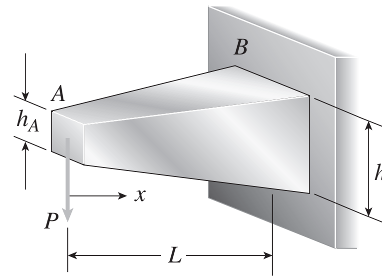
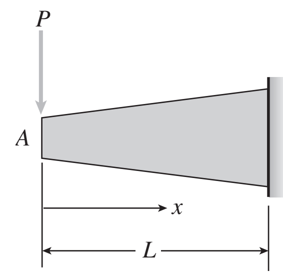
Nonprismatic Beams

Problem 5.7-1 A tapered cantilever beam AB of length L has square cross sections and supports a concentrated load P at the free end (see figure on the next page). The width and height of the beam vary linearly from h_A at the free end to h_B at the fixed end.

Determine the distance x from the free end A to the cross section of maximum bending stress if $h_B = 3h_A$. What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?



Solution 5.7-1 **Tapered cantilever beam**



Square cross sections

h_A = height and width at smaller end
h_B = height and width at larger end
h_x = height and width at distance x

$$\frac{h_B}{h_A} = 3$$

$$h_x = h_A + (h_B - h_A)\left(\frac{x}{L}\right) = h_A\left(1 + \frac{2x}{L}\right)$$

$$S_x = \frac{1}{6}(h_x)^3 = \frac{h_A^3}{6}\left(1 + \frac{2x}{L}\right)^3$$

Stress at distance x

$$\sigma_1 = \frac{M_x}{S_x} = \frac{6Px}{(h_A)^3\left(1 + \frac{2x}{L}\right)^3}$$

At end A: $x = 0$ $\quad \sigma_A = 0$

At support B: $x = L$

$$\sigma_B = \frac{2PL}{9(h_A)^3}$$

Cross section of maximum stress

Set $\frac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$$x = \frac{L}{4} \quad \longleftarrow$$

Maximum bending stress

$$\sigma_{max} = (\sigma_1)_{x=L/4} = \frac{4PL}{9(h_A)^3} \quad \longleftarrow$$

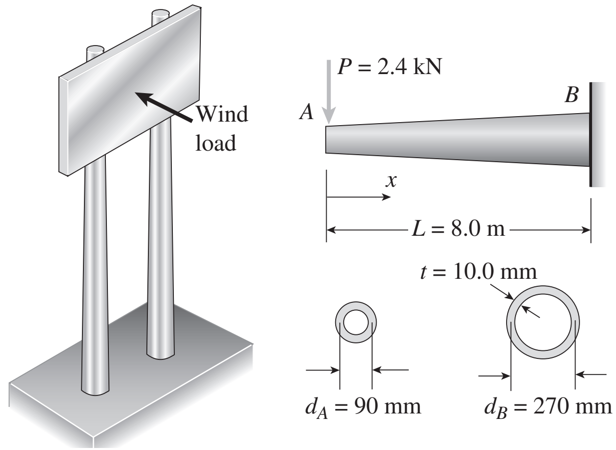
Ratio of σ_{max} to σ_B

$$\frac{\sigma_{max}}{\sigma_B} = 2 \quad \longleftarrow$$

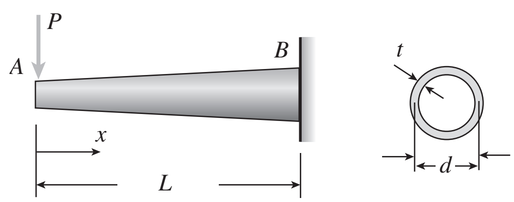
Problem 5.7-2 A tall signboard is supported by two vertical beams consisting of thin-walled, tapered circular tubes (see figure). For purposes of this analysis, each beam may be represented as a cantilever AB of length $L = 8.0$ m subjected to a lateral load $P = 2.4$ kN at the free end. The tubes have constant thickness $t = 10.0$ mm and *average* diameters $d_A = 90$ mm and $d_B = 270$ mm at ends A and B, respectively.

Because the thickness is small compared to the diameters, the moment of inertia at any cross section may be obtained from the formula $I = \pi d^3 t/8$ (see Case 22, Appendix D), and therefore the section modulus may be obtained from the formula $S = \pi d^2 t/4$.

At what distance x from the free end does the maximum bending stress occur? What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?



Solution 5.7-2 Tapered circular tube



$P = 2.4$ kN
$L = 8.0$ m
$t = 10$ mm
d = average diameter

At end A: $d_A = 90$ mm
At support B: $d_B = 270$ mm

AT DISTANCE x:

$$d_x = d_A + (d_B - d_A)\left(\frac{x}{L}\right) = 90 + 180\frac{x}{L} = 90\left(1 + \frac{2x}{L}\right)$$

$$S_x = \frac{\pi}{4}(d_x)^2(t) = \frac{\pi}{4}(90)^2\left(1 + \frac{2x}{L}\right)^2(10)$$

$$= 20{,}250\pi\left(1 + \frac{2x}{L}\right)^2 \qquad S_x = \text{mm}^3$$

$M_x = Px = 2400x \qquad x = \text{meters}, M_x = \text{N} \cdot \text{m}$

$$\sigma_1 = \frac{M_x}{S_x} = \frac{2400x}{20.25\pi\left(1 + \frac{2x}{L}\right)^2} \qquad L = \text{meters}, \sigma_1 = \text{MPa}$$

AT END A: $x = 0 \qquad \sigma_1 = \sigma_A = 0$

AT SUPPORT B: $\quad x = L = 8.0$ m
$\sigma_1 = \sigma_B = 33.53$ MPa

CROSS SECTION OF MAXIMUM STRESS

Set $\frac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$$x = \frac{L}{2} = 4.0 \text{ m} \quad \longleftarrow$$

MAXIMUM BENDING STRESS

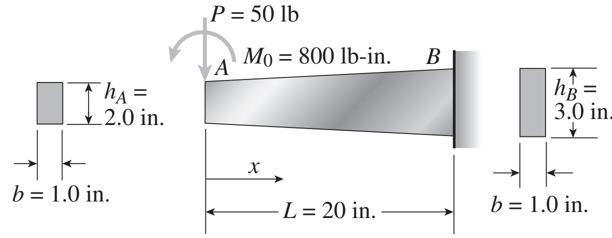
$$\sigma_{max} = (\sigma_1)_{x=L/2} = \frac{2400(4.0)}{(20.25\,\pi)(1 + 1)^2}$$

$$= 37.73 \text{ MPa} \quad \longleftarrow$$

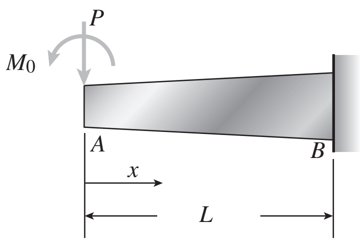
RATIO OF σ_{max} *to* σ_B $\qquad \frac{\sigma_{max}}{\sigma_B} = \frac{9}{8} = 1.125 \quad \longleftarrow$

Problem 5.7-3 A tapered cantilever beam AB having rectangular cross sections is subjected to a concentrated load $P = 50$ lb and a couple $M_0 = 800$ lb-in. acting at the free end (see figure). The width b of the beam is constant and equal to 1.0 in., but the height varies linearly from $h_A = 2.0$ in. at the loaded end to $h_B = 3.0$ in. at the support.

At what distance x from the free end does the maximum bending stress σ_{max} occur? What is the magnitude σ_{max} of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?



Solution 5.7-3 Tapered cantilever beam



$P = 50$ lb
$M_0 = 800$ lb-in.
$L = 20$ in.
$h_A = 2.0$ in.
$h_B = 3.0$ in.
$b = 1.0$ in.

