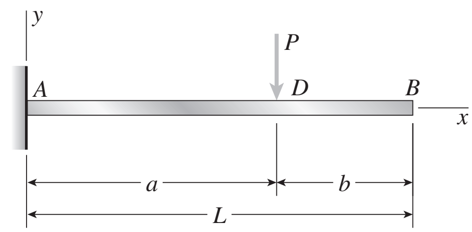
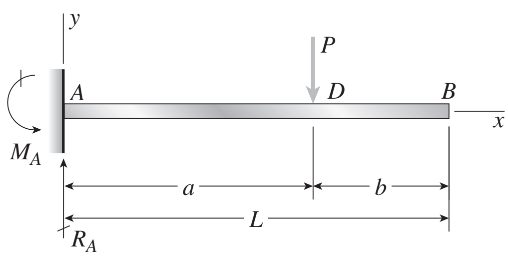
Representation of Loads on Beams by Discontinuity Functions

Problem 9.11-1 through 9.11-12 A beam and its loading are shown in the figure. Using discontinuity functions, write the expression for the intensity $q(x)$ of the equivalent distributed load acting on the beam (include the reactions in the expression for the equivalent load).



Solution 9.11-1 Cantilever beam



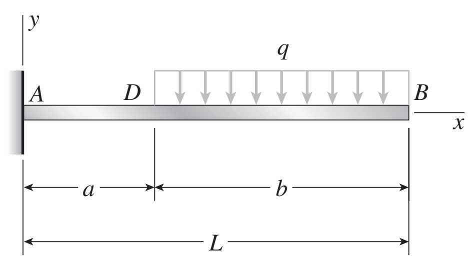
From equilibrium:

$R_A = P \quad M_A = Pa$

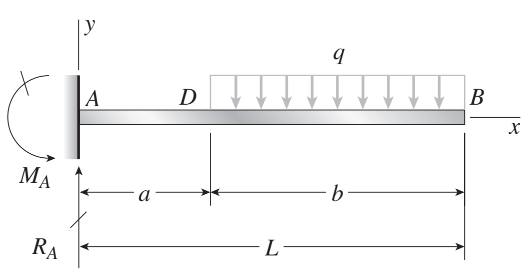
Use Table 9-2.

$$q(x) = -R_A \langle x \rangle^{-1} + M_A \langle x \rangle^{-2} + P \langle x - a \rangle^{-1}$$
$$= -P \langle x \rangle^{-1} + Pa \langle x \rangle^{-2} + P \langle x - a \rangle^{-1} \quad \leftarrow$$

Problem 9.11-2



Solution 9.11-2 Cantilever beam

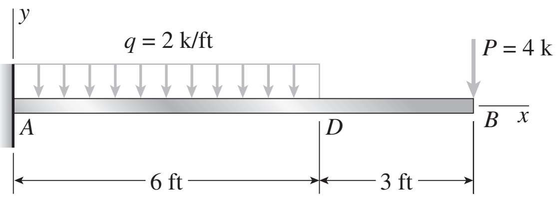


From equilibrium: $R_A = qb \quad M_A = \frac{qb}{2}(2a + b)$

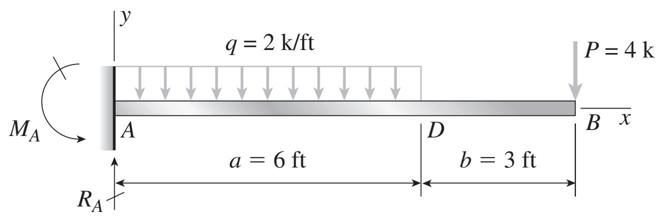
Use Table 9-2.

$$q(x) = -R_A \langle x \rangle^{-1} + M_A \langle x \rangle^{-2} + q \langle x - a \rangle^{0} - q \langle x - L \rangle^{0}$$
$$= -qb \langle x \rangle^{-1} + \frac{qb}{2}(2a + b) \langle x \rangle^{-2}$$
$$+ q \langle x - a \rangle^{0} - q \langle x - L \rangle^{0} \quad \leftarrow$$

