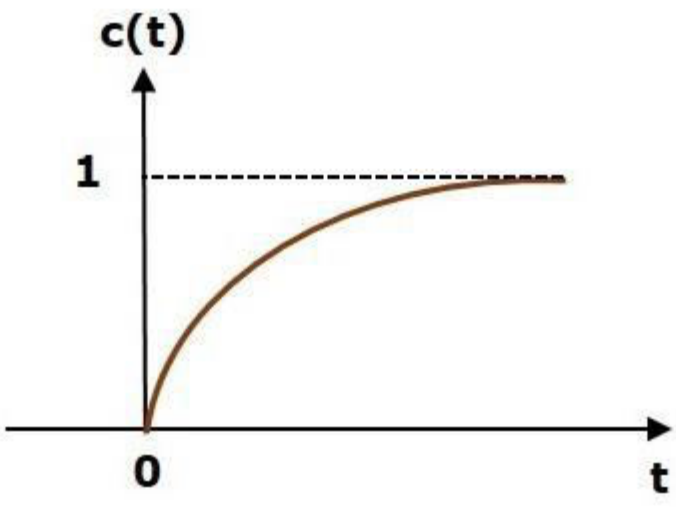
UNIT-IV

Stability of control systems:
Routh-Hurwitz criterion- root locus- rules for the construction of root loci- introduction to proportional- derivative and integral controllers.

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systems based on Stability :

We can classify the systems based on stability as follows.

1) Absolutely stable system

2) Conditionally stable system

3)Marginally stable system

Absolutely Stable System::

If the system is stable for all the range of system component values, then it is known as the absolutely stable system. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of ‘s’ plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the ‘s’ plane.

Conditionally Stable System::

If the system is stable for a certain range of system component values, then it is known as conditionally stable system.

Marginally Stable System::

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as marginally stable system. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

Routh-Hurwitz Stability Criterion::

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable.
But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

Necessary Condition for Routh-Hurwitz Stability::

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Sufficient Condition for Routh-Hurwitz Stability::

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Routh Array Method::

If all the roots of the characteristic equation exist to the left half of the ‘s’ plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the ‘s’ plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable.

But, it is difficult to find the roots of the characteristic equation as order increases.
So, to overcome this problem there we have the Routh array method.
In this method, there is no need to calculate the roots of the characteristic equation.
First formulate the Routh table and find the number of the sign changes in the first column of the Routh table.
The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the ‘s’ plane and the control system is unstable.

The following table shows the Routh array of the nth order characteristic polynomial.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0$$

s^n	a_0	a_2	a_4	a_6	$\ldots$	$\ldots$
s^{n-1}	a_1	a_3	a_5	a_7	$\ldots$	$\ldots$
s^{n-2}	$b_1 = \frac{a_1a_2 - a_3a_0}{a_1}$	$b_2 = \frac{a_1a_4 - a_5a_0}{a_1}$	$b_3 = \frac{a_1a_6 - a_7a_0}{a_1}$	$\ldots$	$\ldots$	$\ldots$
s^{n-3}	$c_1 = \frac{b_1a_3 - b_2a_1}{b_1}$	$c_2 = \frac{b_1a_55 - b_3a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
s^1	$\vdots$	$\vdots$				
s^0	a_n					

1) find the stability of the control system having characteristic equation,

$$s^4+3s^3+3s^2+2s+1=0$$

Sol::

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4+3s^3+3s^2+2s+1$ are positive. So, the control system satisfies the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\frac{(3\times3)-(2\times1)}{3}=\frac{7}{3}$	$\frac{(3\times1)-(0\times1)}{3}=\frac{3}{3}$ $=1$	
s^1	$\frac{\left(\frac{7}{3}\times2\right)-(1\times3)}{\frac{7}{3}}$ $=\frac{5}{7}$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive.
There is no sign change in the first column of the Routh array.
All the 3 roots of the characteristic equation will lie on left half of the S-plane.
So, the control system is stable.

<u>Special Cases of Routh Array ::</u>

The two special cases are −

1) The first element of any row of the Routh's array is zero.

2) All the elements of any row of the Routh's array are zero.

1) First Element of any row of the Routh's array is zero

If any row of the Routh's array contains only the first element as zero then replace the first element with a small positive integer, ϵ. And then continue the process of completing the Routh's table. Now, find the number of sign changes in the first column of the Routh's table by substituting ϵϵ tends to zero.

find the stability of the control system having characteristic equation,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4 + 2s^3 + s^2 + 2s + 1 = 0$ are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^4	1	1	1
s^3	$\not{2}\ 1$	$\not{2}\ 1$	
s^2	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(0\times1)}{1} = 1$	
s^1			
s^0			

The first element of row s^2 is zero. So, replace it by ϵ and continue the process of completing the Routh table.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$\frac{(\epsilon \times 1)-(1 \times 1)}{\epsilon} = \frac{\epsilon - 1}{\epsilon}$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.

As ϵ tends to zero, the Routh table becomes like this.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

2) All the Elements of any row of the Routh's array are zero

In this case, follow these two steps −

Write the auxilary equation, A(s) of the row, which is just above the row of zeros.

Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

find the stability of the control system having characteristic equation,

$$s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

Step 1 − Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the given characteristic polynomial are positive. So, the control system satisfied the necessary condition.

Step 2 − Form the Routh array for the given characteristic polynomial.

s^5	1	1	1
s^4	~~3~~ 1	~~3~~ 1	~~3~~ 1
s^3	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(1\times1)}{1} = 0$	
s^2			
s^1			
s^0			

The row s^4 elements have the common factor of 3. So, all these elements are divided by 3.