UNITS: pounds and inches

AT DISTANCE x:

$$h_x = h_A + (h_B - h_A)\frac{x}{L} = 2 + (1)\left(\frac{x}{L}\right) = 2 + \frac{x}{L}$$

$$S_x = \frac{bh_x^2}{6} = \frac{b}{6}\left(2 + \frac{x}{L}\right)^2 = \frac{1}{6}\left(2 + \frac{x}{L}\right)^2$$

$$M_x = Px + M_0 = (50)(x) + 800 = 50(16 + x)$$

$$\sigma_1 = \frac{M_x}{S_x} = \frac{50(16 + x)(6)}{\left(2 + \frac{x}{L}\right)^2} = \frac{120{,}000(16 + x)}{(40 + x)^2}$$

AT END A: $x = 0$ $\quad \sigma_1 = \sigma_A = 1200$ psi

AT SUPPORT B: $x = L = 20$ in. $\quad \sigma_1 = \sigma_B = 1200$ psi

CROSS SECTION OF MAXIMUM STRESS

Set $\frac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$x = 8.0$ in. ←

MAXIMUM BENDING STRESS

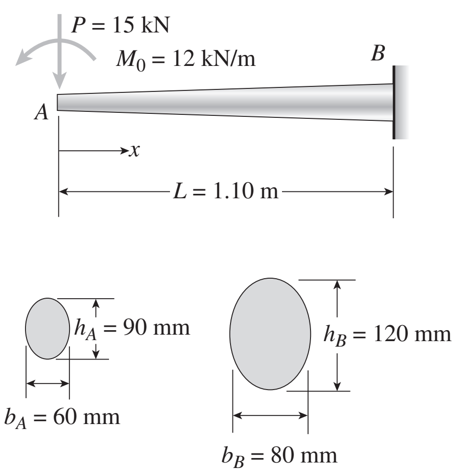
$$\sigma_{max} = (\sigma_1)_{x=8.0} = \frac{(120{,}000)(24)}{(48)^2} = 1250 \text{ psi} \quad \leftarrow$$

RATIO OF σ_{max} to σ_B

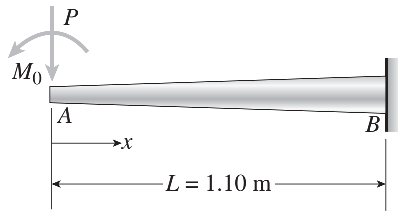
$$\frac{\sigma_{max}}{\sigma_B} = \frac{1250}{1200} = \frac{25}{24} = 1.042 \quad \leftarrow$$

Problem 5.7-4 The spokes in a large flywheel are modeled as beams fixed at one end and loaded by a force P and a couple M_0 at the other (see figure). The cross sections of the spokes are elliptical with major and minor axes (height and width, respectively) having the lengths shown in the figure. The cross-sectional dimensions vary linearly from end A to end B.

Considering only the effects of bending due to the loads P and M_0, determine the following quantities: (a) the largest bending stress σ_A at end A; (b) the largest bending stress σ_B at end B; (c) the distance x to the cross section of maximum bending stress; and (d) the magnitude σ_{max} of the maximum bending stress.



Solution 5.7-4 **Elliptical spokes in a flywheel**



$P = 15\text{ kN} = 15{,}000\text{ N}$
$M_0 = 12\text{ kN}\cdot\text{m} = 12{,}000\text{ N}\cdot\text{m}$
$L = 1.1\text{ mm}$

UNITS: Newtons, meters

AT END A: $b_A = 0.06\text{ m}$, $h_A = 0.09\text{ m}$

AT SUPPORT B: $b_B = 0.08\text{ m}$, $h_B = 0.12\text{ m}$

AT DISTANCE x:

$$b_x = b_A + (b_B - b_A)\frac{x}{L} = 0.06 + 0.02\frac{x}{L} = 0.02\left(3 + \frac{x}{L}\right)$$

$$h_x = h_A + (h_B - h_A)\frac{x}{L} = 0.09 + 0.03\frac{x}{L} = 0.03\left(3 + \frac{x}{L}\right)$$

Case 16, Appendix D: $I = \frac{\pi}{64}(bh^3)$

$$I_x = \frac{\pi}{64}(b_x)(h_x)^3 \qquad S_x = \frac{I_x}{h_x/2} = \frac{\pi b_x h_x^2}{32}$$

$$S_x = \frac{\pi}{32}(0.02)\left(3 + \frac{x}{L}\right)(0.03)^2\left(3 + \frac{x}{L}\right)^2$$

$$= \frac{9\pi}{16 \times 10^6}\left(3 + \frac{x}{L}\right)^3$$

$M_x = M_0 + Px = 12{,}000\text{ N}\cdot\text{m} + (15{,}000\text{ N})x$
$= 15{,}000(0.8 + x)$

$$\sigma_1 = \frac{M_x}{S_x} = \frac{15{,}000(0.8 + x)(16 \times 10^6)}{9\pi\left(3 + \frac{x}{L}\right)^3}$$

$$= \frac{(80 \times 10^9)(0.8 + x)}{3\pi\left(3 + \frac{x}{L}\right)^3}$$

(a) AT END A: $x = 0$

$$\sigma_A = (\sigma_1)_{x=0} = \frac{(80 \times 10^9)(0.8)}{(3\pi)(27)} = 251.5 \times 10^6\text{ N/m}^2$$

$= 251.5\text{ MPa}$ ←

(b) AT END B: $x = L = 1.1\text{ m}$

$$\sigma_B = (\sigma_1)_{x=L} = \frac{(80 \times 10^9)(0.8 + 1.1)}{(3\pi)(3 + 1)^3}$$

$= 252.0 \times 10^6\text{ N/m}^2 = 252.0\text{ MPa}$ ←

(c) CROSS SECTION OF MAXIMUM STRESS

Set $\frac{d\sigma_1}{dx} = 0$ Evaluate the derivative, set it equal to zero, and solve for x.

$x = 0.45\text{ m}$ ←

(d) MAXIMUM BENDING STRESS

$$\sigma_{max} = (\sigma_1)_{x=0.45} = \frac{(80 \times 10^9)(0.8 + 0.45)}{(3\pi)\left(3 + \frac{0.45}{1.1}\right)^3}$$

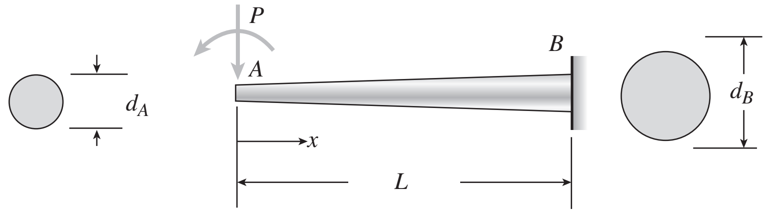
$= 267.8 \times 10^6\text{ N/m}^2 = 267.8\text{ MPa}$ ←

Problem 5.7-5 Refer to the tapered cantilever beam of solid circular cross section shown in Fig. 5-24 of Example 5-9.

(a) Considering only the bending stresses due to the load P, determine the range of values of the ratio d_B/d_A for which the maximum normal stress occurs at the support.

(b) What is the maximum stress for this range of values?