Problem 9.11-3



Solution 9.11-3 Cantilever beam



$q = 2 \text{ k/ft} = \frac{1}{6} \text{ k/in.}$

$a = 6 \text{ ft} = 72 \text{ in.}$
$b = 3 \text{ ft} = 36 \text{ in.}$
$L = 9 \text{ ft} = 108 \text{ in.}$

FROM EQUILIBRIUM:

$R_A = 16 \text{ k} \qquad M_A = 864 \text{ k-in.}$

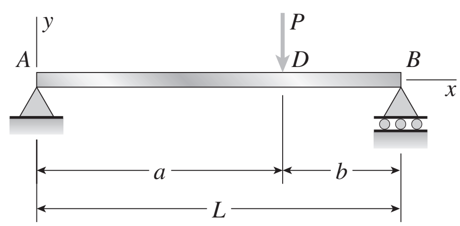
USE TABLE 9-2. Units: kips, inches

$$q(x) = -R_A \langle x \rangle^{-1} + M_A \langle x \rangle^{-2} + q \langle x \rangle^{0} - q \langle x - a \rangle^{0} + P \langle x - L \rangle^{-1}$$

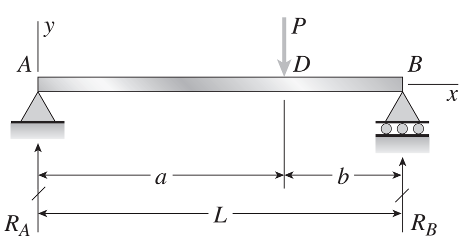
$$= -16 \langle x \rangle^{-1} + 864 \langle x \rangle^{-2} + \frac{1}{6} \langle x \rangle^{0} - \frac{1}{6} \langle x - 72 \rangle^{0} + 4 \langle x - 108 \rangle^{-1} \longleftarrow$$

(Units: x = in., q = k/in.)

Problem 9.11-4



Solution 9.11-4 Simple beam



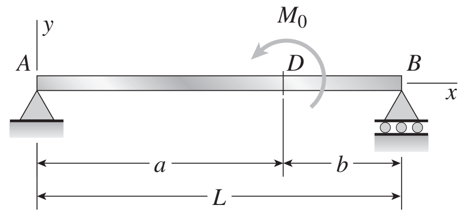
FROM EQUILIBRIUM: $R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$

USE TABLE 9-2.

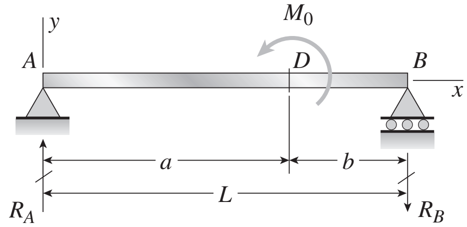
$$q(x) = -R_A \langle x \rangle^{-1} + P \langle x - a \rangle^{-1} - R_B \langle x - L \rangle^{-1}$$

$$= -\frac{Pb}{L} \langle x \rangle^{-1} + P \langle x - a \rangle^{-1} - \frac{Pa}{L} \langle x - L \rangle^{-1} \longleftarrow$$

Problem 9.11-5



Solution 9.11-5 Simple beam



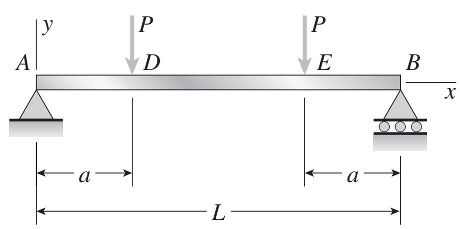
FROM EQUILIBRIUM: $R_A = \frac{M_0}{L} \quad R_B = \frac{M_0}{L}$ (downward)

USE TABLE 9-2.

