omega_d} e^{-\sigma t} \sin(\omega_d t + \theta) - \omega_n e^{-\sigma t} \cos(\omega_d t + \theta)$

.

$$\Rightarrow \tan(\omega_d t_p + \theta) = \frac{\omega_d}{\sigma} = \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\theta\,,\ \Rightarrow \omega_d t_p = k\pi \quad k = 0,1,2,\cdots$$

Peak overshoot occurs at *k = 1*. $\Rightarrow t_p = \pi/\omega_d = \pi/\omega_n\sqrt{1-\zeta^2}$.

3. ***Settling time,*** t_s***:*** For 2% tolerance band, $\frac{\omega_n}{\omega_d}e^{-\sigma t_s} = 0.02$, $\Rightarrow t_s \cong \frac{4}{\sigma} = 4T$.

4. ***Steady-state error,*** e_{ss}***:*** It is found previously that steady-state error for step input is zero. Let us now consider ramp input, $r(t) = tu(t)$.

Then, $e_{ss} = \lim_{s\to 0} s\{R(s) - Y(s)\} = \lim_{s\to 0} s\{\frac{1}{s^2} - \frac{1}{s^2}\cdot\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\}$

$$e_{ss} = \lim_{s\to 0}\frac{1}{s}\{1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\} = \lim_{s\to 0}\frac{1}{s}\left\{\frac{s^2 + 2\zeta\omega_n s + \cancel{\omega_n^2} - \cancel{\omega_n^2}}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\} = \frac{2\zeta\omega_n}{\omega_n^2} = \frac{2\zeta}{\omega_n}\,.$$

Therefore, the steady-state error due to ramp input is $\frac{2\zeta}{\omega_n}$.

Steady-state error and error constants

The steady-state performance of a stable control system is generally judged by its steady-state error to step, ramp and parabolic inputs. For a unity feedback system,

$$E(s) = \frac{R(s)}{1+G(s)}\,, \Rightarrow e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}\,.$$

It is seen that steady-state error depends upon the input $R(s)$ and the forward transfer function $G(s)$. The steady-state errors for different inputs are derived as follows:

1. For unit-step input: $r(t) = u(t), R(s) = \frac{1}{s}$

$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+G(0)} = \frac{1}{1+k_p}$; k_p is called position error constant.

2. For unit-ramp input: $r(t) = tu(t), R(s) = \frac{1}{s^2}$

$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{s\left[1+G(s)\right]} = \lim_{s\to 0}\frac{1}{sG(s)} = \frac{1}{k_v}$; k_v is called velocity error constant.

3. For unit-parabolic input: $r(t) = t^2/2, R(s) = \frac{1}{s^3}$

$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{1}{s^2\left[1+G(s)\right]} = \lim_{s\to 0}\frac{1}{s^2 G(s)} = \frac{1}{k_a}$; k_a is called acceleration error const.

3.3 Types of Feedback Control System

The open-loop transfer function of a system can be written as,

$$G(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots}{s^n(s+p_1)(s+p_2)(s+p_3)\cdots} = \frac{K'(T_{z1}s+1)(T_{z2}s+1)(T_{z3}s+1)\cdots}{s^n(T_{p1}s+1)(T_{p2}s+1)(T_{p3}s+1)\cdots}$$

If n = 0, the system is called type-0 system, if n = 1, the system is called type-1 system, if n = 2, the system is called type-2 system, etc. Steady-state errors for various inputs and system types are tabulated below.

Type of input	Steady-state error		
	Type-0 system	Type-1 system	Type-2 system
Unit-step	$1/(1+K_p)$	0	0
Unit-ramp	∞	$1/K_v$	0
Unit-parabolic	∞	∞	$1/K_a$
	$K_p = \lim_{s\to 0} G(s)$	$K_v = \lim_{s\to 0} sG(s)$	$K_a = \lim_{s\to 0} s^2G(s)$

The error constants for non-unity feedback systems may be obtained by replacing G(s) by G(s)H(s). Systems of type higher than 2 are not employed due to two reasons:

1. The system is difficult to stabilize.
2. The dynamic errors for such systems tend to be larger than those types-0, -1 and -2.

Effect of Adding a Zero to a System

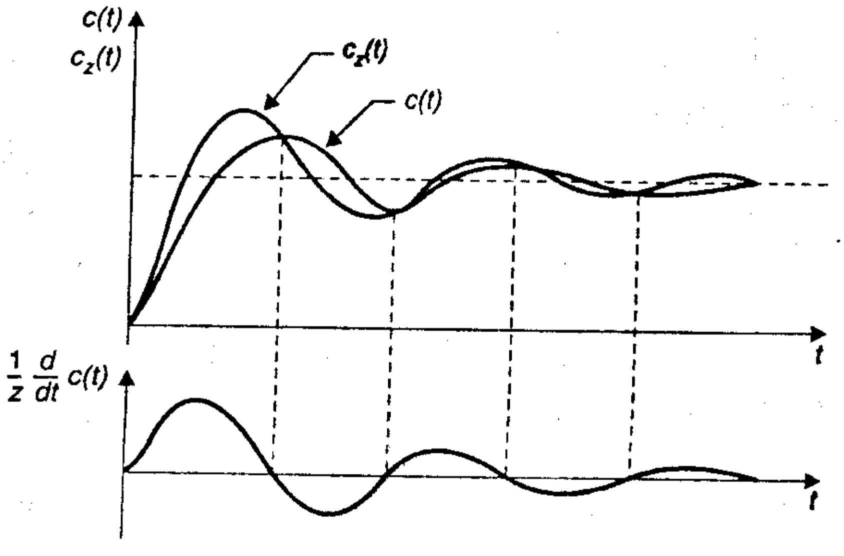
Let a zero at s = -z be added to a second order system. Then we have,

$$\frac{C(s)}{R(s)} = \frac{(s+z)\omega_n^2/z}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} + \frac{s}{z}\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right).$$

The multiplication term is adjusted to make the steady-state gain of the system unity. This gives $c_{ss} = 1$ when the input is unit step. Let $c_z(t)$ be the response of the system given by the above equation and $c(t)$ is the response without adding the pole. Manipulation of the above equation gives,

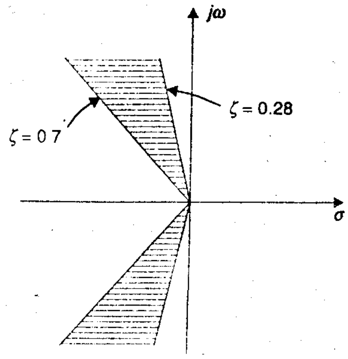
$$c_z(t) = c(t) + \frac{1}{z}\frac{d}{dt}c(t).$$

The effect of added derivative term is to produce a pronounced early peak to the system response which will be clear from the figure in the next page. Closer the zero to origin, the more pronounce the peaking phenomenon. Due to this fact, *the zeros on the real axis near the origin are generally avoided in design*. However, in a sluggish system the artful introduction of a zero at the proper position can improve the transient response. We can see from equation (03) that as z increases, i.e., the zero moves further into the left half of the s-plane, its effect becomes less pronounced.



Design Specifications of Second-order Systems

A control system is generally required to meet three time response specifications: steady-state accuracy, damping factor ζ (or peak overshoot, M_p) and settling time t_s. Steady-state accuracy requirement is met by suitable choice of *Kp, Kv, or Ka* depending on the type of the system. For most control systems ζ in the range of 0.7 – 0.28 (or peak overshoot of 5 – 40%) is considered acceptable. For this range of ζ, the closed-loop pole locations are restricted to the shaded region of the s-plane as shown in Figure.



For the antenna position control system, $\omega_n = \sqrt{k_1 k_2 k_m / \tau_m}$; $\zeta = \dfrac{1}{2\omega_n \tau_m}$; $e_{ss}\big|_{\text{ramp}} = \dfrac{2\zeta}{\omega_n}$; $t_s = \dfrac{4}{\zeta\omega_n}$. Here, k_2 is only the adjustable parameter. If we increase k_2 , ω_n will increase and thus settling time will decrease. At the same time, ζ will decrease, this indicates the increase in peak overshoot. Thus by merely increasing gain, we cannot improve both transient and steady-state error specifications. We need to add additional components to the system. These are called compensators. It will allow improvement of both transient and steady-state specifications.

UNIT-IV

Routh Hurwitz Stability Criterion

After reading the theory of network synthesis, we can easily say that any pole of the system lies on the right hand side of the origin of the s plane, it makes the system unstable. On the basis of this condition A. Hurwitz and E.J.Routh started investigating the necessary and sufficient conditions of stability of a system. We will discuss two criteria for stability of the system. A first criterion is given by A. Hurwitz and this criterion is also known as **Hurwitz Criterion for stability** or **Routh Hurwitz Stability Criterion**.

Hurwitz Criterion

With the help of characteristic equation, we will make a number of Hurwitz determinants in order to find out the stability of the system. We define characteristic equation of the system as

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots\cdots\cdots\cdots\cdots a_{n-1} s^1 + a_n$$

Now there are n determinants for n^{th} order characteristic equation.
Let us see how we can write determinants from the coefficients of the characteristic equation. The step by step procedure for k^{th} order characteristic equation is written below:
Determinant one : The value of this determinant is given by $|a_1|$ where a_1 is the coefficient of s^{n-1} in the characteristic equation.

Determinant two : The value of this determinant is given by $\begin{bmatrix} a_1 & a_3 \\ a_0 & a_2 \end{bmatrix}$ Here number of elements in each row is equal to determinant number and we have determinant number here is two. The first row consists of first two odd coefficients and second row consists of first two even coefficients.

Determinant three : The value of this determinant is given by $\begin{bmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix}$ Here number of elements in each row is equal to determinant number and we have determinant number here is three. The first row consists of first three odd coefficients, second row consists of first three even coefficients and third row consists of first element as zero and rest of two elements as first two odd coefficients.