Solution 5.7-5 **Tapered cantilever beam**



FROM EQ. (5-32), EXAMPLE 5-9

$$\sigma_1 = \frac{32Px}{\pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^3} \qquad \text{Eq. (1)}$$

FIND THE VALUE OF X THAT MAKES σ_1 A MAXIMUM

$$\text{Let } \sigma_1 = \frac{\mu}{\upsilon} \qquad \frac{d\sigma_1}{dx} = \frac{\upsilon\left(\frac{du}{dx}\right) - u\left(\frac{d\upsilon}{dx}\right)}{\upsilon^2} = \frac{N}{D}$$

$$N = \pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^3[32P]$$

$$-[32Px][\pi][3]\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^2\left[\frac{1}{L}(d_B - d_A)\right]$$

After simplification:

$$N = 32\pi P\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^2\left[d_A - 2(d_B - d_A)\frac{x}{L}\right]$$

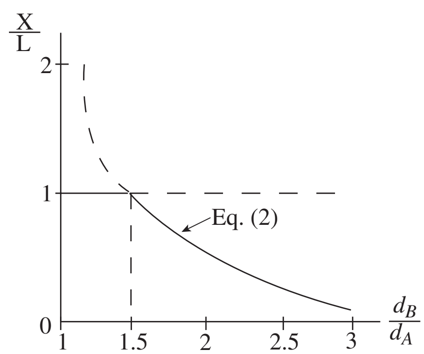
$$D = \pi^2\left[d_A + (d_B - d_A)\frac{x}{L}\right]^6$$

$$\frac{d\sigma_1}{dx} = \frac{N}{D} = \frac{32P\left[d_A - 2(d_B - d_A)\frac{x}{L}\right]}{\pi\left[d_A + (d_B - d_A)\left(\frac{x}{L}\right)\right]^4}$$

$$\frac{d\sigma_1}{dx} = 0 \qquad d_A - 2(d_B - d_A)\left(\frac{x}{L}\right) = 0$$

$$\therefore \frac{x}{L} = \frac{d_A}{2(d_B - d_A)} = \frac{1}{2\left(\frac{d_B}{d_A} - 1\right)} \qquad \text{Eq. (2)}$$

(a) GRAPH OF X/L VERSUS d_B/d_A (EQ. 2)



Maximum bending stress occurs at the support when $1 \le \frac{d_B}{d_A} \le 1.5$ ←

(b) MAXIMUM STRESS (AT SUPPORT B)

Substitute $x/L = 1$ into Eq. (1):

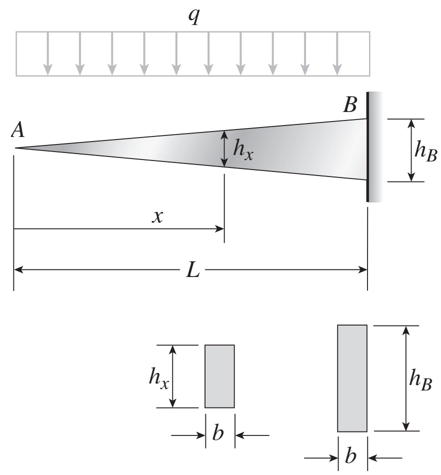
$$\sigma_{max} = \frac{32PL}{\pi d_B^3} \quad \leftarrow$$

Fully Stressed Beams

Problems 5.7-6 to 5.7-8 pertain to fully stressed beams of rectangular cross section. Consider only the bending stresses obtained from the flexure formula and disregard the weights of the beams.

Problem 5.7-6 A cantilever beam *AB* having rectangular cross sections with constant width b and varying height h_x is subjected to a uniform load of intensity q (see figure).

How should the height h_x vary as a function of x (measured from the free end of the beam) in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-6 **Fully stressed beam with constant width and varying height**

h_x = height at distance x
h_B = height at end B
b = width (constant)

AT DISTANCE x: $M = \dfrac{qx^2}{2}$ $\quad S = \dfrac{bh_x^2}{6}$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qx^2}{bh_x^2}$$

$$h_x = x\sqrt{\frac{3q}{b\sigma_{\text{allow}}}}$$

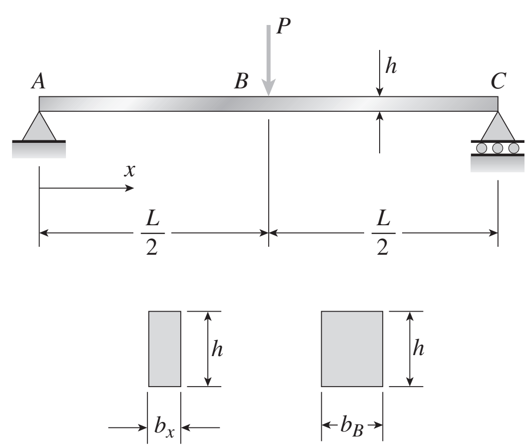
AT THE FIXED END $(x = L)$:

$$h_B = L\sqrt{\frac{3q}{b\sigma_{\text{allow}}}}$$

Therefore, $\dfrac{h_x}{h_B} = \dfrac{x}{L}$ $\quad h_x = \dfrac{h_B x}{L}$ ←

Problem 5.7-7 A simple beam *ABC* having rectangular cross sections with constant height h and varying width b_x supports a concentrated load P acting at the midpoint (see figure).

How should the width b_x vary as a function of x in order to have a fully stressed beam? (Express b_x in terms of the width b_B at the midpoint of the beam.)



Solution 5.7-7 Fully stressed beam with constant height and varying width

$h =$ height of beam (constant)

$b_x =$ width at distance x from end $A\left(0 \le x \le \dfrac{L}{2}\right)$

$b_B =$ width at midpoint $B \quad (x = L/2)$

AT DISTANCE x $\quad M = \dfrac{Px}{2} \quad S = \dfrac{1}{6} b_x h^2$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3Px}{b_x h^2} \qquad b_x = \frac{3Px}{\sigma_{\text{allow}} h^2}$$

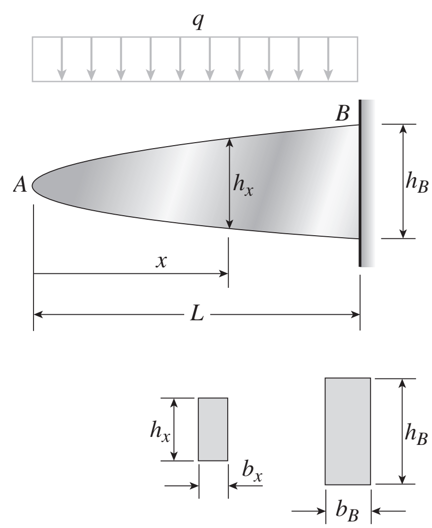
AT MIDPOINT B $(x = L/2)$

$$b_B = \frac{3PL}{2\sigma_{\text{allow}} h^2}$$

Therefore, $\dfrac{b_x}{b_b} = \dfrac{2x}{L}$ and $b_x = \dfrac{2b_B x}{L}$ ←

NOTE: The equation is valid for $0 \le x \le \dfrac{L}{2}$ and the beam is symmetrical about the midpoint.

Problem 5.7-8 A cantilever beam AB having rectangular cross sections with varying width b_x and varying height h_x is subjected to a uniform load of intensity q (see figure). If the width varies linearly with x according to the equation $b_x = b_B x/L$, how should the height h_x vary as a function of x in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-8 Fully stressed beam with varying width and varying height

$h_x =$ height at distance x
$h_B =$ height at end B
$b_x =$ width at distance x
$b_B =$ width at end B

$$b_x = b_B\left(\frac{x}{L}\right)$$

AT DISTANCE x

$$M = \frac{qx^2}{2} \qquad S = \frac{b_x h_x^2}{6} = \frac{b_B x}{6L}(h_x)^2$$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qLx}{b_B h_x^2}$$

$$h_x = \sqrt{\frac{3qLx}{b_B \sigma_{\text{allow}}}}$$

AT THE FIXED END $(x = L)$

$$h_B = \sqrt{\frac{3qL^2}{b_B \sigma_{\text{allow}}}}$$

Therefore, $\dfrac{h_x}{h_B} = \sqrt{\dfrac{x}{L}} \qquad h_x = h_B\sqrt{\dfrac{x}{L}}$ ←

Shear Stresses in Rectangular Beams

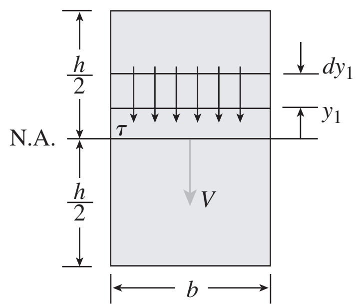
Problem 5.8-1 The shear stresses τ in a rectangular beam are given by Eq. (5-39):

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

in which V is the shear force, I is the moment of inertia of the cross-sectional area, h is the height of the beam, and y_1 is the distance from the neutral axis to the point where the shear stress is being determined (Fig. 5-30).