Special case (ii) − All the elements of row s^3 are zero. So, write the auxiliary equation, A(s) of the row s^4.

$$A(s) = s^4 + s^2 + 1$$

$$\frac{\mathrm{d}A(s)}{\mathrm{d}s} = 4s^3 + 2s$$

Place these coefficients in row s^3.

s^5	1	1	1
s^4	1	1	1
s^3	~~4~~ 2	~~2~~ 1	
s^2	$\frac{(2\times1)-(1\times1)}{2} = 0.5$	$\frac{(2\times1)-(0\times1)}{2} = 1$	
s^1	$\frac{(0.5\times1)-(1\times2)}{0.5} = \frac{-1.5}{0.5}$ $= -3$		
s^0	1		

Step 3 − Verify the sufficient condition for the Routh-Hurwitz stability.
There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

with all zeros, th
is stable.

b. If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

c. If there is a row of all zeros after letting $\in \to 0$, then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial and divide the characteristic equation by auxiliary polynomial to eliminate the imaginary roots. The routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpreted as explained in method-2 of case-II polynomial

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EXAMPLE 5.1

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is $s^4+8s^3+18s^2+16s+5=0$

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	18	5	.... Row-1
s^3	:	8	16		.... Row-2

The elements of s^3 row can be divided by 8 to simplify the computations.

s^4	:	1	18	5	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	16	5		.... Row-3
s^1	:	1.7			.... Row-4
s^0	:	5			.... Row-5
		↑ Column-1			

$$s^2 : \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$$

$$s^2 : 16 \quad 5$$

$$s^1 : \frac{16\times2-5\times1}{16}$$

$$s^1 : 1.6875 \approx 1.7$$

$$s^0 : \frac{1.7\times5-0\times16}{1.7}$$

$$s^0 : 5$$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

EXAMPLE 5.2

Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

The characteristic equation of the system is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

The given characteristic polynomial is 6^{th} order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^6	:	1	8	20	16	.... Row-1
s^5	:	2	12	16		.... Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.

s^6	:	1	8	20	16	....Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-3
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

Column-1

$$s^4 : \frac{1\times8-6\times1}{1} \quad \frac{1\times20-8\times1}{1} \quad \frac{1\times16-0\times1}{1}$$

s^4 : 2 12 16

divide by 2

s^4 : 1 6 8

$$s^3 : \frac{1\times6-6\times1}{1} \quad \frac{1\times8-8\times1}{1}$$

s^3 : 0 0

The auxiliary equation is $A = s^4+6s^2+8$. On differentiating A with respect to s we get

$$\frac{dA}{ds} = 4s^3 + 12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row s^3 : 4 12

divide by 4

s^3 : 1 3

$$s^2 : \frac{1\times6-3\times1}{1} \quad \frac{1\times8-0\times1}{1}$$

s^2 : 3 8

$$s^1 : \frac{3\times3-8\times1}{3}$$

s^1 : 0.33

$$s^0 : \frac{0.33\times8-0\times3}{0.33}$$

s^0 : 8

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is

$$s^4+6s^2+8=0$$

Let $s^2 = x$

$$\therefore x^2+6x+8=0$$

The roots of quadratic are, $x = \frac{-6\pm\sqrt{6^2-4\times8}}{2}$

$$= -3\pm1 = -2 \text{ or } -4$$

The roots of auxiliary polynomial is, $s = \pm\sqrt{x} = \pm\sqrt{-2}$ and $\pm\sqrt{-4}$

$$= +j\sqrt{2},\ -j\sqrt{2},\ +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

[illegible]

Construct Routh array and determine the stability of the system represented by the characteristic equation $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is

$$s^5+s^4+2s^3+2s^2+3s+5=0$$

The given characteristic polynomial is 5^{th} order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{lllll}
s^5 : & 1 & 2 & 3 & \text{.... Row-1} \\
s^4 : & 1 & 2 & 5 & \text{.... Row-2} \\
s^3 : & \in & -2 & & \text{.... Row-3} \\
s^2 : & \dfrac{2\in+2}{\in} & 5 & & \text{.... Row-4} \\
s^1 : & \dfrac{-(5\in^2+4\in+4)}{2\in+2} & & & \text{.... Row-5} \\
s^0 : & 5 & & & \text{.... Row-6}
\end{array}$$

$$s^3 : \frac{1\times 2-2\times 1}{1} \quad \frac{1\times 3-5\times 1}{1}$$

$$s^3 : \quad 0 \quad -2$$

Replace :0 by $\in$

$$s^3 : \in \quad -2$$

$$s^2 : \frac{\in\times 2-(-2\times 1)}{\in} \quad \frac{\in\times 5-0\times 1}{\in}$$

$$s^2 : \frac{2\in+2}{\in} \quad 5$$

$$s^1 : \frac{\frac{2\in+2}{\in}\times(-2)-(5\times\in)}{\frac{2\in+2}{\in}}$$

$$s^1 : \frac{-(5\in^2+4\in+4)}{2\in+2}$$

$$s^0 : \frac{\frac{-(5\in^2+4\in+4)}{2\in+2}\times 5-0\times\frac{2\in+2}{\in}}{\frac{-(5\in^2+4\in+4)}{2\in+2}}$$

$$s^0 : 5$$

On letting $\in\to 0$, we get

$$\begin{array}{lllll}
s^5 : & 1 & 2 & 3 & \text{.... Row-1} \\
s^4 : & 1 & 2 & 5 & \text{.... Row-2} \\
s^3 : & 0 & -2 & & \text{.... Row-3} \\
s^2 : & \infty & 5 & & \text{.... Row-4} \\
s^1 : & -2 & & & \text{.... Row-5} \\
s^0 : & 5 & & & \text{.... Row-6}
\end{array}$$

Column-1

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

RESULT

1. The system is unstable.
2. Two roots are lying on the right half of s-plane and three roots are lying on the left half of s-plane.

EXAMPLE 5.4

By routh stability criterion determine the stability of the system represented by the characteristic equation $9s^5-20s^4+10s^3-s^2-9s-10=0$. Comment on the location of roots of characteristic equation.