Determinant four: The value of this determinant is given by, $\begin{bmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix}$ Here number of elements in each row is equal to determinant number and we have determinant number here is four. The first row consists of first three four coefficients, second row consists of first four even coefficients, third row consists of first element as zero and rest of three elements as first three odd coefficients the fourth row consists of first element as zero and rest of three elements as first three even coefficients.
By following the same procedure we can generalize the determinant formation. The general

form of determinant is given below:
$$\begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot & a_{2k-1} \\ a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot & a_{2k-2} \\ 0 & a_1 & a_3 & a_5 & \cdot & \cdot & \cdot & a_{2k-3} \\ 0 & a_0 & a_2 & a_4 & \cdot & \cdot & \cdot & a_{2k-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & ak \end{bmatrix}$$
Now in order to check the stability of the above system, calculate the value of each determinant. The system will be stable if and only if the value of each determinant is greater than zero, i.e. the value of each determinant should be positive. In all the other cases the system will not be stable.

Routh Stability Criterion

This criterion is also known as modified Hurwitz Criterion of stability of the system. We will study this criterion in two parts. Part one will cover necessary condition for stability of the system and part two will cover the sufficient condition for the stability of the system. Let us again consider the characteristic equation of the system as $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots\cdots\cdots\cdots a_{n-1} s^1 + a_n$ 1) Part one (necessary condition for stability of the system): In this we have two conditions which are written below:

1. All the coefficients of the characteristic equation should be positive and real.
2. All the coefficients of the characteristic equation should be non zero.

2) Part two (sufficient condition for stability of the system): Let us first construct routh array. In order to construct the routh array follow these steps:

- The first row will consist of all the even terms of the characteristic equation. Arrange them from first (even term) to last (even term). The first row is written below: a_0 a_2 $a_4 a_6$............
- The second row will consist of all the odd terms of the characteristic equation. Arrange them from first (odd term) to last (odd term). The first row is written below: a_1 a_3 $a_5 a_7$...........
- The elements of third row can be calculated as: (1) First element : Multiply a_0 with the diagonally opposite element of next column (i.e. a_3) then subtract this from the product of a_1 and a_2 (where a_2 is diagonally opposite element of next column) and then finally divide the result so obtain with a_1. Mathematically we write as first element

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

(2) Second element : Multiply a_0 with the diagonally opposite element of next to next column (i.e. a_5) then subtract this from the product of a_1 and a_4 (where, a_4 is diagonally opposite element of next to next column) and then finally divide the result so obtain with a_1. Mathematically we write as second element $b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$ Similarly, we can calculate all the elements of the third row.

(d) The elements of fourth row can be calculated by using the following procedure: (1) First element : Multiply b_1 with the diagonally opposite element of next column (i.e. a_3) then subtract this from the product of a_1 and b_2 (where, b_2 is diagonally opposite element of next column) and then finally divide the result so obtain with b_1. Mathematically we write as first element $c_1 = \frac{a_1 b_2 - b_1 a_3}{b_1}$

(2) Second element : Multiply b_1 with the diagonally opposite element of next to next column

(i.e. a_5) then subtract this from the product of a_1 and b_3 (where, b_3 is diagonally opposite element of next to next column) and then finally divide the result so obtain with a_1. Mathematically we write as second element $$c_2 = \frac{a_1 b_3 - b_1 a_5}{b_1}$$ Similarly, we can calculate all the elements of the fourth row.
Similarly, we can calculate all the elements of all the rows.
Stability criteria if all the elements of the first column are positive then the system will be stable. However if anyone of them is negative the system will be unstable.
Now there are some special cases related to Routh Stability Criteria which are discussed below:
(1)Caseone:
If the first term in any row of the array is zero while the rest of the row has at least one non zero term.
In this case we will assume a very small value (ε) which is tending to zero in place of zero. By replacing zero with (ε) we will calculate all the elements of the Routh array. After calculating all the elements we will apply the limit at each element containing (ε). On solving the limit at every element if we will get positive limiting value then we will say the given system is stable otherwise in all the other condition we will say the given system is not stable.
(2)Casesecond:
When all the elements of any row of the Routh array are zero. In this case we can say the system has the symptoms of marginal stability. Let us first understand the physical meaning of having all the elements zero of any row. The physical meaning is that there are symmetrically located roots of the characteristic equation in the s plane. Now in order to find out the stability in this case we will first find out auxiliary equation. Auxiliary equation can be formed by using the elements of the row just above the row of zeros in the Routh array. After finding the auxiliary equation we will differentiate the auxiliary equation to obtain elements of the zero row. If there is no sign change in the new routh array formed by using auxiliary equation, then in this we say the given system is limited stable. While in all the other cases we will say the given system is unstable.

3.5 Root Locus Technique in Control System | Root Locus Plot

The **root locus technique in control system** was first introduced in the year 1948 by Evans. Any physical system is represented by a transfer function in the form of

$$G(s) = k \times \frac{numerator\ of\ s}{denomerator\ of\ s}$$

We can find poles and zeros from G(s). The location of poles and zeros are crucial keeping view stability, relative stability, transient response and error analysis. When the system put to service stray inductance and capacitance get into the system, thus changes the location of poles and zeros. In **root locus technique in control system** we will evaluate the position of the roots, their locus of movement and associated information. These information will be used to comment upon the system performance.
Now before I introduce what is a root locus technique, it is very essential here to discuss a few of the advantages of this technique over other stability criteria. Some of the advantages of root locus technique are written below.

Advantages of Root Locus Technique

1. Root locus technique in control system is easy to implement as compared to other methods.
2. With the help of root locus we can easily predict the performance of the whole system.
3. Root locus provides the better way to indicate the parameters.

Now there are various terms related to root locus technique that we will use frequently in this article.

1. Characteristic Equation Related to Root Locus Technique : 1 + G(s)H(s) = 0 is known as characteristic equation. Now on differentiating the characteristic equation and on equating dk/ds equals to zero, we can get break away points.
2. Break away Points : Suppose two root loci which start from pole and moves in opposite direction collide with each other such that after collision they start moving in different directions in the symmetrical way. Or the break away points at which multiple roots of the characteristic equation 1 + G(s)H(s)= 0 occur. The value of K is maximum at the points where the branches of root loci break away. Break away points may be real, imaginary or complex.
3. Break in Point : Condition of break in to be there on the plot is written below : Root locus must be present between two adjacent zeros on the real axis.
4. Centre of Gravity : It is also known centroid and is defined as the point on the plot from where all the asymptotes start. Mathematically, it is calculated by the difference of summation of poles and zeros in the transfer function when divided by the difference of total number of poles and total number of zeros. Centre of gravity is always real and it is denoted by σ_A

   $$\sigma_A = \frac{(Sum\ of\ real\ parts\ of\ poles) - (Sum\ of\ real\ parts\ of\ zeros)}{N - M}$$
   .
   Where, N is number of poles and M is number of zeros.
5. Asymptotes of Root Loci : Asymptote originates from the center of gravity or centroid and goes to infinity at definite some angle. Asymptotes provide direction to the root locus when they depart break away points.
6. Angle of Asymptotes : Asymptotes makes some angle with the real axis and this angle can be calculated from the given formula,

   $$Angle\ of\ asymptotes = \frac{(2p+1) \times 180}{N - M}$$

   Where, p = 0, 1, 2 .... (N-M-1)
   N is the total number of poles
   M is the total number of zeros.
7. Angle of Arrival or Departure : We calculate angle of departure when there exists complex poles in the system. Angle of departure can be calculated as 180-{(sum of angles to a complex pole from the other poles)-(sum of angle to a complex pole from the zeros)}.
8. Intersection of Root Locus with the Imaginary Axis : In order to find out the point of intersection root locus with imaginary axis, we have to use Routh Hurwitz criterion. First, we find the auxiliary equation then the corresponding value of K will give the value of the point of intersection.
9. Gain Margin : We define gain margin as a by which the design value of the gain factor can be multiplied before the system becomes unstable. Mathematically it is given by the formula

   $$Gain\ margin = \frac{Value\ of\ K\ at\ the\ imaginary\ axes\ cross\ over}{Design\ value\ of\ K}$$

10. Phase Margin : Phase margin can be calculated from the given formula:

    $$Phase\ margin = 180 + \angle(G(jw)H(jw))$$

11. Symmetry of Root Locus : Root locus is symmetric about the x axis or the real axis.

How to determine the value of K at any point on the root loci? Now there are two ways of determining the value of K, each way is described below.

1. Magnitude Criteria : At any points on the root locus we can apply magnitude criteria as, $|G(s)H(s)| = 1$ Using this formula we can calculate the value of K at any desired point.
2. Using Root Locus Plot : The value of K at any s on the root locus is given by
$$K = \frac{product\ of\ all\ of\ the\ vector\ lengths\ drawn\ from\ the\ poles\ of\ G(s)H(s)\ to\ s}{product\ of\ all\ of\ the\ vector\ lengths\ drawn\ from\ the\ zeros\ of\ G(s)H(s)\ to\ s}$$

3.6 Root Locus Plot

This is also known as root locus technique in control system and is used for determining the stability of the given system. Now in order to determine the stability of the system using the root locus technique we find the range of values of K for which the complete performance of the system will be satisfactory and the operation is stable.
Now there are some results that one should remember in order to plot the root locus. These results are written below:

1. Region where root locus exists : After plotting all the poles and zeros on the plane, we can easily find out the region of existence of the root locus by using one simple rule which is written below,
   Only that segment will be considered in making root locus if the total number of poles and zeros at the right hand side of the segment is odd.
2. How to calculate the number of separate root loci ? : A number of separate root loci are equal to the total number of roots if number of roots are greater than the number of poles otherwise number of separate root loci is equal to the total number of poles if number of roots are greater than the number of zeros.

Procedure to Plot Root Locus

Keeping all these points in mind we are able to draw the **root locus plot** for any kind of system. Now let us discuss the procedure of making a root locus.