By integrating over the cross-sectional area, show that the resultant of the shear stresses is equal to the shear force V.

Solution 5.8-1 **Resultant of the shear stresses**



$$I = \frac{bh^3}{12}$$

$$\tau = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

$V =$ shear force acting on the cross section
$R =$ resultant of shear stresses τ

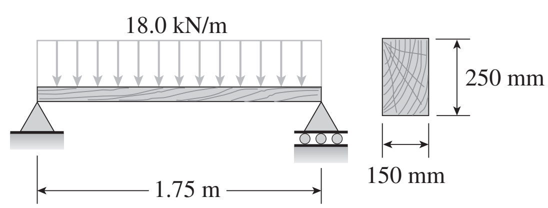
$$R = \int_{-h/2}^{h/2} \tau b dy_1 = 2\int_0^{h/2} \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right) b dy_1$$

$$= \frac{12V}{bh^3}(b)\int_0^{h/2}\left(\frac{h^2}{4} - y_1^2\right) dy_1$$

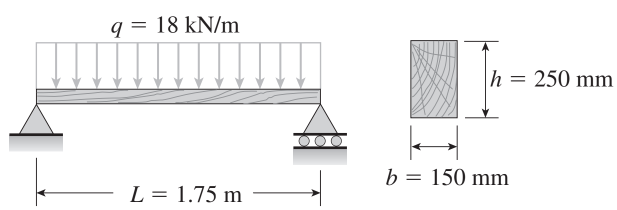
$$= \frac{12V}{h^3}\left(\frac{2h^3}{24}\right) = V$$

$\therefore R = V$ Q.E.D. ←

Problem 5.8-2 Calculate the maximum shear stress τ_{max} and the maximum bending stress σ_{max} in a simply supported wood beam (see figure) carrying a uniform load of 18.0 kN/m (which includes the weight of the beam) if the length is 1.75 m and the cross section is rectangular with width 150 mm and height 250 mm.



Solution 5.8-2 **Wood beam with a uniform load**



Maximum shear stress

$$V = \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3qL}{4bh} = \frac{3(18\text{ kN/m})(1.75\text{ m})}{4(150\text{ mm})(250\text{ mm})}$$

$= 630$ kPa ←

Maximum bending stress

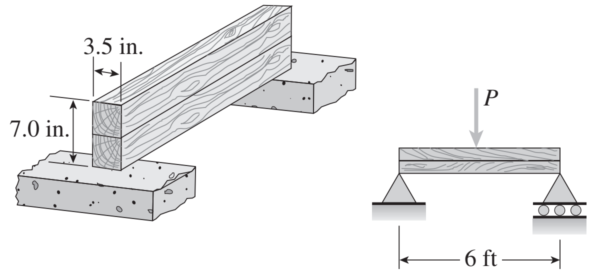
$$M = \frac{qL^2}{8} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M}{S} = \frac{3qL^2}{4bh^2} = \frac{3(18\text{ kN/m})(1.75\text{ m})^2}{4(150\text{ mm})(250\text{ mm})^2}$$

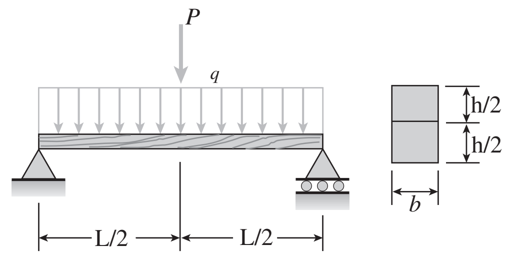
$= 4.41$ MPa ←

Problem 5.8-3 Two wood beams, each of square cross section (3.5 in. × 3.5 in., actual dimensions) are glued together to form a solid beam of dimensions 3.5 in. × 7.0 in. (see figure). The beam is simply supported with a span of 6 ft.

What is the maximum load P_{max} that may act at the midpoint if the allowable shear stress in the glued joint is 200 psi? (Include the effects of the beam's own weight, assuming that the wood weighs 35 lb/ft³.)



Solution 5.8-3 **Simple beam with a glued joint**



$L = 6 \text{ ft} = 72 \text{ in.} \quad b = 3.5 \text{ in.} \quad h = 7.0 \text{ in.}$

$\tau_{allow} = 200 \text{ psi}$

$$\gamma = (35 \text{ lb/ft}^3)\left(\frac{1}{1728}\frac{\text{ft}^3}{\text{in}^3}\right) = \frac{35}{1728} \text{ lb/in.}^3$$

$q =$ weight of beam per unit distance $= \gamma bh$

Maximum load P_{max}

$$V = \frac{P}{2} + \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3\left(\frac{P}{2} + \frac{qL}{2}\right)}{2bh} = \frac{3}{4bh}(P + qL)$$

$$P_{max} = \frac{4}{3}bh\tau - qL = \frac{4}{3}bh\tau - \gamma bhL$$

$$= bh\left(\frac{4}{3}\tau - \gamma L\right)$$

Substitute numerical values:

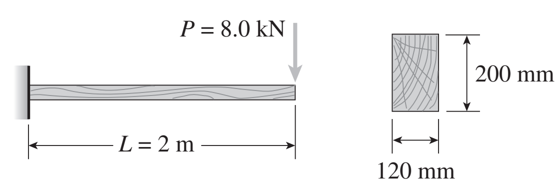
$$P_{max} = (3.5 \text{ in.})(7.0 \text{ in.}) \times \left[\frac{4}{3}(200 \text{ psi}) - \left(\frac{35}{1728} \text{ lb/in.}^2\right)(72 \text{ in.})\right]$$

$$= 6500 \text{ lb} \leftarrow$$

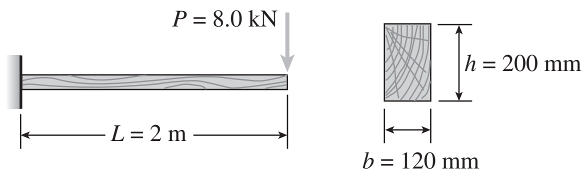
(This result is based solely on the shear stress.)

Problem 5.8-4 A cantilever beam of length $L = 2$ m supports a load $P = 8.0$ kN (see figure). The beam is made of wood with cross-sectional dimensions 120 mm × 200 mm.

Calculate the shear stresses due to the load P at points located 25 mm, 50 mm, 75 mm, and 100 mm from the top surface of the beam. From these results, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-4 Shear stresses in a cantilever beam



Distance from the top surface beam	y_1 (mm)	τ (MPa)	τ (kPa)
0	100	0	0
25	75	0.219	219
50	50	0.375	375
75	25	0.469	469
100 (N.A.)	0	0.500	500

Eq. (5-39): $\tau = \dfrac{V}{2I}\left(\dfrac{h^2}{4} - y_1^2\right)$

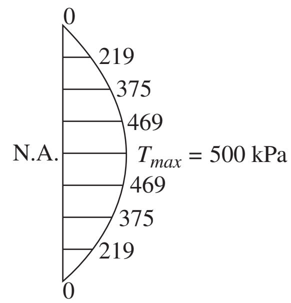
$V = P = 8.0 \text{ kN} = 8{,}000 \text{ N} \qquad I = \dfrac{bh^3}{12} = 80 \times 10^6 \text{ mm}^4$

$h = 200 \text{ mm} \qquad (y_1 = \text{mm})$

$\tau = \dfrac{8{,}000}{2(80 \times 10^6)}\left[\dfrac{(200)^2}{4} - y_1^2\right] \qquad (\tau = \text{N/mm}^2 = \text{MPa})$

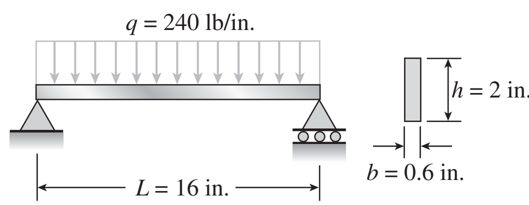
$\tau = 50 \times 10^{-6}(10{,}000 - y_1^2) \qquad (y_1 = \text{mm}; \tau = \text{MPa})$

GRAPH OF SHEAR STRESS τ

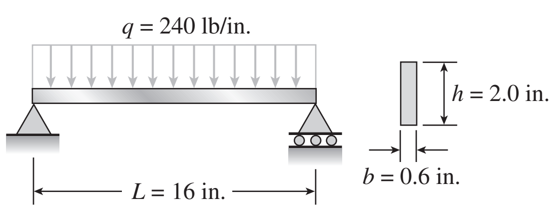


Problem 5.8-5 A steel beam of length $L = 16$ in. and cross-sectional dimensions $b = 0.6$ in. and $h = 2$ in. (see figure) supports a uniform load of intensity $q = 240$ lb/in., which includes the weight of the beam.

