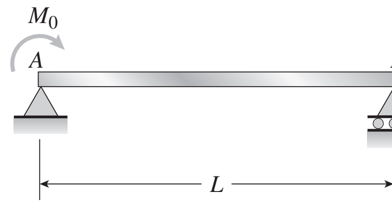
Castigliano's Theorem

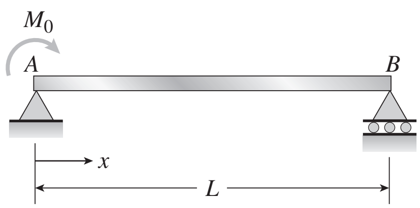
The beams described in the problems for Section 9.9 have constant flexural rigidity EI.

Problem 9.9-1 A simple beam AB of length L is loaded at the left-hand end by a couple of moment M_0 (see figure).

Determine the angle of rotation θ_A at support A. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-1 Simple beam with couple M_0



$$R_A = \frac{M_0}{L} \quad \text{(downward)}$$

$$M = M_0 - R_A x = M_0 - \frac{M_0 x}{L}$$

$$= M_0\left(1 - \frac{x}{L}\right)$$

STRAIN ENERGY

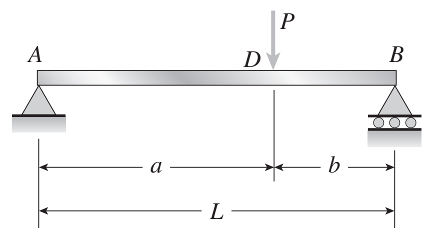
$$U = \int \frac{M^2 dx}{2EI} = \frac{M_0^2}{2EI}\int_0^L \left(1 - \frac{x}{L}\right)^2 dx = \frac{M_0^2 L}{6EI}$$

CASTIGLIANO'S THEOREM $\quad \theta_A = \frac{dU}{dM_0} = \frac{M_0 L}{3EI}$ (clockwise) ←

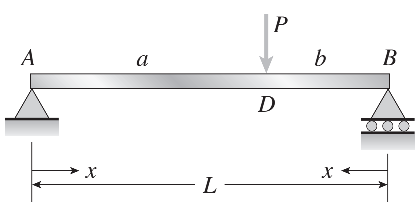
(This result agree with Case 7, Table G-2)

Problem 9.9-2 The simple beam shown in the figure supports a concentrated load P acting at distance a from the left-hand support and distance b from the right-hand support.

Determine the deflection δ_D at point D where the load is applied. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-2 Simple beam with load P



$$R_A = \frac{Pb}{L} \quad R_B = \frac{Pa}{L}$$

$$M_{AD} = R_A x = \frac{Pbx}{L}$$

$$M_{DB} = R_B x = \frac{Pax}{L}$$

STRAIN ENERGY $\quad U = \int \frac{M^2 dx}{2EI}$

$$U_{AD} = \frac{1}{2EI}\int_0^a \left(\frac{Pbx}{L}\right)^2 dx = \frac{P^2 a^3 b^2}{6EIL^2}$$

$$U_{DB} = \frac{1}{2EI}\int_0^b \left(\frac{Pax}{L}\right)^2 dx = \frac{P^2 a^2 b^3}{6EIL^2}$$

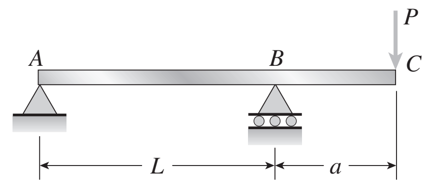
$$U = U_{AD} + U_{DB} = \frac{P^2 a^2 b^2}{6LEI}$$

CASTIGLIANO'S THEOREM

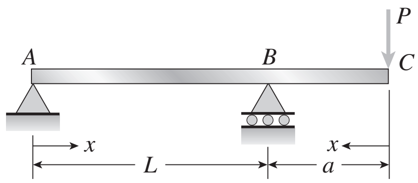
$$\delta_D = \frac{dU}{dP} = \frac{Pa^2 b^2}{3LEI} \quad \text{(downward)} \leftarrow$$

Problem 9.9-3 An overhanging beam ABC supports a concentrated load P at the end of the overhang (see figure). Span AB has length L and the overhang has length a.

Determine the deflection δ_C at the end of the overhang. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-3 **Overhanging beam**



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$M_{AB} = -R_A x = -\frac{Pax}{L}$$

$$M_{CB} = -Px$$

STRAIN ENERGY $\quad U = \int \frac{M^2 dx}{2EI}$

$$U_{AB} = \frac{1}{2EI}\int_0^L \left(-\frac{Pax}{L}\right)^2 dx = \frac{P^2a^2L}{6EI}$$

$$U_{CB} = \frac{1}{2EI}\int_0^a (-Px)^2 dx = \frac{P^2a^3}{6EI}$$

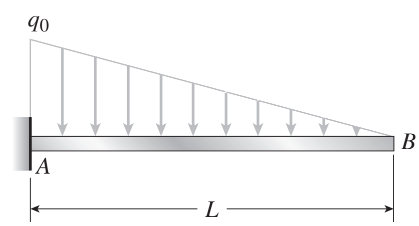
$$U = U_{AB} + U_{CB} = \frac{P^2a^2}{6EI}(L + a)$$

CASTIGLIANO'S THEOREM

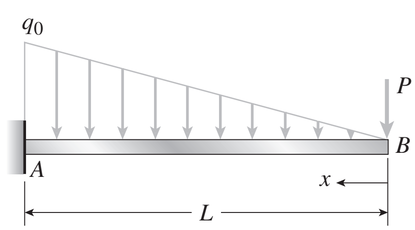
$$\delta_C = \frac{dU}{dP} = \frac{Pa^2}{3EI}(L + a) \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.9-4 The cantilever beam shown in the figure supports a triangularly distributed load of maximum intensity q_0.