1. Find out all the roots and poles from the open loop transfer function and then plot them on the complex plane.
2. All the root loci starts from the poles where k = 0 and terminates at the zeros where K tends to infinity. The number of branches terminating at infinity equals to the difference between the number of poles & number of zeros of G(s)H(s).
3. Find the region of existence of the root loci from the method described above after finding the values of M and N.
4. Calculate break away points and break in points if any.
5. Plot the asymptotes and centroid point on the complex plane for the root loci by calculating the slope of the asymptotes.
6. Now calculate angle of departure and the intersection of root loci with imaginary axis.
7. Now determine the value of K by using any one method that I have described above.

By following above procedure you can easily draw the **root locus plot** for any open loop transfer function.

8. Calculate the gain margin.
9. Calculate the phase margin.
10. You can easily comment on the stability of the system by using Routh array.

Types of Controllers | Proportional Integral and Derivative Controllers

Before I introduce you about various controllers in detail, it is very essential to know the uses of controllers in the theory of control systems. The important uses of the **controllers**are written below:

1. Controllers improve steady state accuracy by decreasing the steady state errors.
2. As the steady state accuracy improves, the stability also improves.
3. They also help in reducing the offsets produced in the system.
4. Maximum overshoot of the system can be controlled using these controllers.
5. They also help in reducing the noise signals produced in the system.
6. Slow response of the over damped system can be made faster with the help of these controllers.

Now what are controllers? A controller is one which compares controlled values with the desired values and has a function to correct the deviation produced.

3.7 Types of Controllers

Let us classify the controllers. There are mainly two **types of controllers** and they are written below:

Continuous Controllers: The main feature of continuous controllers is that the controlled variable (also known as the manipulated variable) can have any value within the range of controller's output. Now in the continuous controller's theory, there are three basic modes on which the whole control action takes place and these modes are written below. We will use the combination of these modes in order to have a desired and accurate output.

1. **Proportional controllers**.
2. **Integral controllers**.
3. **Derivative controllers**.

Combinations of these three controllers are written below:

4. Proportional and integral controllers.
5. Proportional and derivative controllers.

Now we will discuss each of these modes in detail.

Proportional Controllers

We cannot use **types of controllers** at anywhere, with each type controller, there are certain conditions that must be fulfilled. With **proportional controllers** there are two conditions and these are written below:

1. Deviation should not be large, it means there should be less deviation between the input and output.
2. Deviation should not be sudden.

Now we are in a condition to discuss proportional controllers, as the name suggests in a proportional controller the output (also called the actuating signal) is directly proportional to the error signal. Now let us analyze proportional controller mathematically. As we know in proportional controller output is directly proportional to error signal, writing this mathematically we have,

$A(t) \propto e(t)$ Removing the sign of proportionality we have, $A(t) = K_p \times e(t)$

Where, K_p is proportional constant also known as controller gain. It is recommended that K_p should be kept greater than unity. If the value of K_p is greater than unity, then it will amplify the error signal and thus the amplified error signal can be detected easily.

Advantages of Proportional Controller

Now let us discuss some advantages of proportional controller.

1. Proportional controller helps in reducing the steady state error, thus makes the system more stable.
2. Slow response of the over damped system can be made faster with the help of these controllers.

Disadvantages of Proportional Controller

Now there are some serious disadvantages of these controllers and these are written as follows:

1. Due to presence of these controllers we some offsets in the system.
2. Proportional controllers also increase the maximum overshoot of the system.

Integral Controllers

As the name suggests in **integral controllers** the output (also called the actuating signal) is directly proportional to the integral of the error signal. Now let us analyze integral controller mathematically. As we know in an integral controller output is directly proportional to the integration of the error signal, writing this mathematically we have,

$$A(t) \propto \int_0^t e(t)dt$$

Removing the sign of proportionality we have,

$$A(t) = K_i \times \int_0^t e(t)dt$$

Where, K_i is integral constant also known as controller gain. Integral controller is also known as reset controller.

Advantages of Integral Controller

Due to their unique ability they can return the controlled variable back to the exact set point following a disturbance that's why these are known as reset controllers.

Disadvantages of Integral Controller

It tends to make the system unstable because it responds slowly towards the produced error.

Derivative Controllers

We never use **derivative controllers** alone. It should be used in combinations with other modes of controllers because of its few disadvantages which are written below:

1. It never improves the steady state error.
2. It produces saturation effects and also amplifies the noise signals produced in the system.

Now, as the name suggests in a derivative controller the output (also called the actuating signal) is directly proportional to the derivative of the error signal. Now let us analyze derivative controller mathematically. As we know in a derivative controller output is directly proportional to the derivative of the error signal, writing this mathematically we have,

$$A(t) \propto \frac{de(t)}{dt}$$

Removing the sign of proportionality we have,

$$A(t) = K_d \times \frac{de(t)}{dt}$$

Where, K_d is proportional constant also known as controller gain. Derivative controller is also known as rate controller.

Advantages of Derivative Controller

The major advantage of derivative controller is that it improves the transient response of the system.

Proportional and Integral Controller

As the name suggests it is a combination of proportional and an integral controller the output (also called the actuating signal) is equal to the summation of proportional and integral of the error signal. Now let us analyze proportional and integral controller mathematically. As we know in a proportional and integral controller output is directly proportional to the summation of proportional of error and integration of the error signal, writing this mathematically we have,

$$A(t) \propto \int_0^t e(t)dt + A(t) \propto e(t)$$

Removing the sign of proportionality we have,

$$A(t) = K_i \int_0^t e(t)dt + K_p e(t)$$

Where, K_i and k_p proportional constant and integral constant respectively. Advantages and disadvantages are the combinations of the advantages and disadvantages of proportional and integral controllers.

Proportional and Derivative Controller

As the name suggests it is a combination of proportional and a derivative controller the output (also called the actuating signal) is equals to the summation of proportional and derivative of the error signal. Now let us analyze proportional and derivative controller mathematically. As we know in a proportional and derivative controller output is directly proportional to summation of proportional of error and differentiation of the error signal, writing this mathematically we have,

$$A(t) \propto \frac{de(t)}{dt} + A(t) \propto e(t)$$

Removing the sign of proportionality we have,

$$A(t) = K_d \frac{de(t)}{dt} + K_p e(t)$$

Where, K_d and k_p proportional constant and derivative constant respectively. Advantages and disadvantages are the combinations of advantages and disadvantages of proportional and derivative controllers

UNIT V
Frequency Domain Analysis

The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The steady state response of a system for an input sinusoidal signal is known as the **frequency response**. In this chapter, we will focus only on the steady state response.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles.

Let the input signal be −

$$r(t)=A\sin(\omega_0 t)$$

The open loop transfer function will be −

$$G(s)=G(j\omega)$$

We can represent $G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega)=|G(j\omega)|\angle G(j\omega)$$

Substitute, $\omega=\omega_0$ in the above equation.

$$G(j\omega_0)=|G(j\omega_0)|\angle G(j\omega_0)$$

The output signal is

$$c(t)=A|G(j\omega_0)|\sin(\omega_0 t+\angle G(j\omega_0))$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at $\omega=\omega_0$.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at $\omega=\omega_0$.

Where,

- **A** is the amplitude of the input sinusoidal signal.
- **ω_0** is angular frequency of the input sinusoidal signal.

We can write, angular frequency ω_0 as shown below.

$$\omega_0=2\pi f_0$$

Here, f_0 is the frequency of the input sinusoidal signal. Similarly, you can follow the same procedure for closed loop control system.

Frequency Domain Specifications

The frequency domain specifications are **resonant peak, resonant frequency and bandwidth**.

Consider the transfer function of the second order closed loop control system as,

$$T(s)=\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

Substitute, $s=j\omega$ in the above equation.

$$T(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\delta\omega_n(j\omega)+\omega_n^2}$$

$$\Rightarrow T(j\omega)=\frac{\omega_n^2}{-\omega^2+2j\delta\omega\omega_n+\omega_n^2}=\frac{\omega_n^2}{\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega)=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Let, $\frac{\omega}{\omega_n}=u$ Substitute this value in the above equation.

$$T(j\omega)=\frac{1}{(1-u^2)+j(2\delta u)}$$

Magnitude of $T(j\omega)$ is -

$$M=|T(j\omega)|=\frac{1}{\sqrt{(1-u^2)^2+(2\delta u)^2}}$$

Phase of $T(j\omega)$ is -

$$\angle T(j\omega) = -\tan^{-1}\left(\frac{2\delta u}{1-u^2}\right)$$

Resonant Frequency

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r . At $\omega=\omega_r$, the first derivate of the magnitude of $T(j\omega)$ is zero.

Differentiate M with respect to u .

$$\frac{\text{d}M}{\text{d}u} = -\frac{1}{2}\left[(1-u^2)^2+(2\delta u)^2\right]^{\frac{-3}{2}}\left[2(1-u^2)(-2u)+2(2\delta u)(2\delta)\right]$$

$$\Rightarrow \frac{\text{d}M}{\text{d}u} = -\frac{1}{2}\left[(1-u^2)^2+(2\delta u)^2\right]^{\frac{-3}{2}}\left[4u(u^2-1+2\delta^2)\right]$$

Substitute, $u=u_r$ and $\frac{\text{d}M}{\text{d}u}=0$ in the above equation.

$$0 = -\frac{1}{2}\left[(1-u_r^2)^2+(2\delta u_r)^2\right]^{\frac{-3}{2}}\left[4u_r(u_r^2-1+2\delta^2)\right]$$

$$\Rightarrow 4u_r(u_r^2-1+2\delta^2)=0$$

$$\Rightarrow u_r^2-1+2\delta^2=0$$

$$\Rightarrow u_r^2=1-2\delta^2$$

$$\Rightarrow u_r=\sqrt{1-2\delta^2}$$

Substitute, $u_r=\frac{\omega_r}{\omega_n}$ in the above equation.

$$\frac{\omega_r}{\omega_n}=\sqrt{1-2\delta^2}$$

$$\Rightarrow \omega_r=\omega_n\sqrt{1-2\delta^2}$$

Resonant Peak

It is the peak (maximum) value of the magnitude of $T(j\omega)$. It is denoted by M_r .