Calculate the shear stresses in the beam (at the cross section of maximum shear force) at points located 1/4 in., 1/2 in., 3/4 in., and 1 in. from the top surface of the beam. From these calculations, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-5 Shear stresses in a simple beam



Distance from the top surface (in.)	y_1 (in.)	τ (psi)
0	1.00	0
0.25	0.75	1050
0.50	0.50	1800
0.75	0.25	2250
1.00 (N.A.)	0	2400

Eq. (5-39): $\tau = \dfrac{V}{2I}\left(\dfrac{h^2}{4} - y_1^2\right)$

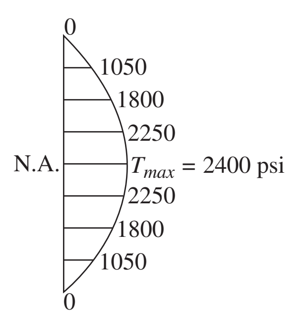
$V = \dfrac{qL}{2} = 1920 \text{ lb} \qquad I = \dfrac{bh^3}{12} = 0.4 \text{ in.}^4$

UNITS: pounds and inches

$\tau = \dfrac{1920}{2(0.4)}\left[\dfrac{(2)^2}{4} - y_1^2\right] = (2400)(1 - y_1^2)$

$(\tau = \text{psi}; y_1 = \text{in.})$

GRAPH OF SHEAR STRESS τ



Problem 5.8-6 A beam of rectangular cross section (width b and height h) supports a uniformly distributed load along its entire length L. The allowable stresses in bending and shear are σ_{allow} and τ_{allow}, respectively.

(a) If the beam is simply supported, what is the span length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?

(b) If the beam is supported as a cantilever, what is the length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?

Solution 5.8-6 Beam of rectangular cross section

$b =$ width $\quad h =$ height $\quad L =$ length

UNIFORM LOAD $\quad q =$ intensity of load

ALLOWABLE STRESSES $\quad \sigma_{allow}$ and τ_{allow}

(a) SIMPLE BEAM

BENDING

$$M_{max} = \frac{qL^2}{8} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{3qL^2}{4bh^2}$$

$$q_{allow} = \frac{4\sigma_{allow}\, bh^2}{3L^2} \qquad (1)$$

SHEAR

$$V_{max} = \frac{qL}{2} \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3qL}{4bh}$$

$$q_{allow} = \frac{4\tau_{allow}\, bh}{3L} \qquad (2)$$

Equate (1) and (2) and solve for L_0:

$$L_0 = h\left(\frac{\sigma_{allow}}{\tau_{allow}}\right) \longleftarrow$$

(b) CANTILEVER BEAM

BENDING

$$M_{max} = \frac{qL^2}{2} \qquad S = \frac{bh^2}{6}$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{3qL^2}{bh^2} \qquad q_{allow} = \frac{\sigma_{allow}\, bh^2}{3L^2} \qquad (3)$$

SHEAR

$$V_{max} = qL \qquad A = bh$$

$$\tau_{max} = \frac{3V}{2A} = \frac{3qL}{2bh}$$

$$q_{allow} = \frac{2\tau_{allow}\, bh}{3L} \qquad (4)$$

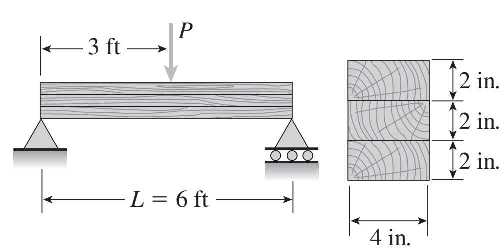
Equate (3) and (4) and solve for L_0:

$$L_0 = \frac{h}{2}\left(\frac{\sigma_{allow}}{\tau_{allow}}\right) \longleftarrow$$

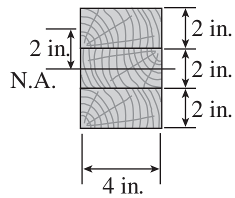
NOTE: If the actual length is less than L_0, the shear stress governs the design. If the length is greater than L_0, the bending stress governs.

Problem 5.8-7 A laminated wood beam on simple supports is built up by gluing together three 2 in. × 4 in. boards (actual dimensions) to form a solid beam 4 in. × 6 in. in cross section, as shown in the figure. The allowable shear stress in the glued joints is 65 psi and the allowable bending stress in the wood is 1800 psi.

If the beam is 6 ft long, what is the allowable load P acting at the midpoint of the beam? (Disregard the weight of the beam.)



Solution 5.8-7 Laminated wood beam on simple supports



$L = 6 \text{ ft} = 72 \text{ in.}$
$\tau_{allow} = 65 \text{ psi}$
$\sigma_{allow} = 1800 \text{ psi}$

Allowable load based upon shear stress in the glued joints

$$\tau = \frac{VQ}{Ib} \qquad Q = (4 \text{ in.})(2 \text{ in.})(2 \text{ in.}) = 16 \text{ in.}^3$$

$$V = \frac{P}{2} \qquad I = \frac{bh^3}{12} = \frac{1}{12}(4 \text{ in.})(6 \text{ in.})^3 = 72 \text{ in.}^4$$

$$\tau = \frac{(P/2)(16 \text{ in.}^3)}{(72 \text{ in.}^4)(4 \text{ in.})} = \frac{P}{36} \qquad (P = \text{lb}; \tau = \text{psi})$$

$$P_1 = 36\tau_{allow} = 36\,(65 \text{ psi}) = 2340 \text{ lb}$$

Allowable load based upon bending stress

$$\sigma = \frac{M}{S} \qquad M = \frac{PL}{4} = P\left(\frac{72 \text{ in.}}{4}\right) = 18P \text{ (lb-in.)}$$

$$S = \frac{bh^2}{6} = \frac{1}{6}(4 \text{ in.})(6 \text{ in.})^2 = 24 \text{ in.}^3$$

$$\sigma = \frac{(18P \text{ lb-in.})}{24 \text{ in.}^3} = \frac{3P}{4} \qquad (P = \text{lb}; \sigma = \text{psi})$$

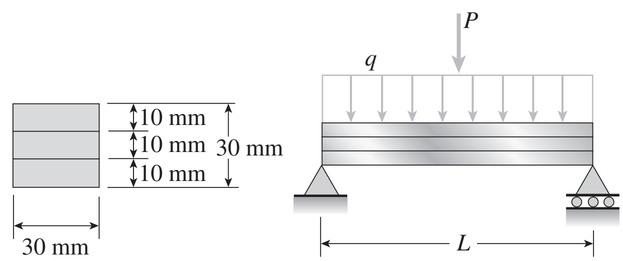
$$P_2 = \frac{4}{3}\sigma_{allow} = \frac{4}{3}(1800 \text{ psi}) = 2400 \text{ lb}$$

Allowable load
Shear stress in the glued joints governs.