SOLUTION

The characteristic polynomial of the system is $9s^5-20s^4+10s^3-s^2-9s-10=0$

On examining the coefficients of the characteristic polynomial, it is found that some of the coefficients are negative and so some roots will lie on the right of s-plane. Hence the system is unstable. The routh array can be constructed to find the number of roots lying on right half of s-plane.

$$9s^5-20s^4+10s^3-s^2-9s-10=0$$

The given characteristic polynomial is 5^{th} order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^5	:	9	10	−9	.... Row-1
s^4	:	−20	−1	−10	.... Row-2
s^3	:	9.55	−13.5		.... Row-3
s^2	:	−29.3	−10		.... Row-4
s^1	:	−16.8			.... Row-5
s^0	:	−10			.... Row-6

↑ Column-1

$s^3: \dfrac{-20\times10-(-1)\times9}{-20} \quad \dfrac{-20\times(-9)-(-10)\times9}{-20}$

$s^3: 9.55 \quad -13.5$

$s^2: \dfrac{9.55\times(-1)-(-13.5)\times(-20)}{9.55} \quad \dfrac{9.55\times(-10)}{9.55}$

$s^2: -29.3 \quad -10$

$s^1: \dfrac{-29.3\times(-13.5)-(-10)\times9.55}{-29.3}$

$s^1: -16.8$

$s^0: \dfrac{-16.8\times(-10)}{-16.8}$

$s^0: -10$

By examining the elements of I[st] column of routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is unstable
2. Three roots are lying on the right half and two roots are lying on the left half of s-plane.

EXAMPLE 5.5

The characteristic polynomial of a system is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7[th] order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations,

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5
s^3	:	2	1			.... Row-5
s^2	:	0.5	1			.... Row-6
s^1	:	−3				.... Row-7
s^0	:	1				.... Row-8

← Column-1

$s^5 : \frac{3\times24-8\times1}{3} \quad \frac{3\times24-8\times1}{3} \quad \frac{3\times23-5\times1}{3}$

$s^5 : 21.33 \quad 21.33 \quad 21.33$

Divide by 21.33

$s^5 : 1 \quad 1 \quad 1$

$s^4 : \frac{1\times8-1\times3}{1} \quad \frac{1\times8-1\times3}{1} \quad \frac{1\times5-0\times3}{1}$

$s^4 : 5 \quad 5 \quad 5$

Divide by 5

$s^5 : 1 \quad 1 \quad 1$

$s^3 : \frac{1\times1-1\times1}{1} \quad \frac{1\times1-1\times1}{1}$

$s^3 : 0 \quad 0$

The auxiliary polynomial is

$$A = s^4 + s^2 + 1$$

Differentiate A with respect to s.

$$\frac{dA}{ds} = 4s^3 + 2s$$

$s^3 : 4 \quad 2$

Divide by 2

$s^3 : 2 \quad 1$

$s^2 : \frac{2\times1-1\times1}{2} \quad \frac{2\times1-0\times1}{2}$

$s^2 : 0.5 \quad 1$

$s^1 : \frac{0.5\times1-1\times2}{0.5}$

$s^1 : -3$

$s^0 : \frac{-3\times1}{-3}$

$s^0 : 1$

On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicate the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is $s^4 + s^2 + 1 = 0$

Put $s^2 = x$ in the auxiliary equation,

$$\therefore s^4 + s^2 + 1 = x^2 + x + 1 = 0$$

The roots of quadratic are, $x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$= 1\angle 120^\circ$ or $1\angle -120^\circ$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}$ or $\pm\sqrt{1\angle -120^\circ}$

$= \pm\sqrt{1}\angle 120^\circ/2$ or $\pm\sqrt{1}\angle -120^\circ/2$

$= \pm 1\angle 60^\circ$ or $\pm 1\angle -60^\circ$

$= \pm(0.5 + j0.866)$ or $\pm(0.5 - j0.866)$

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on the left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh array. The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

Determine the range of K for stability of unity feedback system whose open transfer function is $G(s) = \frac{K}{s(s+1)(s+2)}$.

SOLUTION

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$= \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+}$$

The characteristic equation is

$$s(s+1)(s+2)+K=0$$
$$s(s^2+3s+2)+K=0$$
$$s^3+3s^2+2s+K=0$$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^3	:	1	2
s^2	:	3	K
s^1	:	$\frac{6-K}{3}$	
s^0	:	K	

Column-1

$$s^1 : \frac{3\times2-K\times1}{3}$$
$$s^1 : \frac{6-K}{3}$$
$$s^0 : \frac{\frac{6-K}{3}\times K-0\times3}{(6-K)/3}$$
$$s^0 : K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K>0$

From s^1 row, for the system to be stable, $\frac{6-K}{3}>0$

For $\frac{6-K}{3}>0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

RESULT

The value of K is in the range $0<K<6$ for the system to be stable.

EXAMPLE 5.10

The open loop transfer function of a unity feedback control system is given by

$$G(s)=\frac{K}{(s+2)(s+4)(s^2+6s+25)}.$$

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOLUTION

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}}{1+\dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}}$$

$$= \frac{K}{(s+2)\,(s+4)\,(s^2+6s+25)+K}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is

$$(s+2)\,(s+4)\,(s^2+6s+25)+K=0$$

$$(s^2+6s+8)\,(s^2+6s+25)+K=0$$

$$s^4+12s^3+69s^2+198s+200+K=0$$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	69	200+K	..... Row-1
s^3	:	12	198		..... Row-2

Divide s^3 row by 12 to simplify the calculations

		Column-1			
s^4	:	1	69	200+K	..... Row-1
s^3	:	1	16.5		..... Row-2
s^2	:	52.5	200+K		..... Row-3
s^1	:	$\frac{666.25-K}{52.5}$			..... Row-4
s^0	:	200+K			..... Row-5

$s^2 : \dfrac{1\times 69-16.5\times 1}{1} \quad \dfrac{1\times(200+K)}{1}$

$s^2 : 52.5 \quad 200+K$

$s^1 : \dfrac{52.5\times 16.5-(200+K)\times 1}{52.5}$

$s^1 : \dfrac{666.25-K}{52.5}$

$s^0 : \dfrac{\dfrac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$

$s^0 : 200+K$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^1 row, for the system to be stable, $(666.25-K) > 0$

Since $(666.25-K)>0$, K should be lessthan 666.25.