Determine the deflection δ_B at the free end B. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-4 **Cantilever beam with triangular load**



P = fictitious load corresponding to deflection δ_B

$$M = -Px - \frac{q_0 x^3}{6L}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI}\int_0^L \left(-Px - \frac{q_0 x^3}{6L}\right)^2 dx$$

$$= \frac{P^2L^3}{6EI} + \frac{Pq_0L^4}{30EI} + \frac{q_0^2 L^5}{42EI}$$

CASTIGLIANO'S THEOREM

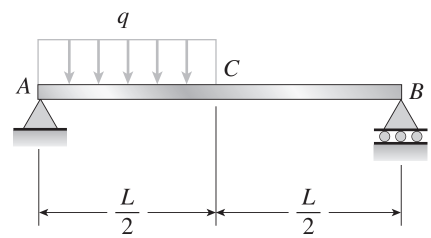
$$\delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{q_0 L^4}{30EI} \quad \text{(downward)}$$

(This result agrees with Cases 1 and 8 of Table G-1.)

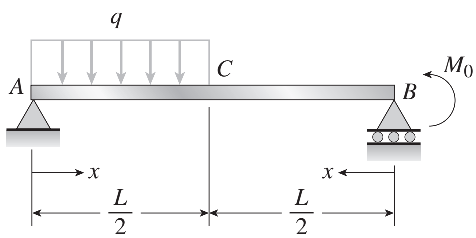
SET $P = 0$: $\quad \delta_B = \dfrac{q_0 L^4}{30EI} \quad \longleftarrow$

Problem 9.9-5 A simple beam *ACB* supports a uniform load of intensity q on the left-hand half of the span (see figure).

Determine the angle of rotation θ_B at support B. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-5 Simple beam with partial uniform load



M_0 = fictitious load corresponding to angle of rotation θ_B

$$R_A = \frac{3qL}{8} + \frac{M_0}{L} \quad R_B = \frac{qL}{8} - \frac{M_0}{L}$$

Bending moment and partial derivative for segment *AC*

$$M_{AC} = R_A x - \frac{qx^2}{2} = \left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}$$

$$\left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = \frac{x}{L}$$

Bending moment and partial derivative for segment *CB*

$$M_{CB} = R_B x + M_0 = \left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0$$

$$\left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -\frac{x}{L} + 1$$

Modified Castigliano's theorem (Eq. 9-88)

$$\theta_B = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{3qL}{8} + \frac{M_0}{L}\right)x - \frac{qx^2}{2}\right]\left[\frac{x}{L}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/2}\left[\left(\frac{qL}{8} - \frac{M_0}{L}\right)x + M_0\right]\left[1 - \frac{x}{L}\right]dx$$

Set fictitious load M_0 equal to zero

$$\theta_B = \frac{1}{EI}\int_0^{L/2}\left(\frac{3qLx}{8} - \frac{qx^2}{2}\right)\left(\frac{x}{L}\right)dx$$

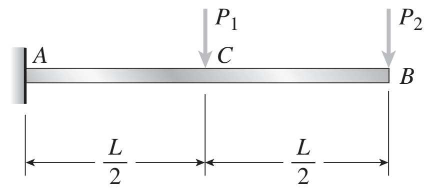
$$+ \frac{1}{EI}\int_0^{L/2}\left(\frac{qLx}{8}\right)\left(1 - \frac{x}{L}\right)dx$$

$$= \frac{qL^3}{128EI} + \frac{qL^3}{96EI} = \frac{7qL^3}{384EI} \quad \text{(counterclockwise)} \longleftarrow$$

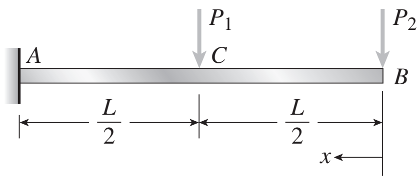
(This result agrees with Case 2, Table G-2.)

Problem 9.9-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2, as shown in the figure.

Determine the deflections δ_C and δ_B at points C and B, respectively. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-6 Cantilever beam with loads P_1 and P_2



Bending moment and partial derivatives for segment CB

$$M_{CB} = -P_2 x \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{CB}}{\partial P_1} = 0 \qquad \frac{\partial M_{CB}}{\partial P_2} = -x$$

Bending moment and partial derivatives for segment AC

$$M_{AC} = -P_1\left(x - \frac{L}{2}\right) - P_2 x \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{AC}}{\partial P_1} = \frac{L}{2} - x \qquad \frac{\partial M_{AC}}{\partial P_2} = -x$$

Modified Castigliano's theorem for deflection δ_C

$$\delta_C = \frac{1}{EI}\int_0^{L/2} (M_{CB})\left(\frac{\partial M_{CB}}{\partial P_1}\right)dx + \frac{1}{EI}\int_{L/2}^{L} (M_{AC})\left(\frac{\partial M_{AC}}{\partial P_1}\right)dx$$

$$= 0 + \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2 x\right]\left(\frac{L}{2} - x\right)dx$$

$$= \frac{L^3}{48EI}(2P_1 + 5P_2) \quad \longleftarrow$$

Modified Castigliano's theorem for deflection δ_B

$$\delta_B = \frac{1}{EI}\int_0^{L/2} (M_{CB})\left(\frac{\partial M_{CB}}{\partial P_2}\right)dx + \frac{1}{EI}\int_{L/2}^{L} (M_{AC})\left(\frac{\partial M_{AC}}{\partial P_2}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/2} (-P_2 x)(-x)\,dx + \frac{1}{EI}\int_{L/2}^{L}\left[-P_1\left(x - \frac{L}{2}\right) - P_2 x\right](-x)\,dx$$

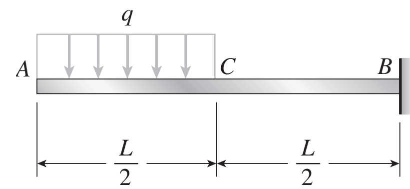
$$= \frac{P_2 L^3}{24EI} + \frac{L^3}{48EI}(5P_1 + 14P_2)$$

$$= \frac{L^3}{48EI}(5P_1 + 16P_2) \quad \longleftarrow$$

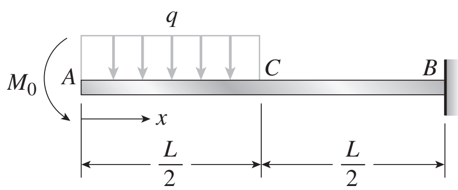
(These results can be verified with the aid of Cases 4 and 5, Table G-1.)