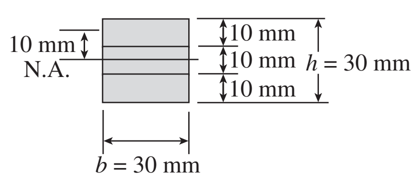
$P_{allow} = 2340 \text{ lb}$ ←

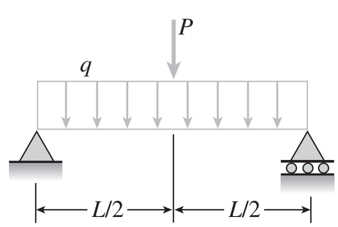
Problem 5.8-8 A laminated plastic beam of square cross section is built up by gluing together three strips, each 10 mm × 30 mm in cross section (see figure). The beam has a total weight of 3.2 N and is simply supported with span length $L = 320$ mm.

Considering the weight of the beam, calculate the maximum permissible load P that may be placed at the midpoint if (a) the allowable shear stress in the glued joints is 0.3 MPa, and (b) the allowable bending stress in the plastic is 8 MPa.



Solution 5.8-8 Laminated plastic beam





$L = 320 \text{ mm}$
$W = 3.2 \text{ N}$

$$q = \frac{W}{L} = \frac{3.2 \text{ N}}{320 \text{ mm}} = 10 \text{ N/m}$$

$$I = \frac{bh^3}{12} = \frac{1}{12}(30 \text{ mm})(30 \text{ mm})^3 = 67{,}500 \text{ mm}^4$$

$$S = \frac{bh^2}{6} = \frac{1}{6}(30 \text{ mm})(30 \text{ mm})^2 = 4500 \text{ mm}^3$$

(a) Allowable load based upon shear in glued joints

$\tau_{allow} = 0.3$ MPa

$$\tau = \frac{VQ}{Ib} \qquad V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + 1.6\,\text{N}$$

(V = newtons; P = newtons)

$Q = (30\text{ mm})(10\text{ mm})(10\text{ mm}) = 3000\text{ mm}^3$

$$\frac{Q}{Ib} = \frac{3000\text{ mm}^3}{(67{,}500\text{ mm}^4)(30\text{ mm})} = \frac{1}{675\text{ mm}^2}$$

$$\tau = \frac{VQ}{Ib} = \frac{P/2 + 1.6\text{ N}}{675\text{ mm}^2} \qquad (\tau = \text{N/mm}^2 = \text{MPa})$$

Solve for P:

$P = 1350\tau_{allow} - 3.2 = 405\text{ N} - 3.2\text{ N} = 402\text{ N}$ ←

(b) Allowable load based upon bending stresses

$\sigma_{allow} = 8$ MPa

$$\sigma = \frac{M_{max}}{S}$$

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = 0.08P + 0.128\ (\text{N} \cdot \text{m})$$

(P = newtons; $M = \text{N} \cdot \text{m}$)

$$\sigma = \frac{(0.08P + 0.128)(\text{N} \cdot \text{m})}{4.5 \times 10^{-6}\text{ m}^3}$$

($\sigma = \text{N/m}^2 = \text{Pa}$)

Solve for P:

$P = (56.25 \times 10^{-6})\,\sigma_{allow} - 1.6$

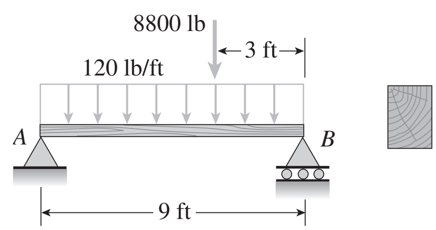
$= (56.25 \times 10^{-6})(8 \times 10^6\text{ Pa}) - 1.6$

$= 450 - 1.6 = 448\text{ N}$ ←

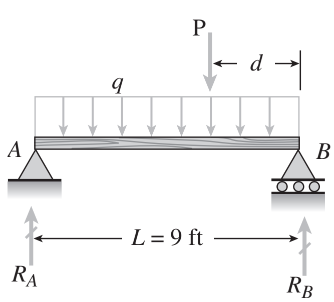
Problem 5.8-9 A wood beam AB on simple supports with span length equal to 9 ft is subjected to a uniform load of intensity 120 lb/ft acting along the entire length of the beam and a concentrated load of magnitude 8800 lb acting at a point 3 ft from the right-hand support (see figure). The allowable stresses in bending and shear, respectively, are 2500 psi and 150 psi.

(a) From the table in Appendix F, select the lightest beam that will support the loads (disregard the weight of the beam).

(b) Taking into account the weight of the beam (weight density = 35 lb/ft³), verify that the selected beam is satisfactory, or, if it is not, select a new beam.



Solution 5.8-9



$q = 120$ lb/ft

$P = 8800$ lb

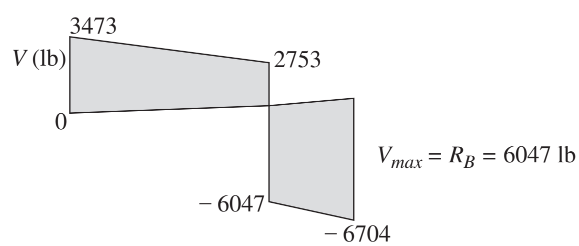
$d = 3$ ft

$\sigma_{allow} = 2500$ psi

$\tau_{allow} = 150$ psi

$$R_A = \frac{qL}{2} + \frac{P}{3} \qquad R_B = \frac{qL}{2} + \frac{2P}{3}$$

(a) Disregarding the weight of the beam



$$R_A = \frac{(120\text{ lb/ft})(9\text{ ft})}{2} + \frac{8800\text{ lb}}{3} = 3473\text{ lb}$$

$$R_B = 540\text{ lb} + \frac{2}{3}(8800\text{ lb}) = 6407\text{ lb}$$

$V_{max} = R_B = 6407$ lb

Maximum bending moment occurs under the concentrated load.

$$M_{max} = R_B d - \frac{qd^2}{2}$$

$$= (6407 \text{ lb})(3 \text{ ft}) - \frac{1}{2}(120 \text{ lb/ft})(3 \text{ ft})^2$$

$$= 18{,}680 \text{ lb-ft} = 224{,}200 \text{ lb-in.}$$

$$\tau_{max} = \frac{3V}{2A} \quad A_{req} = \frac{3V_{max}}{2\tau_{allow}} = \frac{3(6407 \text{ lb})}{2(150 \text{ psi})} = 64.1 \text{ in.}^2$$

$$\sigma = \frac{M}{S} \quad S_{req} = \frac{M_{max}}{\sigma_{allow}} = \frac{224{,}200 \text{ lb-in.}}{2500 \text{ psi}} = 89.7 \text{ in.}^3$$

From Appendix F: Select 8 × 10 in. beam (nominal dimensions) ←

$A = 71.25 \text{ in.}^2 \qquad S = 112.8 \text{ in.}^3$

(b) Considering the weight of the beam

$q_{BEAM} = 17.3$ lb/ft (weight density = 35 lb/ft^3)

$$R_B = 6407 \text{ lb} + \frac{(17.3 \text{ lb/ft})(9 \text{ ft})}{2} = 6407 + 78 = 6485 \text{ lb}$$

$$V_{max} = 6485 \text{ lb} \quad A_{req'd} = \frac{3V_{max}}{2\tau_{allow}} = 64.9 \text{ in.}^2$$

8 × 10 beam is still satisfactory for shear.