At $u=u_r$, the Magnitude of $T(j\omega)$ is -

$$M_r=\frac{1}{\sqrt{(1-u_r^2)^2+(2\delta u_r)^2}}$$

Substitute, $u_r=\sqrt{1-2\delta^2}$ and $1-u_r^2=2\delta^2$ in the above equation.

$$M_r=\frac{1}{\sqrt{(2\delta^2)^2+\left(2\delta\sqrt{1-2\delta^2}\right)^2}}$$

$$\Rightarrow M_r=\frac{1}{2\delta\sqrt{1-\delta^2}}$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio δ . So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth

It is the range of frequencies over which, the magnitude of $T(j\omega)$ drops to 70.7% from its zero frequency value.

At $\omega=0$, the value of u will be zero.

Substitute, $u=0$ in M.

$$M=\frac{1}{\sqrt{(1-0^2)^2+(2\delta(0))^2}}=1$$

Therefore, the magnitude of $T(j\omega)$ is one at $\omega=0$.

At 3-dB frequency, the magnitude of $T(j\omega)$ will be 70.7% of magnitude of $T(j\omega)$ at $\omega=0$.

i.e., at $\omega=\omega_B, M=0.707(1)=\frac{1}{\sqrt{2}}$

$$\Rightarrow M=\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{(1-u_b^2)^2+(2\delta u_b)^2}}$$

$$\Rightarrow 2=(1-u_b^2)^2+(2\delta)^2u_b^2$$

Let, $u_b^2=x$

$$\Rightarrow 2=(1-x)^2+(2\delta)^2x$$

$$\Rightarrow x^2+(4\delta^2-2)x-1=0$$

$$\Rightarrow x=\frac{-(4\delta^2-2)\pm\sqrt{(4\delta^2-2)^2+4}}{2}$$

Consider only the positive value of x.

$$x=1-2\delta^2+\sqrt{(2\delta^2-1)^2+1}$$

$$\Rightarrow x=1-2\delta^2+\sqrt{(2-4\delta^2+4\delta^4)}$$

Substitute, $x=u_b^2=\frac{\omega_b^2}{\omega_n^2}$

$$\frac{\omega_b^2}{\omega_n^2}=1-2\delta^2+\sqrt{(2-4\delta^2+4\delta^4)}$$

$$\Rightarrow \omega_b=\omega_n\sqrt{1-2\delta^2+\sqrt{(2-4\delta^2+4\delta^4)}}$$

$-4\delta^2+4\delta^4)$

Bandwidth ω_b in the frequency response is inversely proportional to the rise time t_r in the time domain transient response.

The Bode plot or the Bode diagram consists of two plots –

- Magnitude plot
- Phase plot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, yaxis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The **magnitude** of the open loop transfer function in dB is -

$$M=20\log|G(j\omega)H(j\omega)|$$

The **phase angle** of the open loop transfer function in degrees is -

$$\phi=\angle G(j\omega)H(j\omega)$$

Note – The base of logarithm is 10.

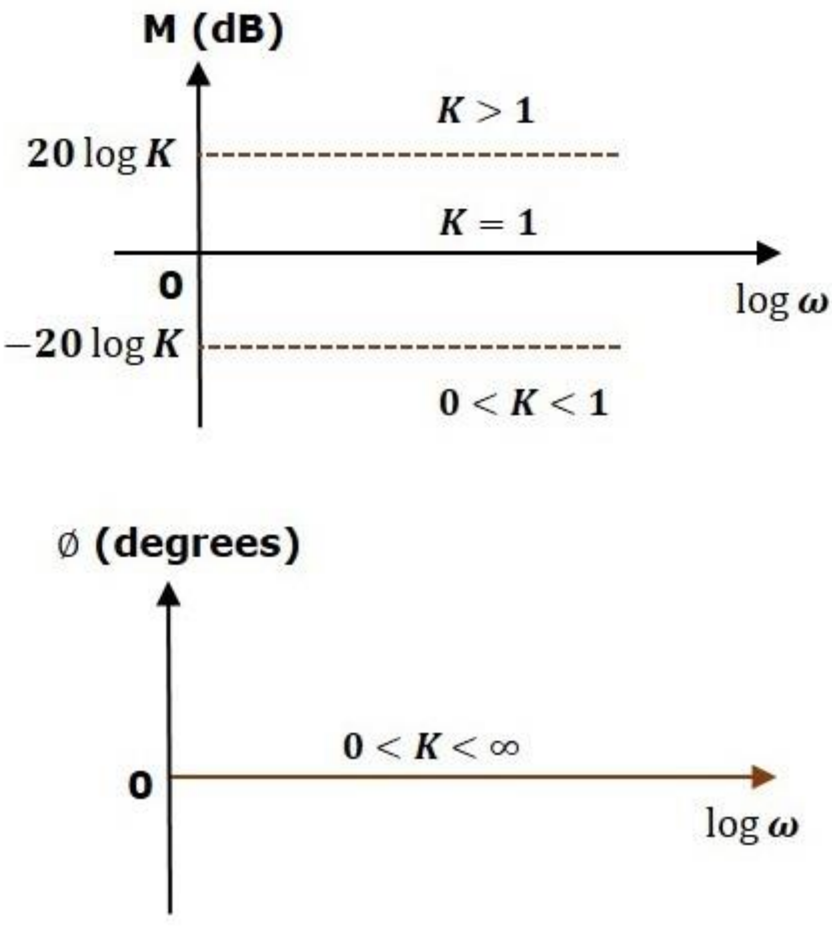
Basic of Bode Plots

The following table shows the slope, magnitude and the phase angle values of the terms present in the open loop transfer function. This data is useful while drawing the Bode plots.

Type of term	G(jω)H(jω)	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	$20\log K$	0
Zero at origin	$j\omega$	20	$20\log\omega$	90
'n' zeros at origin	$(j\omega)^n$	$20n$	$20n\log\omega$	$90n$
Pole at origin	$\frac{1}{j\omega}$	-20	$-20\log\omega$	$-90 \text{ or } 270$
'n' poles at origin	$\frac{1}{(j\omega)^n}$	$-20n$	$-20n\log\omega$	$-90n \text{ or } 270n$
Simple zero	$1+j\omega r$	20	$0 \text{ for } \omega<\frac{1}{r}$ $20\log\omega r \text{ for } \omega>\frac{1}{r}$	$0 \text{ for } \omega<\frac{1}{r}$ $90 \text{ for } \omega>\frac{1}{r}$
Simple pole	$\frac{1}{1+j\omega r}$	-20	$0 \text{ for } \omega<\frac{1}{r}$ $-20\log\omega r \text{ for } \omega>\frac{1}{r}$	$0 \text{ for } \omega<\frac{1}{r}$ $-90 \text{ or } 270 \text{ for } \omega>\frac{1}{r}$
Secon	$\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2j\delta\omega}{\omega_n}\right)$	40	$40\log\omega_n \text{ for } \omega<\omega_n$	$0 \text{ for } \omega<\omega_n$

d order derivative term	$2\omega_n^2+2j\delta\omega\omega_n)$		n $20\log(2\delta\omega_n^2)$ for $\omega=\omega_n$ $40\log\omega$ for $\omega>\omega_n$	90 for $\omega=\omega_n$ 180 for $\omega>\omega_n$
Second order integral term	$\frac{1}{\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2j\delta\omega}{\omega_n}\right)}$	-40	$-40\log\omega_n$ for $\omega<\omega_n$ $-20\log(2\delta\omega_n^2)$ for $\omega=\omega_n$ $-40\log\omega$ for $\omega>\omega_n$	-0 for $\omega<\omega_n$ -90 for $\omega=\omega_n$ -180 for $\omega>\omega_n$

Consider the open loop transfer function $G(s)H(s)=K$.
Magnitude $M=20\log K$ dB
Phase angle $\phi=0$ degrees
If $K=1$, then magnitude is 0 dB.
If $K>1$, then magnitude will be positive.
If $K<1$, then magnitude will be negative.
The following figure shows the corresponding Bode plot.



The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift $20\log K$ dB above the 0 dB line. For the negative values of K, the horizontal line will shift $20\log K$ dB below the 0 dB line. The Zero degrees line itself is the phase plot for all the positive values of K.
Consider the open loop transfer function $G(s)H(s)=s$.

Magnitude $M=20\log\omega$ dB

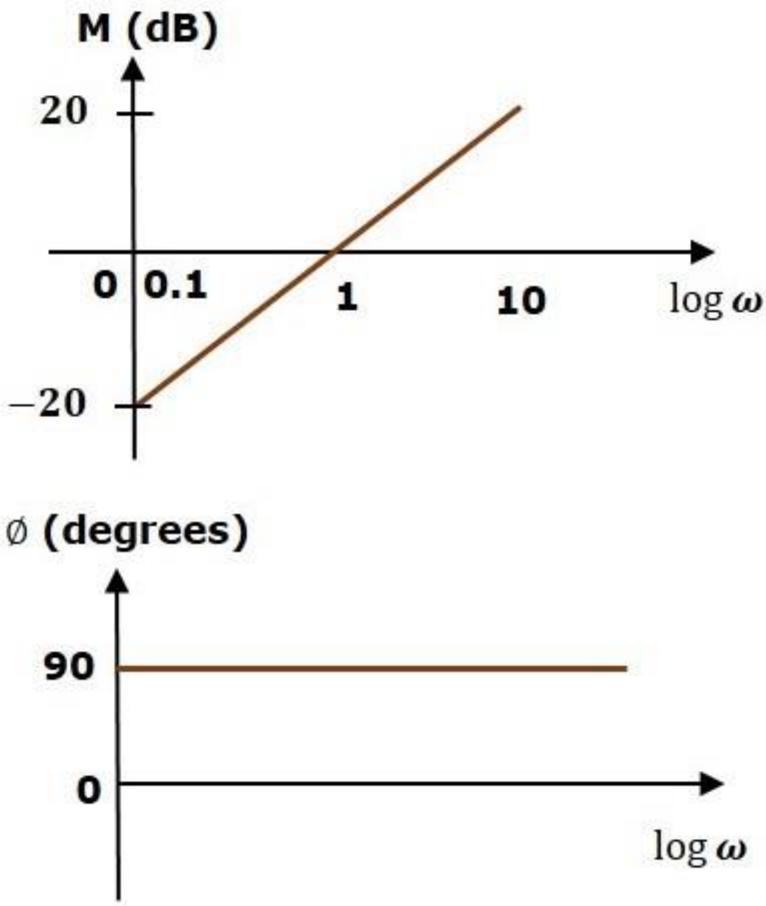
Phase angle $\phi=90^0$

At $\omega=0.1$ rad/sec, the magnitude is -20 dB.