From s^0 row, for the system to be stable, $(200+K) > 0$

Since $(200+K) > 0$, K should be greaterthan -200, but practical values of K starts from 0. Hence K should be greater than 0.

$\therefore$ The range of K for the system to be stable is $0<K<666.25$.

When $K= 666.25$ the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When $K=666.25$, the coefficients of auxiliary equation are given by the s^2 row.

$\therefore$ The auxiliary equation is $52.5s^2+200+K=0$

$$52.5s^2 + 200 + 666.25 = 0$$

$$s^2 = \frac{-200-666.25}{52.5} = -16.5$$

$$s = \pm j4.06$$

When $K = 666.25$, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

$\therefore$ The frequency of oscillation, $\omega = 4.06$ rad/sec.

RESULT

1. The range of K for stability is $0<K<666.25$
2. The system oscillates when $K = 666.25$
3. The frequency of oscillation, $\omega = 4.06$ rad/sec (When $K = 666.25$).

EXAMPLE 5.11

The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K(s+1)}{s^3 + as^2 + 2s + 1}$$

. Determine the value of K and a so that the system oscillates at a frequency of 2 rad/sec.

The above all problems are ROUTH HURWITZ criterion problems from NAGOOR KANI TEXT BOOK

Refer text book for any doubts

Root locus Technique ::

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles are in on left half of the 's' plane or on the right half of the 's' plane or on an imaginary axis. So, we can't find the nature of the control system. To overcome this limitation, there is a technique known as the root locus.

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

The root locus technique was introduced by **W.R.Evans** in 1948 for the analysis of control systems. The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called ***root loci*** (or the path taken by a root of characteristic equation when open loop gain K is varied from 0 to ∞ is called root locus).

VARIOUS STEPS IN THE PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Location of poles and zeros

Draw the real and imaginary axis on an ordinary graph sheet and choose same scales both on real and imaginary axis.

The poles are marked by cross **"X"** and zeros are marked by small circle **"o"**. The number of root locus branches is equal to number of poles of open loop transfer function.

Let, n = number of poles

m = number of finite zeros

Now, **m** root locus branches ends at finite zeros. The remaining **n−m** root locus branches will end at zeros at infinity.

Step 2 : Root locus on real axis

In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Step 3 : Angles of asymptotes and centroid

If n is number of poles and m is number of finite zeros, then n−m root locus branches will terminate at zeros at infinity.

These n–m root locus branches will go along an asymptotic path and meets the asymptotes at infinity. Hence number of asymptotes is equal to number of root locus branches going to infinity. The angles of asymptotes and the centroid are given by the following formulae.

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

$$\text{where, } q = 0, 1, 2, 3, \ldots\ldots(n-m)$$

$$\text{Centroid (meeting point of asymptote with real axis)} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

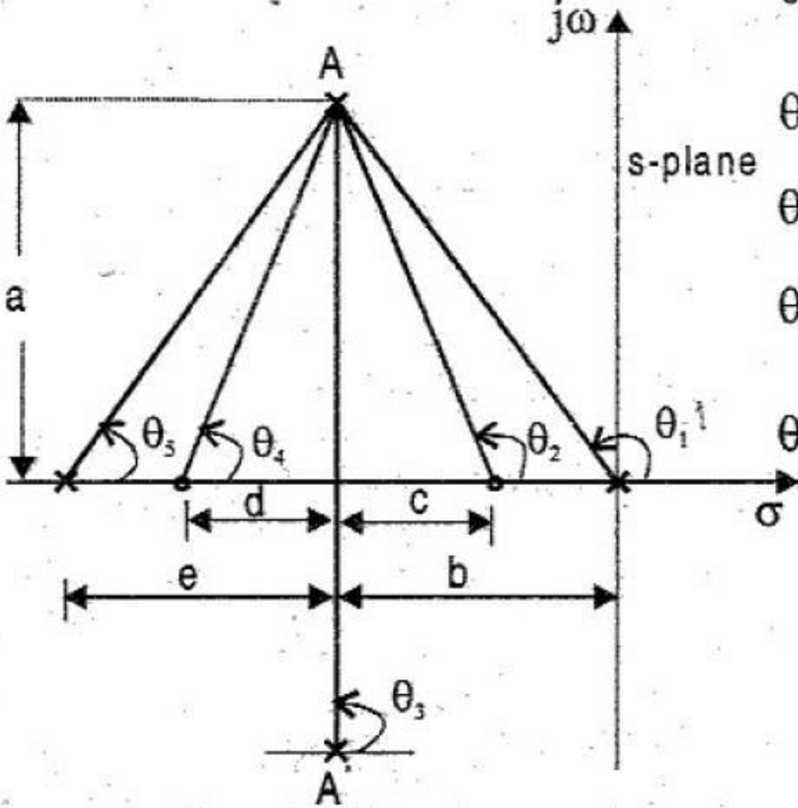
Step 4 : Breakaway and Breakin points

The breakaway or breakin points either lie on real axis or exist as complex conjugate pairs. If there is a root locus on real axis between 2 poles then there exist a breakaway point. If there is a root locus on real axis between 2 zeros then there exist a breakin point. If there is a root locus on real axis between pole and zero then there may be or may not be breakaway or breakin point.

The breakaway and breakin point is given by roots of the equation $dK/ds = 0$. The roots of $dK/ds = 0$ are actual breakaway or breakin point provided for this value of root, the gain K should be positive and real.