Problem 9.9-7 The cantilever beam *ACB* shown in the figure is subjected to a uniform load of intensity q acting between points *A* and *C*.

Determine the angle of rotation θ_A at the free end *A*. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-7 Cantilever beam with partial uniform load



M_0 = fictitious load corresponding to the angle of rotation θ_A

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT *AC*

$$M_{AC} = -M_0 - \frac{qx^2}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = -1$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT *CB*

$$M_{CB} = -M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right) \quad \left(\frac{L}{2} \le x \le L\right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -1$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

$$\theta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right) dx$$

$$= \frac{1}{EI}\int_0^{L/2}\left(-M_0 - \frac{qx^2}{2}\right)(-1)\,dx$$

$$+ \frac{1}{EI}\int_{L/2}^{L}\left[-M_0 - \frac{qL}{2}\left(x - \frac{L}{4}\right)\right](-1)\,dx$$

SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\theta_A = \frac{1}{EI}\int_0^{L/2}\frac{qx^2}{2}\,dx + \frac{1}{EI}\int_{L/2}^{L}\left(\frac{qL}{2}\right)\left(x - \frac{L}{4}\right)dx$$

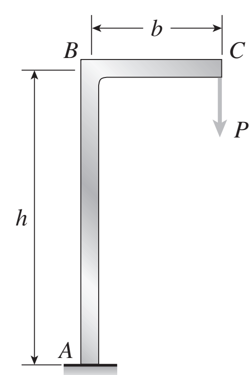
$$= \frac{qL^3}{48EI} + \frac{qL^3}{8EI}$$

$$= \frac{7qL^3}{48EI} \quad \text{(counterclockwise)} \quad \longleftarrow$$

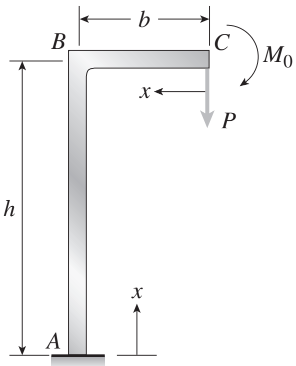
(This result can be verified with the aid of Case 3, Table G-1.)

Problem 9.9-8 The frame *ABC* supports a concentrated load *P* at point *C* (see figure). Members *AB* and *BC* have lengths h and b, respectively.

Determine the vertical deflection δ_C and angle of rotation θ_C at end *C* of the frame. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-8 Frame with concentrated load



P = concentrated load acting at point C (corresponding to the deflection δ_C)

M_0 = fictitious moment corresponding to the angle of rotation θ_C

BENDING MOMENT AND PARTIAL DERIVATIVES FOR MEMBER AB

$$M_{AB} = Pb + M_0 \quad (0 \le x \le h)$$

$$\frac{\partial M_{AB}}{\partial P} = b \quad \frac{\partial M_{AB}}{M_0} = 1$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR MEMBER BC

$$M_{BC} = Px + M_0 \quad (0 \le x \le b)$$

$$\frac{\partial M_{BC}}{\partial P} = x \quad \frac{\partial M_{BC}}{\partial M_0} = 1$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(b)dx + \frac{1}{EI}\int_0^b (Px + M_0)(x)dx$$

Set $M_0 = 0$:

$$\delta_C = \frac{1}{EI}\int_0^h Pb^2 dx + \frac{1}{EI}\int_0^b Px^2 dx$$

$$= \frac{Pb^2}{3\,EI}(3h + b) \quad \text{(downward)} \quad \longleftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_C

$$\theta_C = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_0}\right) dx$$

$$= \frac{1}{EI}\int_0^h (Pb + M_0)(1)dx + \frac{1}{EI}\int_0^b (Px + M_0)(1)\, dx$$

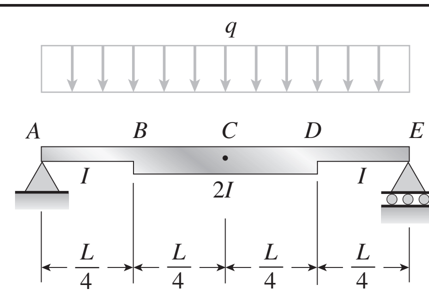
Set $M_0 = 0$:

$$\theta_C = \frac{1}{EI}\int_0^h Pb\, dx + \frac{1}{EI}\int_0^b Px\, dx$$

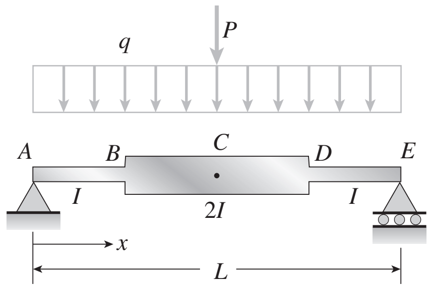
$$= \frac{Pb}{2EI}(2h + b) \quad \text{(clockwise)} \quad \longleftarrow$$

Problem 9.9-9 A simple beam $ABCDE$ supports a uniform load of intensity q (see figure). The moment of inertia in the central part of the beam (BCD) is twice the moment of inertia in the end parts (AB and DE).