At $\omega=1$ rad/sec, the magnitude is 0 dB.

At $\omega=10$ rad/sec, the magnitude is 20 dB.

The following figure shows the corresponding Bode plot.



The magnitude plot is a line, which is having a slope of 20 dB/dec. This line started at $\omega=0.1$ rad/sec having a magnitude of -20 dB and it continues on the same slope. It is touching 0 dB line at $\omega=1$ rad/sec. In this case, the phase plot is 90^0 line.

Consider the open loop transfer function $G(s)H(s)=1+s\tau$.

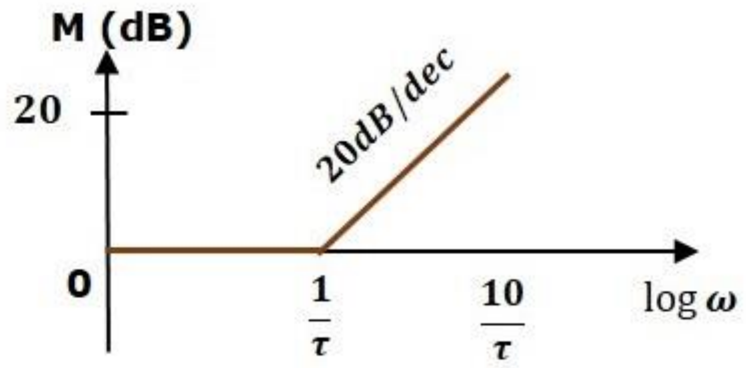
Magnitude $M=20\log\sqrt{1+\omega^2\tau^2}$ dB

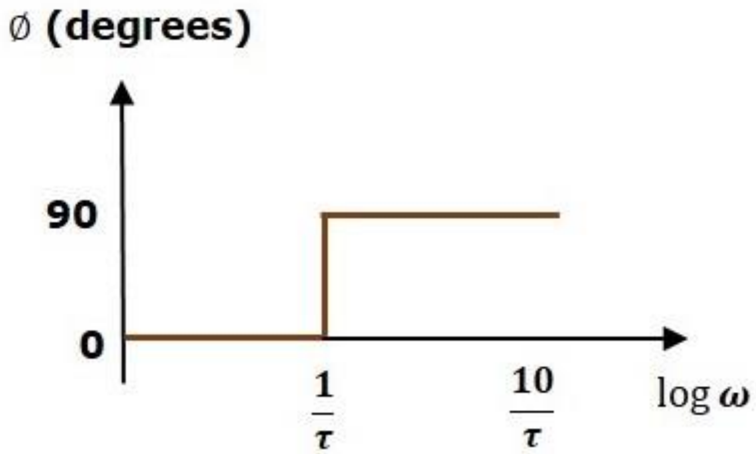
Phase angle $\phi=\tan^{-1}\omega\tau$ degrees

For $\omega<\frac{1}{\tau}$, the magnitude is 0 dB and phase angle is 0 degrees.

For $\omega>\frac{1}{\tau}$, the magnitude is $20\log\omega\tau$ dB and phase angle is 90^0.

The following figure shows the corresponding Bode plot.





The magnitude plot is having magnitude of 0 dB upto $\omega=\frac{1}{\tau}$ω=1τ rad/sec. From $\omega=\frac{1}{\tau}$ω=1τ rad/sec, it is having a slope of 20 dB/dec. In this case, the phase plot is having phase angle of 0 degrees up to $\omega=\frac{1}{\tau}$ω=1τ rad/sec and from here, it is having phase angle of 90^0. This Bode plot is called the **asymptotic Bode plot**. As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

Similarly, you can draw the Bode plots for other terms of the open loop transfer function which are given in the table.

In this chapter, let us understand in detail how to construct (draw) Bode plots.

Rules for Construction of Bode Plots

Follow these rules while constructing a Bode plot.

- Represent the open loop transfer function in the standard time constant form.
- Substitute, $s=j\omega$s=jω in the above equation.
- Find the corner frequencies and arrange them in ascending order.
- Consider the starting frequency of the Bode plot as 1/10th of the minimum corner frequency or 0.1 rad/sec whichever is smaller value and draw the Bode plot upto 10 times maximum corner frequency.
- Draw the magnitude plots for each term and combine these plots properly.
- Draw the phase plots for each term and combine these plots properly.

Note − The corner frequency is the frequency at which there is a change in the slope of the magnitude plot.

Example

Consider the open loop transfer function of a closed loop control system

$$G(s)H(s)=10s(s+2)(s+5)$$ G(s)H(s)=10s(s+2)(s+5)

Let us convert this open loop transfer function into standard time constant form.

$$G(s)H(s)=10s2(s2+1)5(s5+1)$$ G(s)H(s)=10s2(s2+1)5(s5+1)

$$\Rightarrow G(s)H(s)=s(1+s2)(1+s5)$$ ⇒G(s)H(s)=s(1+s2)(1+s5)

So, we can draw the Bode plot in semi log sheet using the rules mentioned earlier.

Stability Analysis using Bode Plots

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency

The frequency at which the phase plot is having the phase of -180^0 is known as **phase cross over frequency**. It is denoted by ω_{pc}ωpc . The unit of phase cross over frequency is **rad/sec**.

Gain Cross over Frequency

The frequency at which the magnitude plot is having the magnitude of zero dB is known as **gain cross over frequency**. It is denoted by ω_{gc}ωgc . The unit of gain cross over frequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

- If the phase cross over frequency ω_{pc}ωpc is greater than the gain cross over frequency ω_{gc}ωgc , then the control system is **stable**.
- If the phase cross over frequency ω_{pc}ωpc is equal to the gain cross over frequency ω_{gc}ωgc , then the control system is **marginally stable**.
- If the phase cross over frequency ω_{pc}ωpc is less than the gain cross over frequency ω_{gc}ωgc , then the control system is **unstable**.

Gain Margin

Gain margin GMGM is equal to negative of the magnitude in dB at phase cross over frequency.

$$GM=20\log(1M_{pc})=20logM_{pc}$$ GM=20log(1Mpc)=20logMpc

Where, M_{pc}Mpc is the magnitude at phase cross over frequency. The unit of gain margin (GM) is **dB**.

Phase Margin

The formula for phase margin PMPM is

$$PM=180_0+\phi_{gc}$$ PM=1800+ϕgc

Where, ϕ_{gc}ϕgc is the phase angle at gain cross over frequency. The unit of phase margin is **degrees**.

The stability of the control system based on the relation between gain margin and phase margin is listed below.

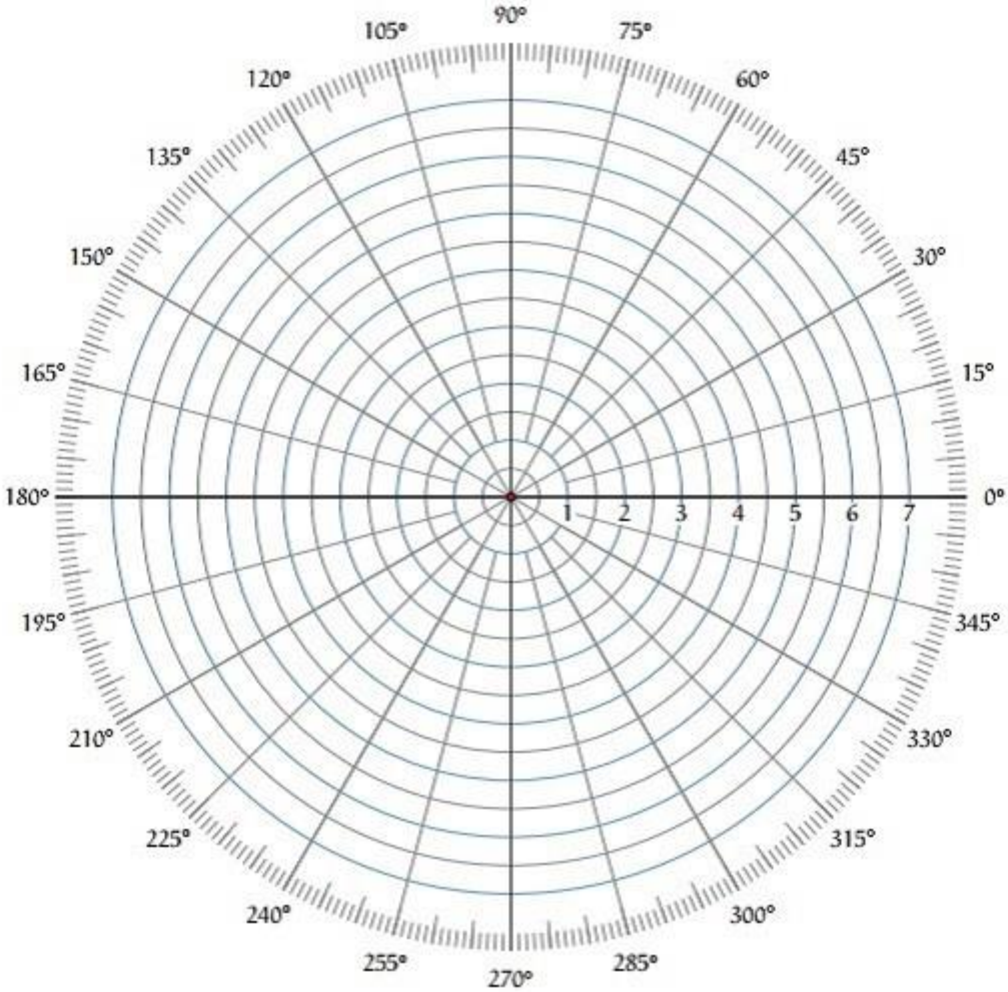
- If both the gain margin GMGM and the phase margin PMPM are positive, then the control system is **stable**.
- If both the gain margin GMGM and the phase margin PMPM are equal to zero, then the control system is **marginally stable**.
- If the gain margin GMGM and / or the phase margin PMPM are/is negative, then the control system is **unstable**.