Step 5 : Angle of Departure and angle of arrival

$$\left.\begin{matrix}\text{Angle of Departure}\\ \text{(from a complex pole A)}\end{matrix}\right\} = 180° - \begin{pmatrix}\text{Sum of angles of vector to the}\\ \text{complex pole A from other poles}\end{pmatrix} + \begin{pmatrix}\text{Sum of angles of vectors to the}\\ \text{complex pole A from zeros}\end{pmatrix}$$



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

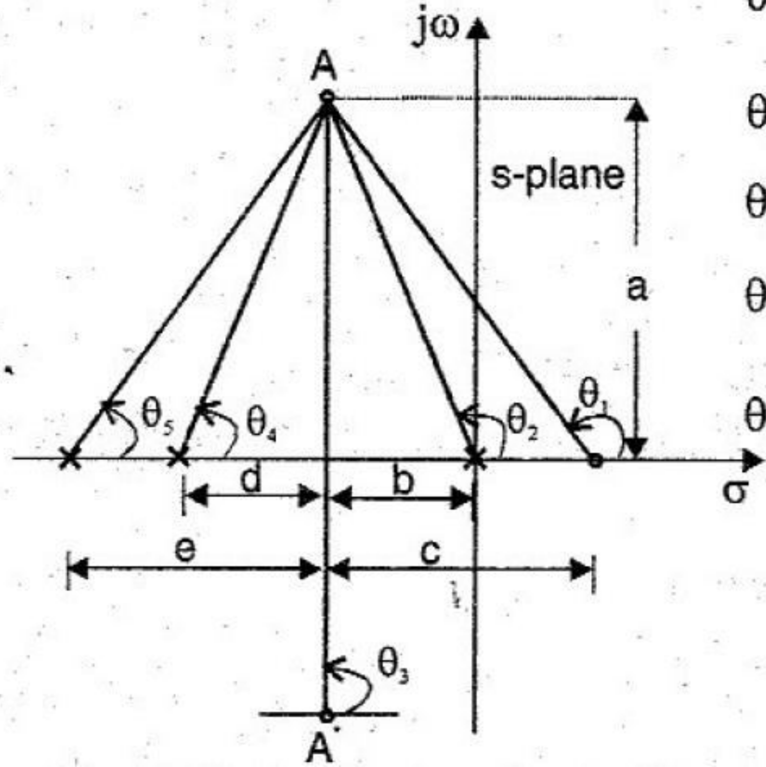
$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Calculation of angle of departure

$$\left.\begin{array}{l}\text{Angle of arrival at a}\\ \text{complex zero A}\end{array}\right\} = 180° - \left(\begin{array}{l}\text{Sum of angles of vectors to the}\\ \text{complex zero A from all other zeros}\end{array}\right) + \left(\begin{array}{l}\text{Sum of angles of vectors to the}\\ \text{complex zero A from poles}\end{array}\right)$$



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Calculation of angle of arrival

Step 6 : Point of intersection of root locus with imaginary axis

Letting $s = j\omega$ in the characteristic equation and separate the real part and imaginary part. Two equations are obtained : *one by equating real part to zero and the other by equating imaginary part to zero.* Solve the two equations for ω and **K**. The values of ω gives the points where the root locus crosses imaginary axis. The value of K gives the value of gain K at there crossing points. Also this value of K is the limiting value of K for stability of the system.

A unity feedback control system has an open loop transfer function, $G(s) = \frac{K}{s\,(s^2+4s+13)}$. Sketch the root locus.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s\,(s^2+4s+13)=0$.

The roots of the quadratic are, $s = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$

$\therefore$ The poles are lying at $s = 0, -2+j3$ and $-2-j3$.

Let us denote the poles as P_1, P_2, and P_3.

Here, $P_1 = 0$, $P_2 = -2+j3$ and $P_3 = -2-j3$.

The poles are marked by X (cross) as shown in fig 1.

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig 1.

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^{\circ}(2q+1)}{n-m} \quad ; \quad q = 0, 1, \ldots\ldots\ldots n-m$$

Here $n = 3$, and $m = 0$. $\therefore q = 0, 1, 2, 3$.

When $q = 0$, $\text{Angles} = \pm\frac{180^{\circ}}{3} = \pm 60^{\circ}$

When $q = 1$, $\text{Angles} = \pm\frac{180^{\circ}\times 3}{3} = \pm 180^{\circ}$

When $q = 2$, $\text{Angles} = \pm\frac{180^{\circ}\times 5}{3} = \pm 300^{\circ} = \mp 60^{\circ}$

When $q = 3$, $\text{Angles} = \pm\frac{180^{\circ}\times 7}{3} = \pm 420^{\circ} = \pm 60^{\circ}$

> ***Note :*** *It is enough if you calculate the required number of angles. Here it is given by first three values of angles. The remaining values will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 1.