Find the deflection δ_C at the midpoint C of the beam. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-9 Nonprismatic beam



P = fictitious load corresponding to the deflection δ_C at the midpoint

$$R_A = \frac{qL}{2} + \frac{P}{2}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR THE LEFT-HAND HALF OF THE BEAM (A TO C)

$$M_{AC} = \frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial P} = \frac{x}{2} \quad \left(0 \le x \le \frac{L}{2}\right)$$

MODIFIED CASTIGLIANO'S THEOREM (EQ. 9-88)

Integrate from A to C and multiply by 2.

$$\delta_C = 2\int \left(\frac{M_{AC}}{EI}\right)\left(\frac{\partial M_{AC}}{\partial P}\right)dx$$

$$= 2\left(\frac{1}{EI}\right)\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

$$+ 2\left(\frac{1}{2EI}\right)\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2}\right)\left(\frac{x}{2}\right)dx$$

SET FICTITIOUS LOAD P EQUAL TO ZERO

$$\delta_C = \frac{2}{EI}\int_0^{L/4}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

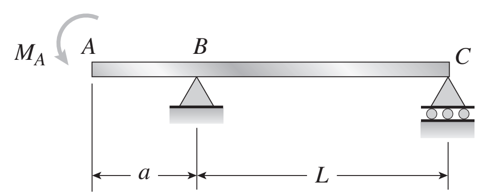
$$+ \frac{1}{EI}\int_{L/4}^{L/2}\left(\frac{qLx}{2} - \frac{qx^2}{2}\right)\left(\frac{x}{2}\right)dx$$

$$= \frac{13\,qL^4}{6{,}144\,EI} + \frac{67\,qL^4}{12{,}288\,EI}$$

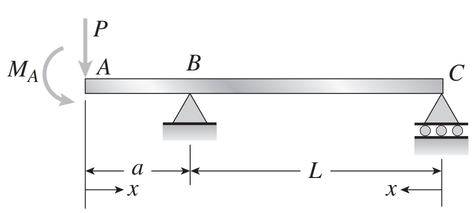
$$\delta_C = \frac{31qL^4}{4096\,EI} \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.9-10 An overhanging beam ABC is subjected to a couple M_A at the free end (see figure). The lengths of the overhang and the main span are a and L, respectively.

Determine the angle of rotation θ_A and deflection δ_A at end A. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-10 Overhanging beam *ABC*



M_A = couple acting at the free end A (corresponding to the angle of rotation θ_A)

P = fictitious load corresponding to the deflection δ_A

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AB

$$M_{AB} = -M_A - Px \quad (0 \le x \le a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = -1 \qquad \frac{\partial M_{AB}}{\partial P} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT BC

Reaction at support C: $R_C = \frac{M_A}{L} + \frac{Pa}{L}$ (downward)

$$M_{BC} = -R_C x = -\frac{M_A x}{L} - \frac{Pax}{L} \quad (0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial M_A} = -\frac{x}{L} \qquad \frac{\partial M_{BC}}{\partial P} = -\frac{ax}{L}$$

(Continued)

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF ROTATION θ_A

$$\theta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial M_A}\right) dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-1)\,dx$$

$$+ \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{x}{L}\right) dx$$

Set $P = 0$:

$$\theta_A = \frac{1}{EI}\int_0^a M_A dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{x}{L}\right) dx$$

$$= \frac{M_A}{3\,EI}(L + 3a) \quad \text{(counterclockwise)} \quad \longleftarrow$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_A

$$\delta_A = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right) dx$$

$$= \frac{1}{EI}\int_0^a (-M_A - Px)(-x) dx$$

$$+ \frac{1}{EI}\int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L}\right)\left(-\frac{ax}{L}\right) dx$$

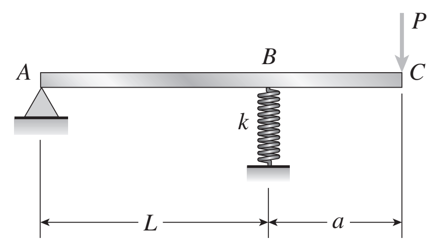
Set $P = 0$:

$$\delta_A = \frac{1}{EI}\int_0^a M_A x dx + \frac{1}{EI}\int_0^L \left(\frac{M_A x}{L}\right)\left(\frac{ax}{L}\right) dx$$

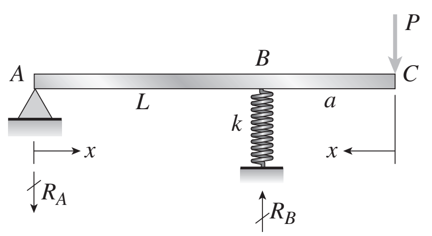
$$= \frac{M_A a}{6EI}(2L + 3a) \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.9-11 An overhanging beam *ABC* rests on a simple support at *A* and a spring support at *B* (see figure). A concentrated load *P* acts at the end of the overhang. Span *AB* has length *L*, the overhang has length *a*, and the spring has stiffness *k*.

Determine the downward displacement δ_C of the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-11 **Beam with spring support**



$$R_A = \frac{Pa}{L} \quad \text{(downward)}$$

$$R_B = \frac{P}{L}(L + a) \quad \text{(upward)}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT *AB*

$$M_{AB} = -R_A x = -\frac{Pax}{L} \qquad \frac{dM_{AB}}{dP} = -\frac{ax}{L} \qquad (0 \le x \le L)$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT *BC*

$$M_{BC} = -Px \qquad \frac{dM_{BC}}{dP} = -x \qquad (0 \le x \le a)$$

STRAIN ENERGY OF THE SPRING (EQ. 2-38a)

$$U_S = \frac{R_B^2}{2k} = \frac{P^2(L + a)^2}{2\,kL^2}$$

STRAIN ENERGY OF THE BEAM (EQ. 9-80a)