In the previous chapters, we discussed the Bode plots. There, we have two separate plots for both magnitude and phase as the function of frequency. Let us now discuss about polar plots. Polar plot is a plot which can be drawn between magnitude and phase. Here, the magnitudes are represented by normal values only.

The polar form of $G(j\omega)H(j\omega)$G(jω)H(jω) is

$G(j\omega)H(j\omega)=|G(j\omega)H(j\omega)|\angle G(j\omega)H(j\omega)$G(jω)H(jω)=|G(jω)H(jω)|∠G(jω)H(jω)

The **Polar plot** is a plot, which can be drawn between the magnitude and the phase angle of $G(j\omega)H(j\omega)$G(jω)H(jω) by varying ωω from zero to ∞. The polar graph sheet is shown in the following figure.



This graph sheet consists of concentric circles and radial lines. The **concentric circles** and the **radial lines** represent the magnitudes and phase angles respectively. These angles are represented by positive values in anti-clock wise direction. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle 270^0 in anti-clock wise direction is equal to the angle -90^0 in clockwise direction.

Rules for Drawing Polar Plots

Follow these rules for plotting the polar plots.

- Substitute, $s=j\omega$s=jω in the open loop transfer function.
- Write the expressions for magnitude and the phase of $G(j\omega)H(j\omega)$G(jω)H(jω) .
- Find the starting magnitude and the phase of $G(j\omega)H(j\omega)$G(jω)H(jω) by substituting $\omega=0$ω=0 . So, the polar plot starts with this magnitude and the phase angle.
- Find the ending magnitude and the phase of $G(j\omega)H(j\omega)$G(jω)H(jω) by substituting $\omega=\infty$ω=∞ . So, the polar plot ends with this magnitude and the phase angle.
- Check whether the polar plot intersects the real axis, by making the imaginary term of $G(j\omega)H(j\omega)$G(jω)H(jω) equal to zero and find the value(s) of ωω .
- Check whether the polar plot intersects the imaginary axis, by making real term of $G(j\omega)H(j\omega)$G(jω)H(jω) equal to zero and find the value(s) of ωω .
- For drawing polar plot more clearly, find the magnitude and phase of $G(j\omega)H(j\omega)$G(jω)H(jω) by considering the other value(s) of ωω .

Example

Consider the open loop transfer function of a closed loop control system.

$$G(s)H(s)=5s(s+1)(s+2)$$

Let us draw the polar plot for this control system using the above rules.

Step 1 − Substitute, $s=j\omega$ in the open loop transfer function.

$$G(j\omega)H(j\omega)=5j\omega(j\omega+1)(j\omega+2)$$

The magnitude of the open loop transfer function is

$$M=5\omega(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})$$

The phase angle of the open loop transfer function is

$$\phi=-90^0-\tan^{-1}\omega-\tan^{-1}\frac{\omega}{2}$$

Step 2 − The following table shows the magnitude and the phase angle of the open loop transfer function at $\omega=0$ rad/sec and $\omega=\infty$ rad/sec.

Frequency (rad/sec)	Magnitude	Phase angle(degrees)
0	∞	-90 or 270
∞	0	-270 or 90

So, the polar plot starts at $(\infty,-90^0)$ and ends at $(0,-270^0)$. The first and the second terms within the brackets indicate the magnitude and phase angle respectively.

Step 3 − Based on the starting and the ending polar co-ordinates, this polar plot will intersect the negative real axis. The phase angle corresponding to the negative real axis is -180^0 or 180^0. So, by equating the phase angle of the open loop transfer function to either -180^0 or 180^0, we will get the ω value as $\sqrt{2}$. By substituting $\omega=\sqrt{2}$ in the magnitude of the open loop transfer function, we will get $M=0.83$. Therefore, the polar plot intersects the negative real axis when $\omega=\sqrt{2}$ and the polar coordinate is $(0.83,-180^0)$.

So, we can draw the polar plot with the above information on the polar graph sheet.

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from −∞ to ∞. That means, Nyquist plots are used to draw the complete frequency response of the open loop transfer function.

Nyquist Stability Criterion

The Nyquist stability criterion works on the **principle of argument**. It states that if there are P poles and Z zeros are enclosed by the 's' plane closed path, then the corresponding $G(s)H(s)$ plane must encircle the origin $P-Z$ times. So, we can write the number of encirclements N as,

$$N=P-Z$$

- If the enclosed 's' plane closed path contains only poles, then the direction of the encirclement in the $G(s)H(s)$ plane will be opposite to the direction of the enclosed closed path in the 's' plane.
- If the enclosed 's' plane closed path contains only zeros, then the direction of the encirclement in the $G(s)H(s)$ plane will be in the same direction as that of the enclosed closed path in the 's' plane.

Let us now apply the principle of argument to the entire right half of the 's' plane by selecting it as a closed path. This selected path is called the **Nyquist** contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are in the left half of the 's' plane. So, the poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

- The Poles of the characteristic equation are same as that of the poles of the open loop transfer function.
- The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

We know that the open loop control system is stable if there is no open loop pole in the the right half of the 's' plane.

i.e., $P=0 \Rightarrow N=-Z$

We know that the closed loop control system is stable if there is no closed loop pole in the right half of the 's' plane.

i.e., $Z=0 \Rightarrow N=P$

Nyquist stability criterion states the number of encirclements about the critical point (1+j0) must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the 's' plane. The shift in origin to (1+j0) gives the characteristic equation plane.

Rules for Drawing Nyquist Plots

Follow these rules for plotting the Nyquist plots.

- Locate the poles and zeros of open loop transfer function $G(s)H(s)$ in 's' plane.
- Draw the polar plot by varying ω from zero to infinity. If pole or zero present at s = 0, then varying ω from 0+ to infinity for drawing polar plot.
- Draw the mirror image of above polar plot for values of ω ranging from $-\infty$ to zero (0^- if any pole or zero present at s=0).
- The number of infinite radius half circles will be equal to the number of poles or zeros at origin. The infinite radius half circle will start at the point where the mirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point (-1+j0) lies outside the encirclement, then the closed loop control system is absolutely stable.

Stability Analysis using Nyquist Plots

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180^0) is known as the **phase cross over frequency**. It is denoted by ω_{pc} .

Gain Cross over Frequency

The frequency at which the Nyquist plot is having the magnitude of one is known as the **gain cross over frequency**. It is denoted by ω_{gc} .

The stability of the control system based on the relation between phase cross over frequency and gain cross over frequency is listed below.

- If the phase cross over frequency ω_{pc} is greater than the gain cross over frequency ω_{gc} , then the control system is **stable**.
- If the phase cross over frequency ω_{pc} is equal to the gain cross over frequency ω_{gc} , then the control system is **marginally stable**.
- If phase cross over frequency ω_{pc} is less than gain cross over frequency ω_{gc} , then the control system is **unstable**.

Gain Margin

The gain margin GM is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

Where, M_{pc} is the magnitude in normal scale at the phase cross over frequency.

Phase Margin

The phase margin PM is equal to the sum of 180^0 and the phase angle at the gain cross over frequency.

$$PM = 180^0 + \phi_{gc}$$

Where, ϕ_{gc} is the phase angle at the gain cross over frequency.

The stability of the control system based on the relation between the gain margin and the phase margin is listed below.

- If the gain margin GM is greater than one and the phase margin PM is positive, then the control system is **stable**.
- If the gain margin GM is equal to one and the phase margin PM is zero degrees, then the control system is **marginally stable**.
- If the gain margin GM is less than one and / or the phase margin PM is negative, then the control system is **unstable**.

UNIT VI

1. Compensator design using Bode plot

A compensator or controller is added to a system to improve its steady state as well as dynamic responses.

Nyquist plot is difficult to modify after introducing controller.

Instead Bode plot is used since two important design criteria, phase margin and gain crossover frequency are visible from the Bode plot along with gain margin.

Points to remember

• Low frequency asymptote of the magnitude curve is indicative of one of the error constants Kp,Kv,Ka depending on the system types.

• Specifications on the transient response can be translated into phase margin (PM), gain margin (GM), gain crossover frequency, bandwidth etc.

• Design using bode plot is simple and straight forward.

• Reconstruction of Bode plot is not a difficult task.

Phase lead, Phase lag and Lag-lead compensators

Phase lead, phase lag and lag-lead compensators are widely used in frequency domain design.

Before going into the details of the design procedure, we must remember the following.

• Phase lead compensation is used to improve stability margins. It increases system bandwidth thus improving the speed of the response.

• Phase lag compensation reduces the system gain at high frequencies without reducing low frequency gain. Thus the total gain/low frequency gain can be increased which in turn will improve the steady state accuracy. High frequency noise can also be attenuated. But stability margin and bandwidth reduce.

• Using a lag lead compensator, where a lag compensator is cascaded with a lead compensator, both steady state and transient responses can be improved.