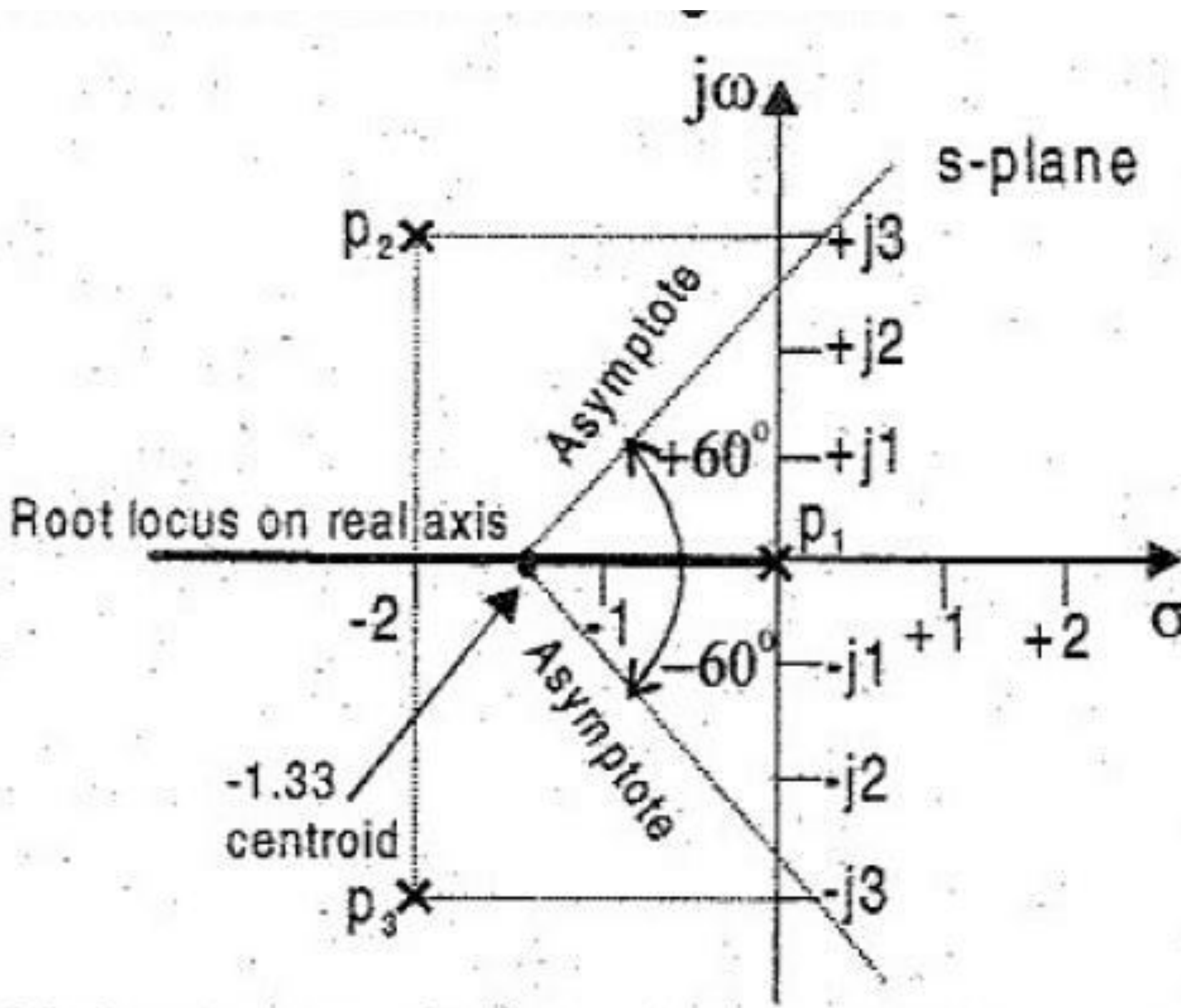


Fig 1. Figure showing the asymptote, root locus on real axis and location of poles and centroid

Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s^2+4s+13)}}{1+\dfrac{K}{s\,(s^2+4s+13)}} = \frac{K}{s\,(s^2+4s+13)+K}$$

The characteristic equation is, $s(s^2+4s+13)+K=0$

$$\therefore \quad s^3+4s^2+13s+K=0 \quad \Rightarrow \quad K=-s^3-4s^2-13s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

Put $\dfrac{dK}{ds}=0$

$$\therefore \quad -(3s^2+8s+13)=0 \quad \Rightarrow \quad (3s^2+8s+13)=0$$

$$\therefore s = \frac{-8 \pm \sqrt{8^2-4\times 13\times 3}}{2\times 3} = -1.33 \pm j1.6$$

Check for K : When, $s=-1.33+j1.6$, the value of K is given by,

$$K=-(s^3+4s^2+13s) = -[(-1.33+j1.6)^3+4\,(-1.33+j1.6)^2+13\,(-1.33+j1.6)]$$

$\neq$ positive and real.

Also it can be shown that when $s = -1.33 - j1.6$ the value of K is not equal to real and positive.

Since the values of K for, $s = -1.33 \pm j1.6$, are not real and positive, these points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step 5 : To find the angle of departure

Let us consider the complex pole p_2 shown in fig 4.22.2. Draw vectors from all other poles to the pole p_2 as shown in fig 4.22.2. Let the angles of these vectors be θ_1 and θ_2.

$$\text{Here, } \theta_1 = 180° - \tan^{-1}(3/2) = 123.7° \;;\quad \theta_2 = 90°$$

Angle of departure from the complex pole $p_2 = 180° - (\theta_1 + \theta_2)$

$$= 180° - (123.7° + 90°)$$

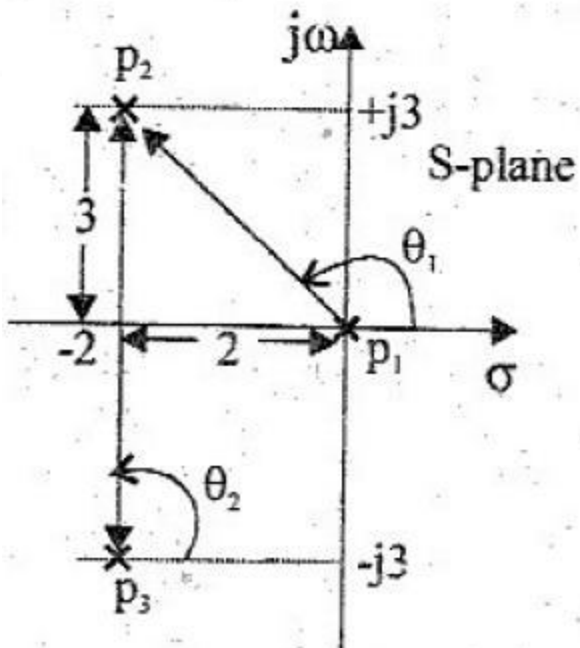


Fig 4.22.2

The angle of departure at complex pole p_3 is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole $p_3 = +33.7°$

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by,

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \quad \Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