$q_{TOTAL} = 120 \text{ lb/ft} + 17.3 \text{ lb/ft} = 137.3 \text{ lb/ft}$

$$M_{max} = R_B d - \frac{qd^2}{2} = (6485 \text{ lb})(3 \text{ ft}) - \frac{1}{2}\left(137.3 \frac{\text{lb}}{\text{ft}}\right)(3 \text{ ft})^2$$

$$= 18{,}837 \text{ lb-ft} = 226{,}050 \text{ lb-in.}$$

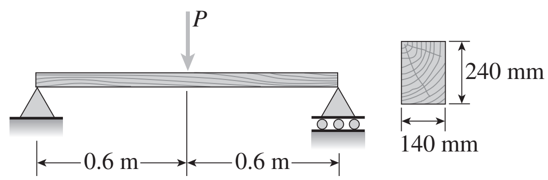
$$S_{req'd} = \frac{M_{max}}{\sigma_{allow}} = \frac{226{,}050 \text{ lb-in.}}{2500 \text{ psi}} = 90.4 \text{ in.}^3$$

8 × 10 beam is still satisfactory for moment.

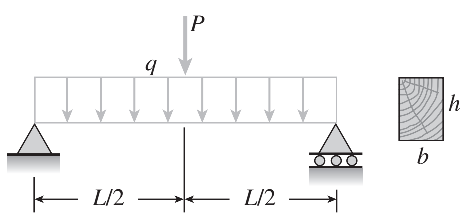
Use 8 × 10 in. beam ←

Problem 5.8-10 A simply supported wood beam of rectangular cross section and span length 1.2 m carries a concentrated load P at midspan in addition to its own weight (see figure). The cross section has width 140 mm and height 240 mm. The weight density of the wood is 5.4 kN/m^3.

Calculate the maximum permissible value of the load P if (a) the allowable bending stress is 8.5 MPa, and (b) the allowable shear stress is 0.8 MPa.



Solution 5.8-10 **Simply supported wood beam**



$b = 140 \text{ mm} \qquad h = 240 \text{ mm} \qquad A = bh = 33{,}600 \text{ mm}^2$

$$S = \frac{bh^2}{6} = 1344 \times 10^3 \text{ mm}^3$$

$\gamma = 5.4 \text{ kN/m}^3$

$L = 1.2 \text{ m} \quad q = \gamma bh = 181.44 \text{ N/m}$

(a) Allowable load P based upon bending stress

$$\sigma_{allow} = 8.5 \text{ MPa} \quad \sigma = \frac{M_{max}}{S}$$

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = \frac{P(1.2 \text{ m})}{4} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})^2}{8}$$

$$= 0.3P + 32.66 \text{ N} \cdot \text{m} \quad (P = \text{newtons}; M = \text{N} \cdot \text{m})$$

$$M_{max} = S\sigma_{allow} = (1344 \times 10^3 \text{ mm}^3)(8.5 \text{ MPa}) = 11{,}424 \text{ N} \cdot \text{m}$$

Equate values of M_{max} and solve for P:

$0.3P + 32.66 = 11{,}424 \qquad P = 37{,}970 \text{ N}$

or $\quad P = 38.0 \text{ kN}$

(b) Allowable load P based upon shear stress

$$\tau_{allow} = 0.8 \text{ MPa} \quad \tau = \frac{3V}{2A}$$

$$V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})}{2}$$

$$= \frac{P}{2} + 108.86 \text{ (N)}$$

$$V = \frac{2A\tau}{3} = \frac{2}{3}(33{,}600 \text{ mm}^2)(0.8 \text{ MPa}) = 17{,}920 \text{ N}$$

Equate values of V and solve for P:

$\frac{P}{2} + 108.86 = 17{,}920 \quad P = 35{,}622 \text{ N}$

or $\quad P = 35.6 \text{ kN}$ ←

Note: The shear stress governs and $P_{allow} = 35.6$ kN

Problem 5.8-11 A square wood platform, 8 ft × 8 ft in area, rests on masonry walls (see figure). The deck of the platform is constructed of 2 in. nominal thickness tongue-and-groove planks (actual thickness 1.5 in.; see Appendix F) supported on two 8-ft long beams. The beams have 4 in. × 6 in. nominal dimensions (actual dimensions 3.5 in. × 5.5 in.).

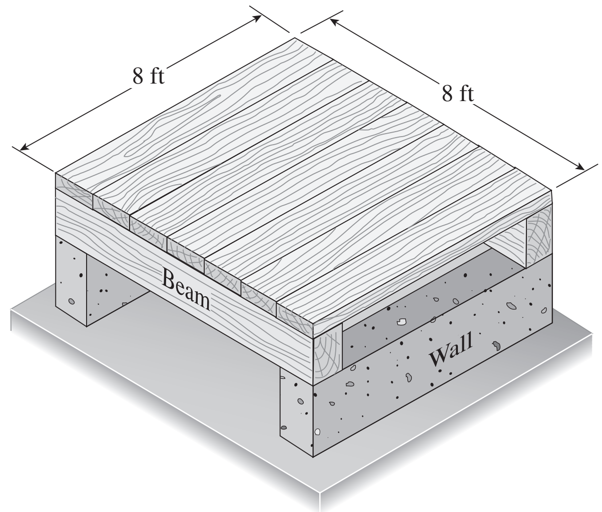
The planks are designed to support a uniformly distributed load w (lb/ft^2) acting over the entire top surface of the platform. The allowable bending stress for the planks is 2400 psi and the allowable shear stress is 100 psi. When analyzing the planks, disregard their weights and assume that their reactions are uniformly distributed over the top surfaces of the supporting beams.

(a) Determine the allowable platform load w_1 (lb/ft^2) based upon the bending stress in the planks.

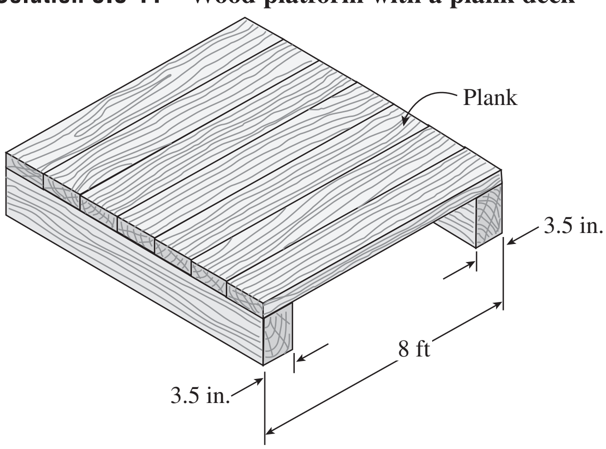
(b) Determine the allowable platform load w_2 (lb/ft^2) based upon the shear stress in the planks.

(c) Which of the preceding values becomes the allowable load w_{allow} on the platform?

(*Hints:* Use care in constructing the loading diagram for the planks, noting especially that the reactions are distributed loads instead of concentrated loads. Also, note that the maximum shear forces occur at the inside faces of the supporting beams.)



Solution 5.8-11 Wood platform with a plank deck



Platform: 8 ft × 8 ft

t = thickness of planks
= 1.5 in.

w = uniform load on the deck (lb/ft^2)

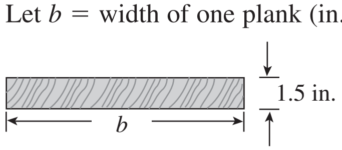
$\sigma_{allow} = 2400$ psi

$\tau_{allow} = 100$ psi

Find w_{allow} (lb/ft^2)

(a) Allowable load based upon bending stress in the planks

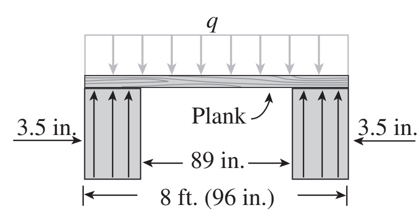
Let b = width of one plank (in.)