Bi-linear transformation transfers the loop transfer function in *z* -plane to *w* -plane.

Since qualitatively *w* -plane is similar to *s* -plane, design technique used in *s* -plane can be employed to design a controller in *w* -plane.

Once the design is done in *w* -plane, controller in *z* -plane can be determined by using the inverse transformation from *w* -plane to *z* -plane.

In the next two lectures we will discuss compensator design in *s* -plane and solve examples to design digital controllers using the same concept.

1.1) Phase lead compensator

If we look at the frequency response of a simple PD controller, it is evident that the magnitude of the compensator continuously grows with the increase in frequency.

The above feature is undesirable because it amplifies high frequency noise that is typically present in any real system.

In lead compensator, a first order pole is added to the denominator of the PD controller at frequencies higher than the corner frequency of the PD controller. Frequency response of a lead compensator is shown in the figure 1.1.

A typical lead compensator has the following transfer function.

$$C(s) = K\frac{\tau s + 1}{\alpha \tau s + 1}, \ \alpha < 1$$

$\frac{1}{\alpha}$ Is the ratio between the pole zero break point (corner) frequencies. Magnitude of the lead compensator is

$$K\frac{\sqrt{1+\omega^2\tau^2}}{\sqrt{1+\alpha^2\omega^2\tau^2}}$$

And the phase contributed by the lead compensator is given by

$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$$

Thus a significant amount of phase is still provided with much less amplitude at high frequencies.

The frequency response of a typical lead compensator is shown in Figure 1 where the magnitude varies from

$20\log_{10} K$ to $20\log_{10}\frac{K}{\alpha}$

and maximum phase is always less than 90° (around 60° in general).

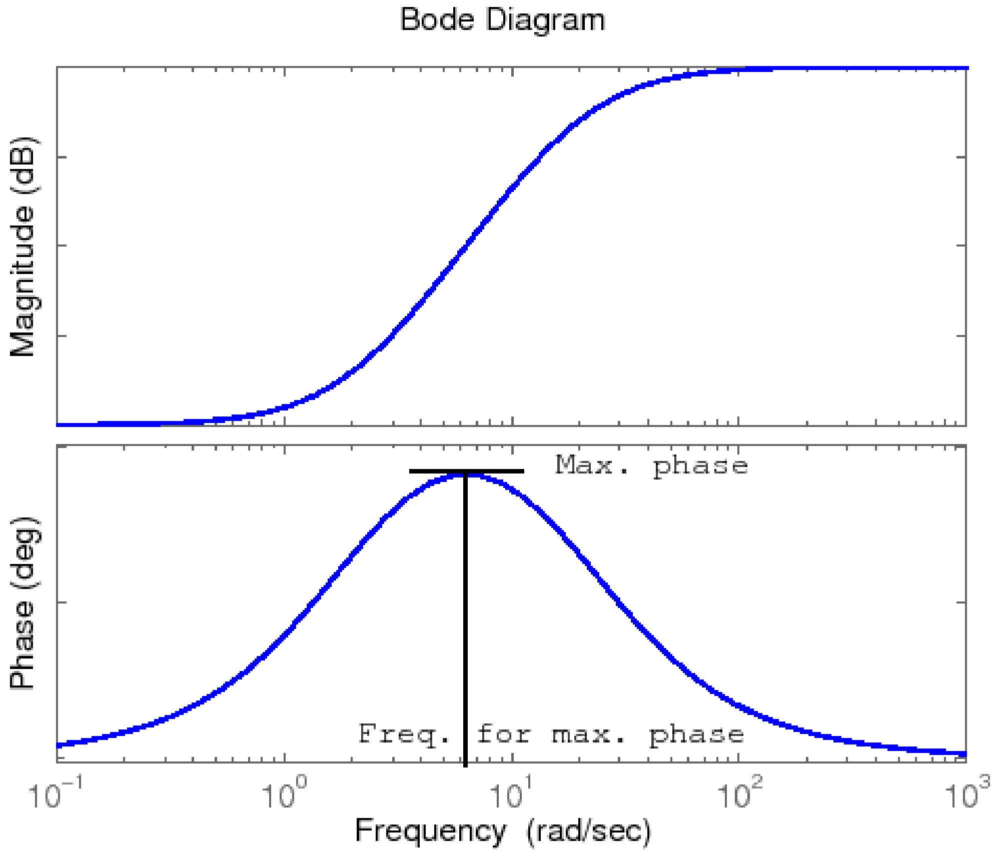


Figure 1.1: Frequency response of a lead compensator

It can be seen that the frequency where the phase is maximum is given by

$$\omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$$

The maximum phase corresponds to

$$\sin \phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

$$\Rightarrow \alpha = \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right)$$

The magnitude of

$C(s)$ at $\omega_{\max}$ is $\frac{K}{\sqrt{\alpha}}$.

1.2) Lag Compensator Design

In the previous lecture we discussed lead compensator design. In this lecture we would see how to design a phase lag compensator

Phase lag compensator

The essential feature of a lag compensator is to provide an increased low frequency gain, thus decreasing the steady state error, without changing the transient response significantly.

For frequency response design it is convenient to use the following transfer function of a lag compensator.

$$C_{lag}(s) = \alpha \frac{\tau s + 1}{\alpha \tau s + 1},$$

Where

$\alpha > 1$

The above expression is only the lag part of the compensator. The overall compensator is

$$C(s) = K C_{lag}(s)$$

$$\text{when, } s \to 0, \quad C_{lag}(s) \to \alpha$$
$$\text{when, } s \to \infty, \quad C_{lag}(s) \to 1$$

Frequency response of a lag compensator is shown in fig: 1.2. Typical objective of lag compensator design is to provide an additional gain of α in the low frequency region and to leave the system with sufficient phase margin.

The frequency response of a lag compensator, with α=4 and τ=3, is shown in Figure 1 where the magnitude varies from

$20\log_{10}\alpha$ dB to 0 dB.

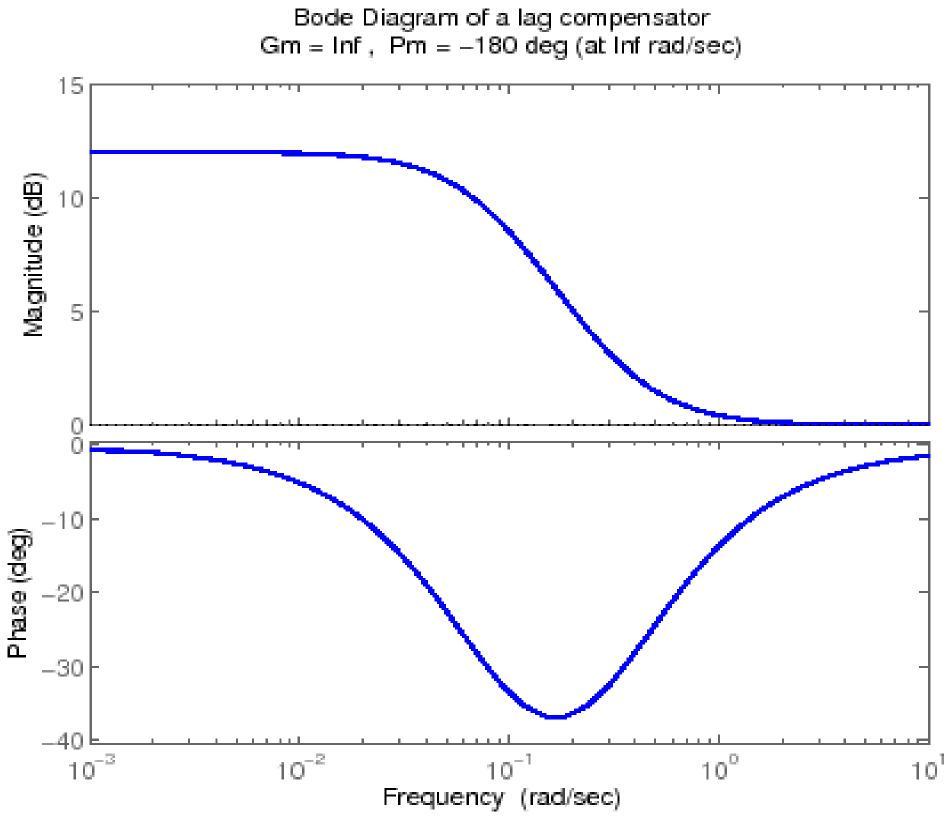


Figure 1.2: Frequency response of a lag compensator

Since the lag compensator provides the maximum lag near the two corner frequencies, to maintain the PM of the system, zero of the compensator should be chosen such that $\omega = 1/\tau$ is much lower than the gain crossover frequency of the uncompensated system.

In general, τ is designed such that $1/\tau$ is at least one decade below the gain crossover frequency of the uncompensated system. Following example will be comprehensive to understand the design procedure.

1.3) *Lag -lead Compensator*

When a single lead or lag compensator cannot guarantee the specified design criteria, a lag- lead compensator is used.

Frequency response of a lag-lead compensator is shown in fig: 1.3 .In lag-lead compensator the lag part precedes the lead part. A continuous time lag-lead compensator is given by

$$C(s) = K \frac{1+\tau_1 s}{1+\alpha_1\tau_1 s}\frac{1+\tau_2 s}{1+\alpha_2\tau_2 s}$$

where,

$$\alpha_1 > 1, \quad \alpha_2 < 1$$

The corner frequencies are

$$\frac{1}{\alpha_1\tau_1}, \frac{1}{\tau_1}, \frac{1}{\tau_2}, \frac{1}{\alpha_2\tau_2}.$$

The frequency response is shown in Figure 1.