On equating imaginary part to zero, we get,

$$-\omega^3 + 13\omega = 0$$

$$-\omega^3 = -13\omega$$

$$\omega^2 = 13 \Rightarrow \quad \omega = \pm\sqrt{13} = \pm 3.6$$

On equating real part to zero,

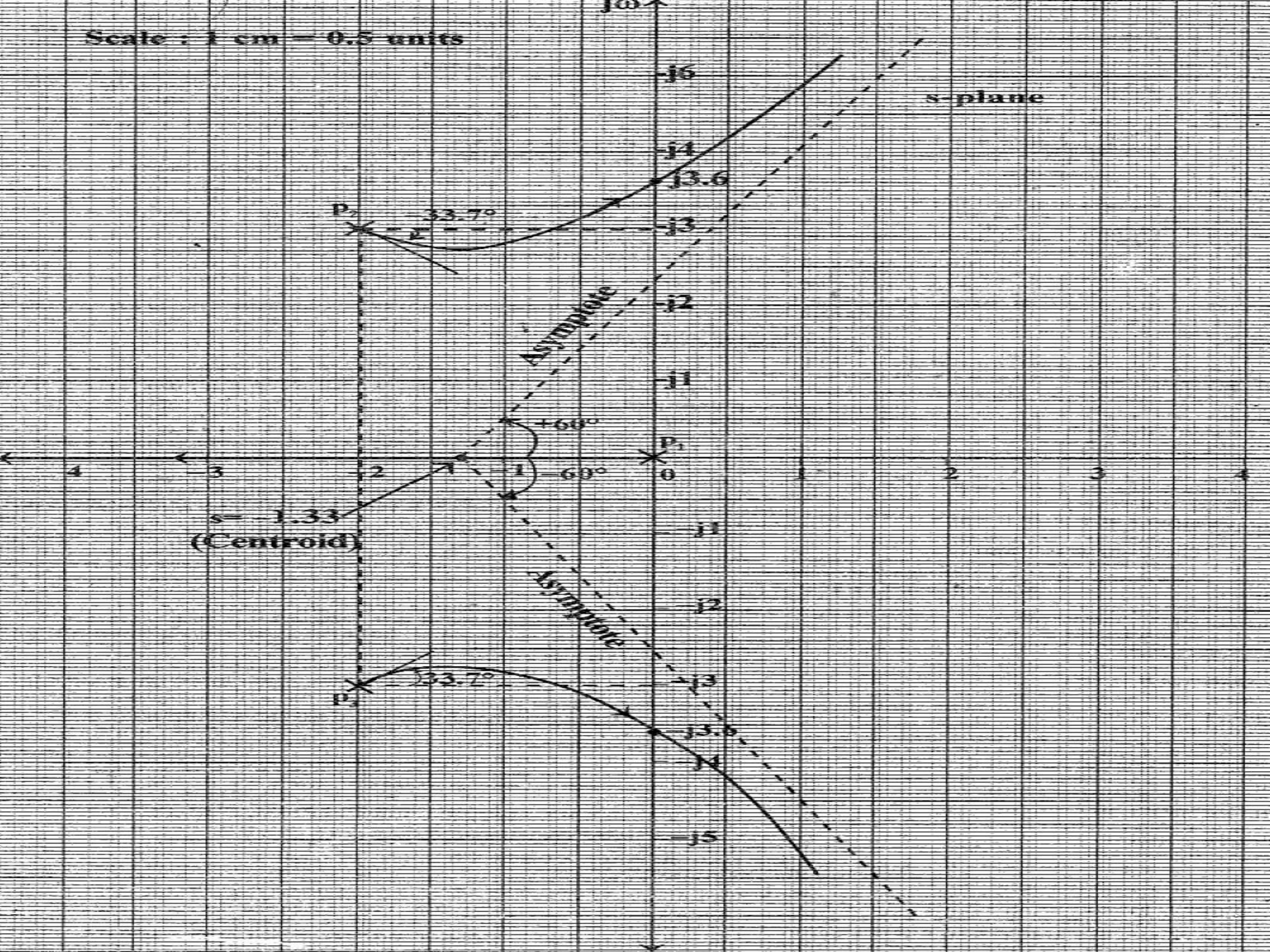
$$-4\omega^2 + K = 0$$

$$K = 4\omega^2$$

$$= 4 \times 13 = 52$$

The crossing point of root locus is ± j3.6. The value of K at this crossing point is K = 52. (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig 4.22.3. The roct locus has three branches one branch starts at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles (along the angle of departure), crosses the imaginary axis at ± j3.6 and travel parallel to asymptotes to meet the zeros at infinity.



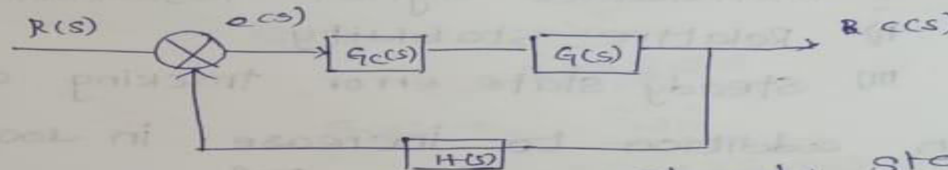
jω
Scale : 1 cm = 0.5 units
s-plane
j6
j4
j3.6
j3
j2
j1
P₂
−33.7°
Asymptote
+60°
−60°
P₁
−4
−3
−2
−1
0
1
2
3
4
s= −1.33
(Centroid)
−j1
−j2
−j3
−j3.6
−j4
−j5
P₃
33.7°
Asymptote

Refer the textbook for remaining problems of root locus

Controllers:-

In feedback control systems, a controller may be introduced to modify the error signal for better control action.

→ A controller with transfer function Gc(s) can be introduced in cascade with open loop transfer function G(s) as shown in the figure.



The controllers modify steady state & transient response of the system. The controllers also modify steady state error of the system.

Classification of controllers:-

The different types of controllers employed in control systems are

1) proportional controller (P- controller)
2) proportional plus integral controller (PI -controller)
3) proportional plus derivative controller (PD ")
4) proportional plus integral plus derivative controller (PID controller)

① Proportional controller:-(P- controller):-

→ The P-controller produces an output signal which is proportional to error signal.

→ The transfer function of proportional controller is kp. The term kp is also known as gain of the controller.

→ The proportional controller amplifies the error signal and increases loop gain of the system.