$A = 1.5b$ (in.^{2})

$$S = \frac{b}{6}(1.5\text{ in.})^2 = 0.375b\ (\text{in.}^3)$$

Free-body diagram of one plank supported on the beams:



Load on one plank:

$$q = \left[\frac{w\ (\text{lb/ft}^2)}{144\text{ in.}^2/\text{ft}^2}\right](b\text{ in.}) = \frac{wb}{144}\ (\text{lb/in.})$$

Reaction $\quad R = q\left(\frac{96\text{ in.}}{2}\right) = \left(\frac{wb}{144}\right)(48) = \frac{wb}{3}$

(R = lb; w = lb/ft^2; b = in.)

M_{max} occurs at midspan.

$$M_{max} = R\left(\frac{3.5\text{ in.}}{2} + \frac{89\text{ in.}}{2}\right) - \frac{q(48\text{ in.})^2}{2}$$

$$= \frac{wb}{3}(46.25) - \frac{wb}{144}(1152) = \frac{89}{12}wb$$

(M = lb-in.; w = lb/ft^2; b = in.)

Allowable bending moment:

$M_{allow} = \sigma_{allow} S = (2400\text{ psi})(0.375b) = 900b$ (lb-in.)

Equate M_{max} and M_{allow} and solve for w:

$\frac{89}{12}wb = 900b \quad w_1 = 121\text{ lb/ft}^2$ ←

(b) ALLOWABLE LOAD BASED UPON SHEAR STRESS IN THE PLANKS

See the free-body diagram in part (a).
V_{max} occurs at the inside face of the support.

$$V_{max} = q\left(\frac{89 \text{ in.}}{2}\right) = 44.5q = (44.5)\left(\frac{wb}{144}\right) = \frac{89\,wb}{288}$$

$(V = \text{lb};\ w = \text{lb/ft}^2;\ b = \text{in.})$

Allowable shear force:

$$\tau = \frac{3V}{2A} \qquad V_{allow} = \frac{2A\tau_{allow}}{3} = \frac{2(1.5b)(100 \text{ psi})}{3} = 100b \text{ (lb)}$$

EQUATE V_{max} AND V_{allow} AND SOLVE FOR w:

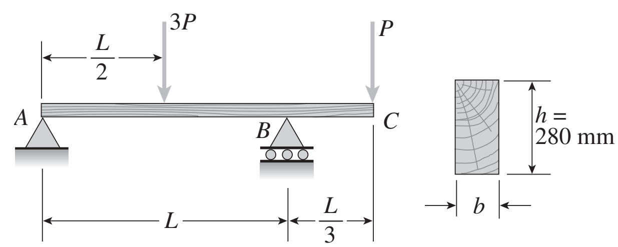
$$\frac{89wb}{288} = 100b \qquad w_2 = 324 \text{ lb/ft}^2 \quad \longleftarrow$$

(c) ALLOWABLE LOAD
Bending stress governs, $w_{allow} = 121 \text{ lb/ft}^2$ ⟵

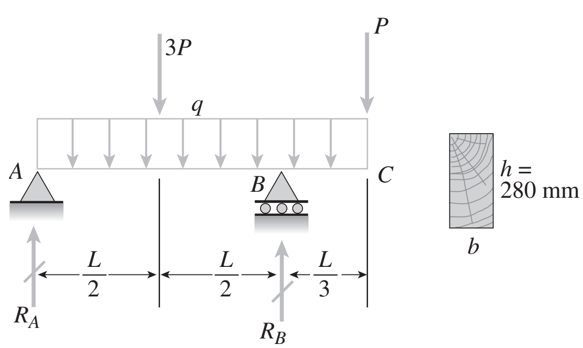
Problem 5.8-12 A wood beam ABC with simple supports at A and B and an overhang BC has height $h = 280$ mm (see figure). The length of the main span of the beam is $L = 3.6$ m and the length of the overhang is $L/3 = 1.2$ m. The beam supports a concentrated load $3P = 15$ kN at the midpoint of the main span and a load $P = 5$ kN at the free end of the overhang. The wood has weight density $\gamma = 5.5$ kN/m³.

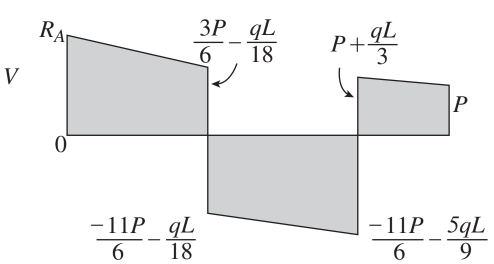
(a) Determine the required width b of the beam based upon an allowable bending stress of 8.2 MPa.

(b) Determine the required width based upon an allowable shear stress of 0.7 MPa.

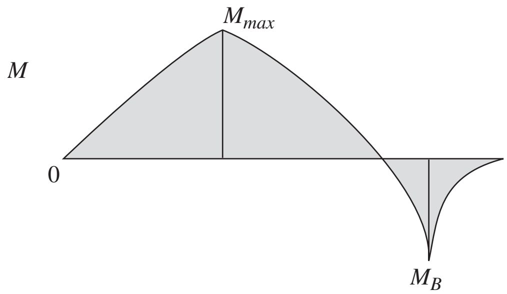


Solution 5.8-12 **Rectangular beam with an overhang**





$L = 3.6$ m
$P = 5$ kN
$\gamma = 5.5$ kN/m³ (for the wood)
$q = \gamma bh$



FIND b

$$R_A = \frac{7P}{6} + \frac{4qL}{9}$$

$$R_B = \frac{17P}{6} + \frac{8qL}{9}$$

$$V_{max} = \frac{11P}{6} + \frac{5qL}{9}$$

$$M_{max} = \frac{7PL}{12} + \frac{7qL^2}{72} \qquad M_B = -\frac{PL}{3} - \frac{qL^2}{18}$$

(a) REQUIRED WIDTH b BASED UPON BENDING STRESS

$$M_{max} = \frac{7PL}{12} + \frac{7qL^2}{72} = \frac{7}{12}(5000\text{ N})(3.6\text{ m}) + \frac{7}{72}(\gamma bh)(3.6\text{ m})^2$$

$$= 10{,}500\text{ N}\cdot\text{m} + \frac{7}{72}(5500\text{ N/m}^3)(\text{b})(0.280\text{ m})(3.6\text{ m})^2$$

$$= 10{,}500 + 1940.4b \quad (b = \text{meters}) \quad (M = \text{newton-meters})$$

$$\sigma = \frac{M_{max}}{S} = \frac{6M_{max}}{bh^2} \qquad \sigma_{allow} = 8.2\text{ MPa}$$

$$M_{max} = \frac{bh^2\sigma_{allow}}{6} = \frac{b}{6}(0.280\text{ m})^2(8.2 \times 10^6\text{ Pa})$$

$$= 107{,}150b$$

EQUATE MOMENTS AND SOLVE FOR b:

$10{,}500 + 1940.4b = 107{,}150b$

$b = 0.0998\text{ m} = 99.8\text{ mm}$ ←

(b) REQUIRED WIDTH b BASED UPON SHEAR STRESS

$$V_{max} = \frac{11P}{6} + \frac{5qL}{9}$$

$$= \frac{11}{6}(5000\text{ N}) + \frac{5}{9}(\gamma bh)(3.6\text{ m})$$

$$= 9167\text{ N} + \frac{5}{9}(5500\text{ N/m}^3)(b)(0.280\text{ m})(3.6\text{ m})$$

$$= 9167 + 3080b \quad (b = \text{meters})$$

$$\tau = \frac{3V_{max}}{2A} = \frac{3V_{max}}{2bh} \qquad (V = \text{newtons})$$

$$\tau_{allow} = 0.7\text{ MPa}$$

$$V_{max} = \frac{2bh\tau_{allow}}{3} = \frac{2b}{3}(0.280\text{ m})(0.7 \times 10^6\text{ N/m}^2)$$

$$= 130{,}670b$$

EQUATE SHEAR FORCES AND SOLVE FOR b:

$9167 + 3080b = 130{,}670b$

$b = 0.0718\text{ m} = 71.8\text{ mm}$ ←

NOTE: Bending stress governs. $b = 99.8\text{ mm}$