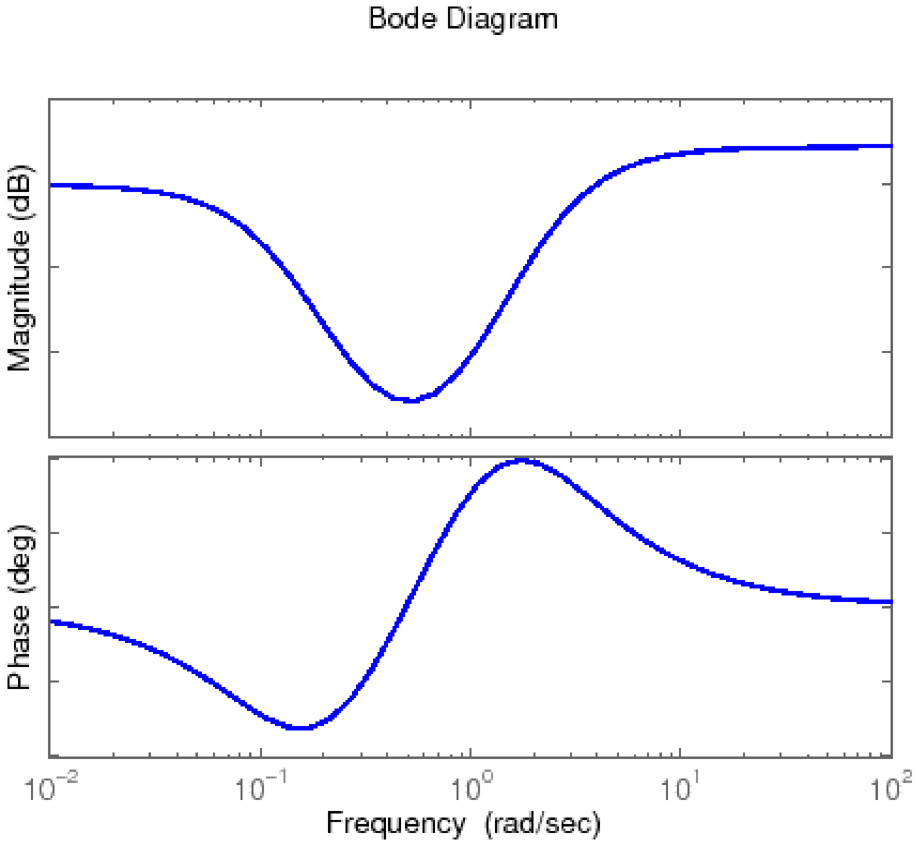


Figure 1.3: Frequency response of a lag-lead compensator

• If it is not specified which type of compensator has to be designed, one should first check the PM and BW of the uncompensated system with adjustable gain *K*.

• If the BW is smaller than the acceptable BW one may go for lead compensator. If the BW is large, lead compensator may not be useful since it provides high frequency amplification.

• One may go for a lag compensator when BW is large provided the open loop system is stable.

• If the lag compensator results in a too low BW (slow speed of response), a lag-lead compensator may be used.

1.4) *Lead or Phase-Lead Compensator Using Root Locus*

A first-order lead compensator can be designed using the root locus. A lead compensator in root locus form is given by

where the magnitude of z is less than the magnitude of p. A phase-lead compensator tends to shift the root locus toward the left half plane. This results in an improvement in the system's stability and an increase in the response speed.

When a lead compensator is added to a system, the value of this intersection will be a larger negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a larger negative number than the added zero. Thus, the result of a lead compensator is that the asymptotes' intersection is moved further into the left half plane, and the entire root locus will be shifted to the left. This can increase the region of stability as well as the response speed.

1.5) *Lag or Phase-Lag Compensator Using Root Locus*

A first-order lag compensator can be designed using the root locus. A lag compensator in root locus form is given by

where the magnitude of z is greater than the magnitude of p. A phase-lag compensator tends to shift the root locus to the right, which is undesirable. For this reason, the pole and zero of a lag compensator must be placed close together (usually near the origin) so they do not appreciably change the transient response or stability characteristics of the system.

When a lag compensator is added to a system, the value of this intersection will be a smaller negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a smaller negative number than the added zero. Thus, the result of a lag compensator is that the asymptotes' intersection is moved closer to the right half plane, and the entire root locus will be shifted to the right.

1.6) *Lead-lag Compensator using either Root Locus or Frequency Response*

A lead-lag compensator combines the effects of a lead compensator with those of a lag compensator. The result is a system with improved transient response, stability and steady- state error. To implement a lead-lag compensator, first design the lead compensator to achieve the desired transient response and stability, and then add on a lag compensator to improve the steady-state response.

3. *Feedback compensation:*

3.1) Necessary of Compensation

1. In order to obtain the desired performance of the system, we use compensating networks. Compensating networks are applied to the system in the form of feed forward path gain adjustment.
2. Compensate an unstable system to make it stable.
3. A compensating network is used to minimize overshoot.
4. These compensating networks increase the steady state accuracy of the system. An important point to be noted here is that the increase in the steady state accuracy brings instability to the system.
5. Compensating networks also introduces poles and zeros in the system thereby causes changes in the transfer function of the system. Due to this, performance specifications of the system change.

3.2) *Methods of Compensation*

1. Connecting compensating circuit between error detector and plants known as series compensation as shown in fig 1.7

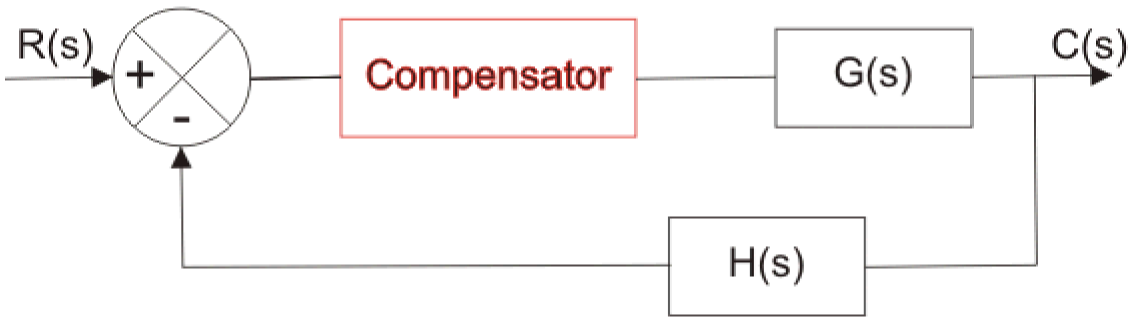


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Fig 1.7: Series Compensator

When a compensator used in a feedback manner as shown in fig 1.8 it is called feed back compensation

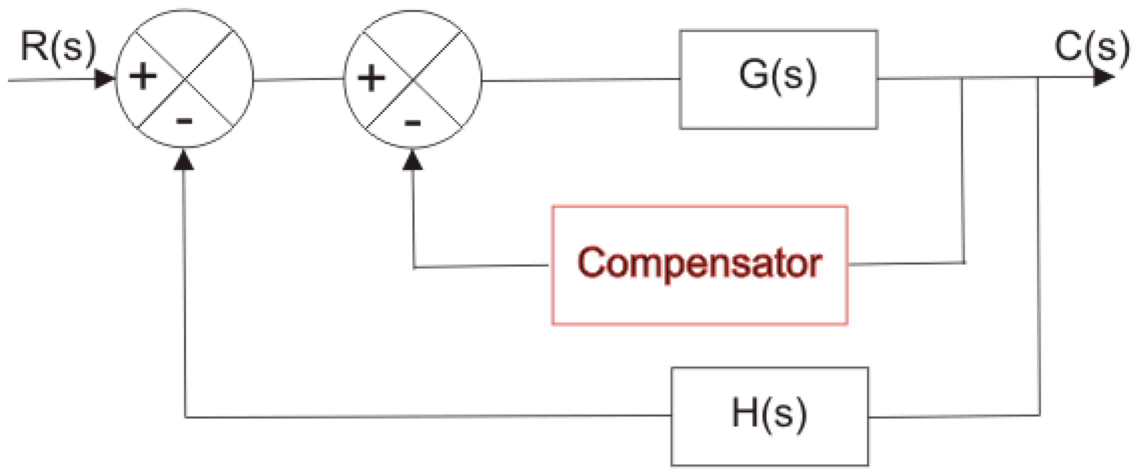

Fig 1.8 feedback compensator

A combination of series and feedback compensator is called load compensator as shown in fig 1.9.

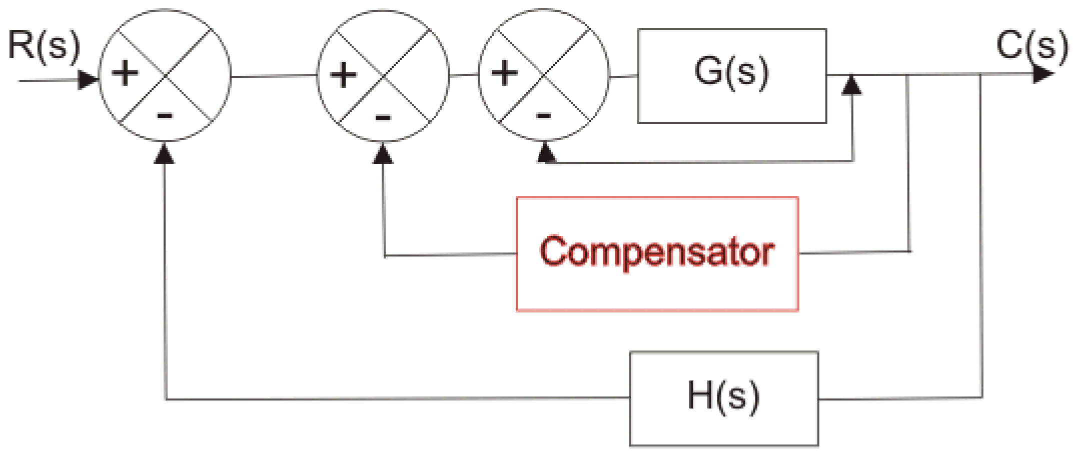

Fig 1.9 Load compensator