→ The P-controller improves the following aspects of the system by increasing loop gain.

i) Distrubance signal rejection
ii) Relative stability
iii) Steady state error tracking accuracy.

→ In addition to increase in loop gain, the P-controller decreases the sensitivity of the system.

** → The P-controller is not used alone as it produces a constant steady state error.

2) PI - Controller :-

→ The PI- controller produces an o/p signal consisting of 2 terms

a) proportional to error signal
b) " " integral of error signal

→ The transfer function of PI controller

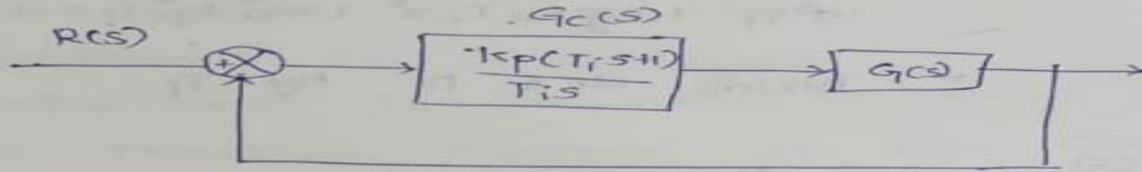
$$= K_p \left[1 + \frac{1}{T_i \, s} \right]$$

$$T/f \text{ function} = K_p \frac{T_i s + 1}{T_i s}$$

Here K_p = Proportional gain

T_i = integral time

→ The block diagram of unity feedback system with PI controller is as shown in the figure.



→ Let open loop transfer function G(s) of a 2° system is

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s}$$ [∵ not take ω_n^2]

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

The closed loop transfer function of the above system is

$$\frac{C(s)}{R(s)} = \frac{G_c(s)\,G(s)}{1 + G_c(s)\,G(s)}$$ [∵ unity feedback]

$$= \frac{\frac{K_p(T_i s+1)}{T_i s} \cdot \frac{\omega_n^2}{s(s+2\xi\omega_n)}}{1 + \frac{K_p(T_i s+1)}{T_i s} \cdot \frac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$\frac{C(s)}{R(s)} = \frac{K_p\,\omega_n^2 (T_i s+1)}{T_i s^2 (s+2\xi\omega_n) + K_p \omega_n^2 \, \text{~~kp~~} (T_i s+1)}$$

$$= \frac{K_p\,\omega_n^2 (T_i s+1)}{s^3 T_i + 2\xi\omega_n T_i s^2 + \omega_n^2 K_p^2 T_i s + K_p^2 \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{K_p \omega_n^2 (T_i s + 1)}{s^3 T_i + 2\xi\omega_n T_i s^2 + \omega_n^2 K_p T_i s + K_p \omega_n^2}$$

= Divide Nr & Dr by T_i

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K_p}{T_i} \omega_n^2 (T_i s + 1)}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 K_p s + \frac{K_p}{T_i} \omega_n^2}$$

$$\frac{K_p}{T_i} = K_i$$

$$\therefore \frac{C(s)}{R(s)} = \frac{K_i \omega_n^2 (1 + T_i s)}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 K_p s + K_i \omega_n^2}$$

From the above closed loop transfer function, it is observed that the PI controller introduces a zero to the system.

→ PI controller increases the order of the system by one.

→ PI controller also increases the type of system by one.

→ The increase in the type reduces steady state error.

→ The increase in the order reduces stability of the system.

② PD - Controller :-

→ The PD- controller produces an o/p signal which consists of two terms.

a) proportional to error signal

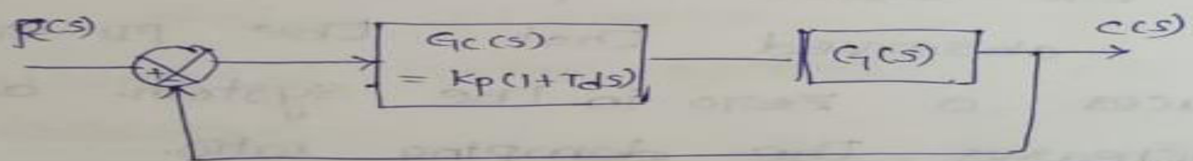
b) proportional to derivative of error signal.

→ The transfer function of PD- controller is $= K_p(1+T_d s)$

Here K_p = propotional gain

T_d = derivative time

→ The block diagram of unity feedback system with PD - controller is as shown in the figure.



→ Let the open loop transfer function of 2° system is

$$G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s}$$

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$$

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{K_p(1+T_d s)\,G(s)}{1+K_p(1+T_d s)\,G(s)}$$

$$= K_p(1+T_d s)\,\omega_n^2$$

$$\frac{C(s)}{R(s)} = \frac{K_p(1+T_d s)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{\dfrac{s(s+2\xi\omega_n) + K_p(1+T_d s)\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$= \frac{K_p\omega_n^2(1+T_d s)}{s^2+2\xi\omega_n s + K_p\omega_n^2 + T_d s K_p\omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{K_p\omega_n^2(1+T_d s)}{s^2 + s(2\xi\omega_n + T_d K_p\omega_n^2) + K_p\omega_n^2}$$

From the closed loop transfer function, it is observed that the PD-controller introduces a zero in the system and it increases the damping ratio.

→ The addition of zero increases peak overshoot and reduces rise time.

→ The increased damping ratio reduces peak overshoot.

→ The PD-controller does n't modify type of the system, & order of the system.

Hence PD-controller does not modify the steady state error.

④ PID-controller:-

The PID-controller produces an o/p signal that consists of three ...

Sree

i) Proportional to error signal
ii) Proportional to integral of error signal.
iii) Proportional to derivative of error signal.

→ The PID- controller have the combined effect of all the three control actions i.e proportional, integral, derivative control actions.

→ The proportional controller stabilizes the gain but produces steady state error.

→ The integral controller reduces (or) eliminates steady state error.

→ The derivative controller reduces rate of change of error.