$$U_B = \int \frac{M^2 dx}{2\,EI}$$

TOTAL STRAIN ENERGY U

$$U = U_B + U_S = \int \frac{M^2 dx}{2\,EI} + \frac{P^2(L + a)^2}{2\,kL^2}$$

APPLY CASTIGLIANO'S THEOREM (EQ. 9-87)

$$\delta_C = \frac{dU}{dP} = \frac{d}{dP}\int \frac{M^2 dx}{2\,EI} + \frac{d}{dP}\left[\frac{P^2(L + a)^2}{2\,kL^2}\right]$$

$$= \frac{d}{dP}\int \frac{M^2 dx}{2\,EI} + \frac{P(L + a)^2}{kL^2}$$

DIFFERENTIATE UNDER THE INTEGRAL SIGN (MODIFIED CASTIGLIANO'S THEOREM)

$$\delta_C = \int \left(\frac{M}{EI}\right)\left(\frac{dM}{dP}\right)dx + \frac{P(L+a)^2}{kL^2}$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{Pax}{L}\right)\left(-\frac{ax}{L}\right)dx +$$

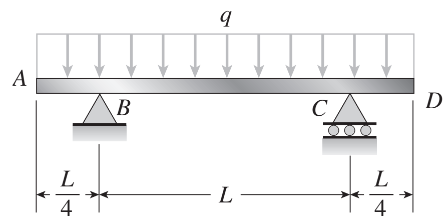
$$+ \frac{1}{EI}\int_0^a (-Px)(-x)dx + \frac{P(L+a)^2}{kL^2}$$

$$= \frac{Pa^2L}{3\,EI} + \frac{Pa^3}{3\,EI} + \frac{P(L+a)^2}{kL^2}$$

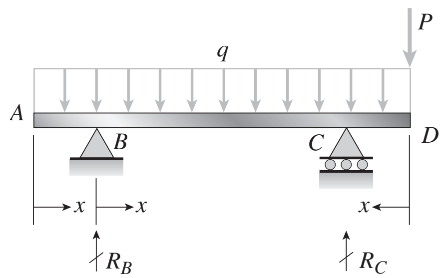
$$\delta_C = \frac{Pa^2(L+a)}{3\,EI} + \frac{P(L+a)^2}{kL^2} \quad \longleftarrow$$

Problem 9.9-12 A symmetric beam $ABCD$ with overhangs at both ends supports a uniform load of intensity q (see figure).

Determine the deflection δ_D at the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-12 **Beam with overhangs**



q = intensity of uniform load

P = fictitious load corresponding to the deflection δ_D

$\frac{L}{4}$ = length of segments AB and CD

L = length of span BC

$$R_B = \frac{3\,qL}{4} - \frac{P}{4} \qquad R_C = \frac{3\,qL}{4} + \frac{5P}{4}$$

BENDING MOMENTS AND PARTIAL DERIVATIVES

SEGMENT AB

$$M_{AB} = -\frac{qx^2}{2} \qquad \frac{\partial M_{AB}}{\partial P} = 0 \qquad \left(0 \le x \le \frac{L}{4}\right)$$

SEGMENT BC

$$M_{BC} = -\left[q\left(x + \frac{L}{4}\right)\right]\left[\frac{1}{2}\left(x + \frac{L}{4}\right)\right] + R_B x$$

$$= -\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3\,qL}{4} - \frac{P}{4}\right)x \quad (0 \le x \le L)$$

$$\frac{\partial M_{BC}}{\partial P} = -\frac{x}{4}$$

SEGMENT CD $\quad M_{CD} = -\frac{qx^2}{2} - Px \quad \left(0 \le x \le \frac{L}{4}\right)$

$$\frac{\partial M_{CD}}{\partial P} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_D

$$\delta_D = \int \left(\frac{M}{EI}\right)\left(\frac{\partial M}{\partial P}\right)dx$$

$$= \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(0)\,dx$$

$$+ \frac{1}{EI}\int_0^L \left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3qL}{4} - \frac{P}{4}\right)x\right] \times$$

$$\left[-\frac{x}{4}\right]dx + \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2} - Px\right)(-x)dx$$

SET $P = 0$:

$$\delta_D = \frac{1}{EI}\int_0^L \left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \frac{3qL}{4}x\right]\left[-\frac{x}{4}\right]dx$$

$$+ \frac{1}{EI}\int_0^{L/4}\left(-\frac{qx^2}{2}\right)(-x)\,dx$$

$$= -\frac{5\,qL^4}{768\,EI} + \frac{qL^4}{2048\,EI} = -\frac{37\,qL^4}{6144\,EI}$$

(Minus means the deflection is opposite in direction to the fictitious load P.)

$$\therefore\ \delta_D = \frac{37\,qL^4}{6144\,EI} \quad \text{(upward)} \quad \longleftarrow$$

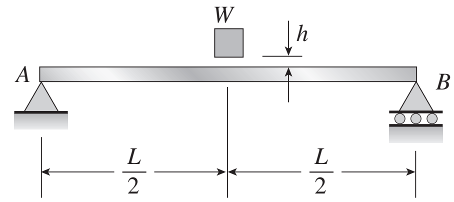
Deflections Produced by Impact

The beams described in the problems for Section 9.10 have constant flexural rigidity EI. Disregard the weights of the beams themselves, and consider only the effects of the given loads.

Problem 9.10-1 A heavy object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure).

Obtain a formula for the maximum bending stress σ_{max} due to the falling weight in terms of h, σ_{st}, and δ_{st}, where σ_{st} is the maximum bending stress and δ_{st} is the deflection at the midpoint when the weight W acts on the beam as a statically applied load.

Plot a graph of the ratio σ_{max}/σ_{st} (that is, the ratio of the dynamic stress to the static stress) versus the ratio h/δ_{st}. (Let h/δ_{st} vary from 0 to 10.)



Solution 9.10-1 **Weight W dropping onto a simple beam**

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

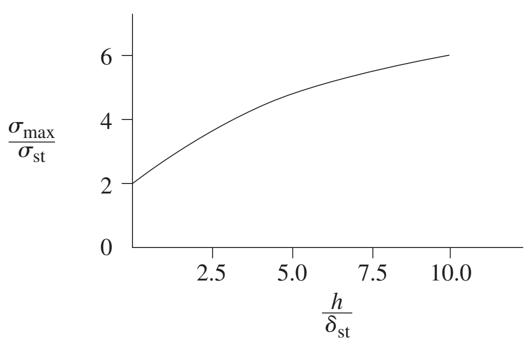
$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\sigma_{max} = \sigma_{st}\left[1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}\right] \longleftarrow$$

$\frac{h}{\delta_{st}}$	$\frac{\sigma_{max}}{\sigma_{st}}$
0	2.00
2.5	3.45
5.0	4.33
7.5	5.00
10.0	5.58

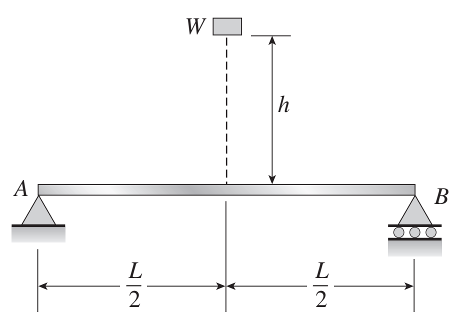
NOTE: $\delta_{st} = \frac{WL^3}{48EI}$ for a simple beam with a load at the midpoint.

GRAPH OF RATIO σ_{max}/σ_{st}



Problem 9.10-2 An object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure). The beam has a rectangular cross section of area A.

Assuming that h is very large compared to the deflection of the beam when the weight W is applied statically, obtain a formula for the maximum bending stress σ_{max} in the beam due to the falling weight.



Solution 9.10-2 **Weight W dropping onto a simple beam**

Height h is very large.

MAXIMUM DEFLECTION (EQ. 9-95)

$$\delta_{max} = \sqrt{2h\delta_{st}}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = \sqrt{\frac{2h}{\delta_{st}}}$$

$$\sigma_{max} = \sqrt{\frac{2h\sigma_{st}^2}{\delta_{st}}} \qquad (1)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \qquad \sigma_{st}^2 = \frac{W^2L^2}{16S^2}$$

$$\delta_{st} = \frac{WL^3}{48EI} \qquad \frac{\sigma_{st}^2}{\delta_{st}} = \frac{3WEI}{S^2L} \qquad (2)$$

For a RECTANGULAR BEAM (with b, depth d):

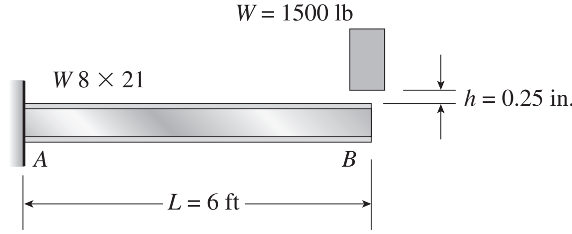
$$I = \frac{bd^3}{12} \qquad S = \frac{bd^2}{6} \qquad \frac{I}{S^2} = \frac{3}{bd} = \frac{3}{A} \qquad (3)$$

Substitute (2) and (3) into (1):

$$\sigma_{max} = \sqrt{\frac{18WhE}{AL}} \quad \longleftarrow$$

Problem 9.10-3 A cantilever beam AB of length $L = 6$ ft is constructed of a W 8×21 wide-flange section (see figure). A weight $W = 1500$ lb falls through a height $h = 0.25$ in. onto the end of the beam.

Calculate the maximum deflection δ_{max} of the end of the beam and the maximum bending stress σ_{max} due to the falling weight. (Assume $E = 30 \times 10^6$ psi.)



Solution 9.10-3 **Cantilever beam**

DATA: $L = 6$ ft $= 72$ in. $\quad W = 1500$ lb
$h = 0.25$ in. $\quad E = 30 \times 10^6$ psi
W 8×21 $\quad I = 75.3$ in.^{4} $\quad S = 18.2$ in.^{3}

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

Simple beam with load at midpoint:

$$k = \frac{48EI}{L^3}$$

Cantilever beam with load at the free end: $k = \dfrac{3EI}{L^3}$

For the cantilever beam in this problem:

$$\delta_{st} = \frac{WL^3}{3EI} = \frac{(1500\text{ lb})(72\text{ in.})^3}{3(30 \times 10^6\text{ psi})(75.3\text{ in.}^4)}$$

$$= 0.08261\text{ in.}$$

Equation (9-94):

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2} = 0.302\text{ in.} \quad \longleftarrow$$

MAXIMUM BENDING STRESS

Consider a cantilever beam with load P at the free end:

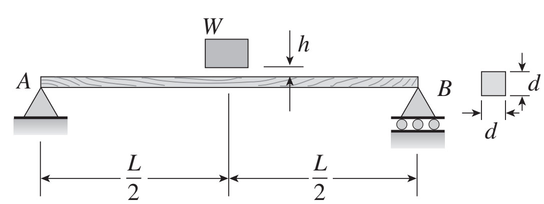
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{PL}{S} \qquad \delta_{max} = \frac{PL^3}{3EI}$$

$$\text{Ratio: } \frac{\sigma_{max}}{\delta_{max}} = \frac{3EI}{SL^2}$$

$$\therefore \sigma_{max} = \frac{3EI}{SL^2}\delta_{max} = 21{,}700\text{ psi} \quad \longleftarrow$$

Problem 9.10-4 A weight $W = 20$ kN falls through a height $h = 1.0$ mm onto the midpoint of a simple beam of length $L = 3$ m (see figure). The beam is made of wood with square cross section (dimension d on each side) and $E = 12$ GPa.

If the allowable bending stress in the wood is $\sigma_{allow} = 10$ MPa, what is the minimum required dimension d?



(Continued)

Solution 9.10-4 Simple beam with falling weight W

DATA: $W = 20$ kN $h = 1.0$ mm $L = 3.0$ m
$E = 12$ GPa $\sigma_{allow} = 10$ MPa

CROSS SECTION OF BEAM (SQUARE)

d = dimension of each side

$$I = \frac{d^4}{12} \quad S = \frac{d^3}{6}$$

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \left(\frac{WL}{4}\right)\left(\frac{6}{d^3}\right) = \frac{3WL}{2d^3} \quad (2)$$

$$\delta_{st} = \frac{WL^3}{48EI} = \frac{WL^3}{48E}\left(\frac{12}{d^4}\right) = \frac{WL^3}{4Ed^4} \quad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1)

$$\frac{2\sigma_{max}d^3}{3WL} = 1 + \left(1 + \frac{8hEd^4}{WL^3}\right)^{1/2}$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{2(10 \text{ MPa})d^3}{3(20 \text{ kN})(3.0 \text{ m})} = 1 + \left[1 + \frac{8(1.0 \text{ mm})(12 \text{ GPa})d^4}{(20 \text{ kN})(3.0 \text{ m})^3}\right]^{1/2}$$

$$\frac{1000}{9}d^3 - 1 = \left[1 + \frac{1600}{9}d^4\right]^{1/2} \quad (d = \text{meters})$$

SQUARE BOTH SIDES, REARRANGE, AND SIMPLIFY

$$\left(\frac{1000}{9}\right)^2 d^3 - \frac{1600}{9}d - \frac{2000}{9} = 0$$

$$2500d^3 - 36d - 45 = 0 \quad (d = \text{meters})$$

SOLVE NUMERICALLY

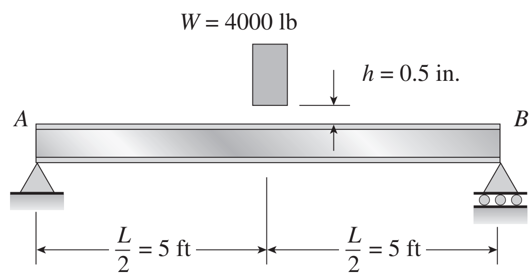
$d = 0.2804$ m $= 280.4$ mm

For minimum value, round upward.

$\therefore d = 281$ mm ←

Problem 9.10-5 A weight $W = 4000$ lb falls through a height $h = 0.5$ in. onto the midpoint of a simple beam of length $L = 10$ ft (see figure).

Assuming that the allowable bending stress in the beam is $\sigma_{allow} = 18{,}000$ psi and $E = 30 \times 10^6$ psi, select the lightest wide-flange beam listed in Table E-1 in Appendix E that will be satisfactory.



Solution 9.10-5 Simple beam of wide-flange shape

DATA: $W = 4000$ lb $h = 0.5$ in.
$L = 10$ ft $= 120$ in.
$\sigma_{allow} = 18{,}000$ psi $E = 30 \times 10^6$ psi

MAXIMUM DEFLECTION (EQ. 9-94)

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2h\delta_{st})^{1/2}$$

or
$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{st} = \frac{M}{S} = \frac{WL}{4S} \quad \delta_{st} = \frac{WL^3}{48EI}$$

$$\frac{\sigma_{max}}{\sigma_{st}} = \sigma_{allow}\left(\frac{4S}{WL}\right) = \frac{4\sigma_{allow}S}{WL} \quad (2)$$

$$\frac{2h}{\delta_{st}} = 2h\left(\frac{48EI}{WL^3}\right) = \frac{96hEI}{WL^3} \quad (3)$$

SUBSTITUTE (2) AND (3) INTO EQ. (1):

$$\frac{4\sigma_{allow}S}{WL} = 1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}$$

REQUIRED SECTION MODULUS

$$S = \frac{WL}{4\sigma_{allow}}\left[1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}\right]$$

SUBSTITUTE NUMERICAL VALUES

$$S = \left(\frac{20}{3}\text{ in.}^3\right)\left[1 + \left(1 + \frac{5\,I}{24}\right)^{1/2}\right] \qquad (4)$$

($S = \text{in.}^3$; $I = \text{in.}^4$)

PROCEDURE

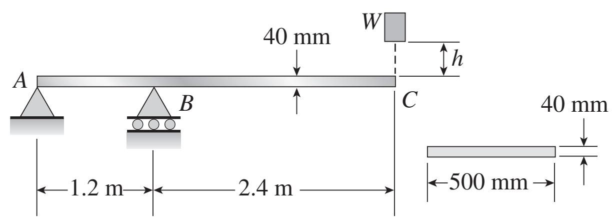
1. Select a trial beam from Table E-1.
2. Substitute I into Eq. (4) and calculate required S.
3. Compare with actual S for the beam.
4. Continue until the lightest beam is found.

Trial beam	Actual I	Actual S	Required S
W 8 × 35	127	31.2	41.6 (NG)
W 10 × 45	248	49.1	55.0 (NG)
W 10 × 60	341	66.7	63.3 (OK)
W 12 × 50	394	64.7	67.4 (NG)
W 14 × 53	541	77.8	77.8 (OK)
W 16 × 31	375	47.2	66.0 (NG)

Lightest beam is W 14 × 53 ⟵

Problem 9.10-6 An overhanging beam ABC of rectangular cross section has the dimensions shown in the figure. A weight $W = 750$ N drops onto end C of the beam.

If the allowable normal stress in bending is 45 MPa, what is the maximum height h from which the weight may be dropped? (Assume $E = 12$ GPa.)



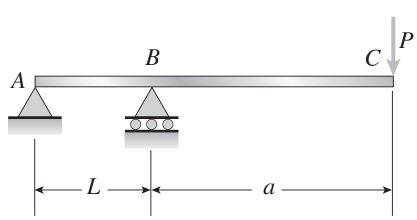
Solution 9.10-6 Overhanging beam

DATA: $W = 750$ N $\quad L_{AB} = 1.2$ in. $\quad L_{BC} = 2.4$ m
$E = 12$ GPa $\quad \sigma_{\text{allow}} = 45$ MPa

$$I = \frac{bd^3}{12} = \frac{1}{12}(500\text{ mm})(40\text{ mm})^3$$
$$= 2.6667 \times 10^6\text{ mm}^4$$
$$= 2.6667 \times 10^{-6}\text{ m}^4$$

$$S = \frac{bd^2}{6} = \frac{1}{6}(500\text{ mm})(40\text{ mm})^2$$
$$= 133.33 \times 10^3\text{ mm}^3$$
$$= 133.33 \times 10^{-6}\text{ m}^3$$

DEFLECTION δ_C AT THE END OF THE OVERHANG



P = load at end C
L = length of span AB
a = length of overhang BC

From the answer to Prob. 9.8-5 or Prob. 9.9-3:

$$\delta_C = \frac{Pa^2(L + a)}{3\,EI}$$

Stiffness of the beam: $$k = \frac{P}{\delta_C} = \frac{3\,EI}{a^2(L + a)} \qquad (1)$$

MAXIMUM DEFLECTION (EQ. 9-94)

Equation (9-94) may be used for any linearly elastic structure by substituting $\delta_{st} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

Simple beam with load at midpoint: $k = \dfrac{48\,EI}{L^3}$

Cantilever beam with load at free end: $k = \dfrac{3\,EI}{L^3}$ Etc.

For the overhanging beam in this problem (see Eq. 1):

$$\delta_{st} = \frac{W}{k} = \frac{Wa^2(L + a)}{3\,EI} \qquad (2)$$

in which $a = L_{BC}$ and $L = L_{AB}$:

$$\delta_{st} = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{3\,EI} \qquad (3)$$

EQUATION (9-94):

$$\delta_{max} = \delta_{st} + (\delta_{st}^2 + 2\,h\,\delta_{st})^{1/2}$$

or

$$\frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \qquad (4)$$

(Continued)

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ.

$$\therefore \frac{\sigma_{max}}{\sigma_{st}} = \frac{\delta_{max}}{\delta_{st}} = 1 + \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2} \quad (5)$$

$$\sigma_{st} = \frac{M}{S} = \frac{WL_{BC}}{S} \quad (6)$$

MAXIMUM HEIGHT h

Solve Eq. (5) for h:

$$\frac{\sigma_{max}}{\sigma_{st}} - 1 = \left(1 + \frac{2h}{\delta_{st}}\right)^{1/2}$$

$$\left(\frac{\sigma_{max}}{\sigma_{st}}\right)^2 - 2\left(\frac{\sigma_{max}}{\sigma_{st}}\right) + 1 = 1 + \frac{2h}{\delta_{st}}$$

$$h = \frac{\delta_{st}}{2}\left(\frac{\sigma_{max}}{\sigma_{st}}\right)\left(\frac{\sigma_{max}}{\sigma_{st}} - 2\right) \quad (7)$$

Substitute δ_{st} from Eq. (3), σ_{st} from Eq. (6), and σ_{allow} for σ_{max}:

$$h = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI}\left(\frac{\sigma_{allow}S}{WL_{BC}}\right)\left(\frac{\sigma_{allow}S}{WL_{BC}} - 2\right) \quad (8)$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (8):

$$\frac{W(L_{BC}^2)\ (L_{AB} + L_{BC})}{6EI} = 0.08100 \text{ m}$$

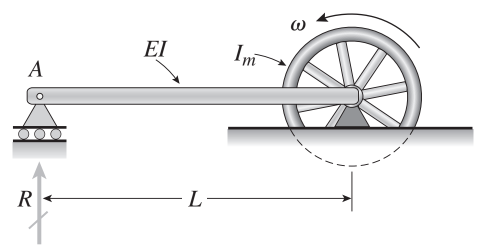
$$\frac{\sigma_{allow}\,S}{WL_{BC}} = \frac{10}{3} = 3.3333$$

$$h = (0.08100 \text{ m})\left(\frac{10}{3}\right)\left(\frac{10}{3} - 2\right) = 0.36 \text{ m}$$

or $h = 360$ mm ←

Problem 9.10-7 A heavy flywheel rotates at an angular speed ω (radians per second) around an axle (see figure). The axle is rigidly attached to the end of a simply supported beam of flexural rigidity EI .and length L (see figure). The flywheel has mass moment of inertia I_m about its axis of rotation.

If the flywheel suddenly freezes to the axle, what will be the reaction R at support A of the beam?



Solution 9.10-7 **Rotating flywheel**

NOTE: We will disregard the mass of the beam and all energy losses due to the sudden stopping of the rotating flywheel. Assume that *all* of the kinetic energy of the flywheel is transformed into strain energy of the beam.

KINETIC ENERGY OF ROTATING FLYWHEEL

$$KE = \frac{1}{2} I_m\, \omega^2$$

STRAIN ENERGY OF BEAM $\quad U = \int \frac{M^2 dx}{2\,EI}$

$M = Rx$, where x is measured from support A.

$$U = \frac{1}{2\,EI}\int_0^L (Rx)^2 dx = \frac{R^2 L^3}{6\,EI}$$

CONSERVATION OF ENERGY

$$KE = U \qquad \frac{1}{2} I_m \omega^2 = \frac{R^2 L^3}{6\,EI}$$

$$R = \sqrt{\frac{3\,EI\,I_m\omega^2}{L^3}}$$ ←

NOTE: The moment of inertia I_m has units of kg · m^2 or N · m · s^2