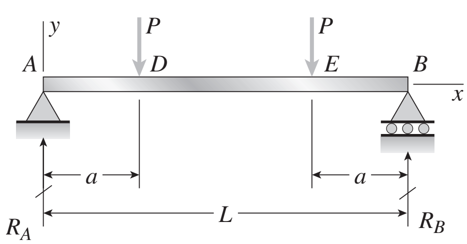
$$q(x) = -R_A \langle x \rangle^{-1} + M_0 \langle x - a \rangle^{-2} + R_B \langle x - L \rangle^{-1}$$

$$= -\frac{M_0}{L} \langle x \rangle^{-1} + M_0 \langle x - a \rangle^{-2} + \frac{M_0}{L} \langle x - L \rangle^{-1} \quad \leftarrow$$

Problem 9.11-6



Solution 9.11-6 Simple beam



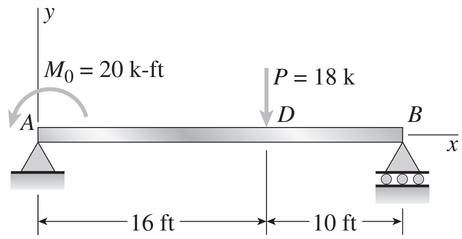
FROM EQUILIBRIUM: $R_A = R_B = P$

USE TABLE 9-2.

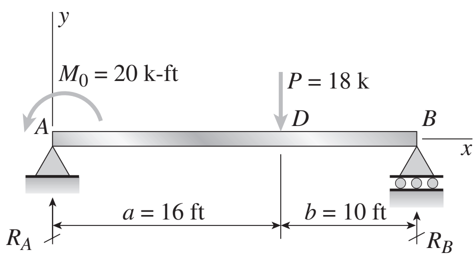
$$q(x) = -R_A \langle x \rangle^{-1} + P \langle x - a \rangle^{-1} + P \langle x - L + a \rangle^{-1} - R_B \langle x - L \rangle^{-1}$$

$$= -P \langle x \rangle^{-1} + P \langle x - a \rangle^{-1} + P \langle x - L + a \rangle^{-1} - P \langle x - L \rangle^{-1} \quad \leftarrow$$

Problem 9.11-7



Solution 9.11-7 **Simple beam**



$M_0 = 20$ k-ft $= 240$ k-in. $\quad P = 18$ k
$a = 16$ ft $= 192$ in. $\quad b = 10$ ft $= 120$ in.
$L = 26$ ft $= 312$ in.

From equilibrium: $R_A = 7.692$ k $\quad R_B = 10.308$ k

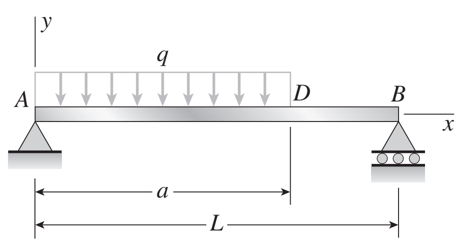
Use Table 9-2. Units: kips, inches

$$q(x) = -R_A\langle x\rangle^{-1} + M_0\langle x\rangle^{-2} + P\langle x-a\rangle^{-1} - R_B\langle x-L\rangle^{-1}$$

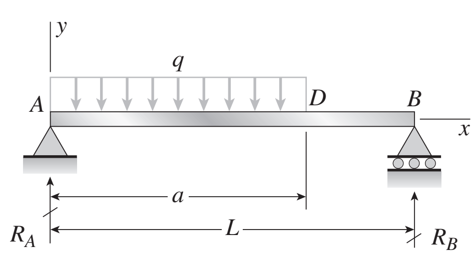
$$= -7.692\langle x\rangle^{-1} + 240\langle x\rangle^{-2} + 18\langle x-192\rangle^{-1} - 10.308\langle x-312\rangle^{-1} \quad \longleftarrow$$

(Units: x = in., q = k/in.)

Problem 9.11-8



Solution 9.11-8 **Simple beam**



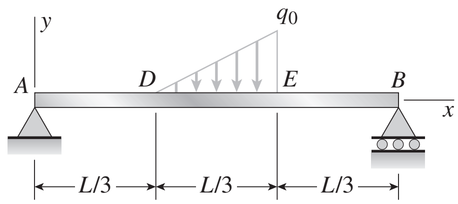
From equilibrium: $R_A = \dfrac{qa}{2L}(2L-a) \qquad R_B = \dfrac{qa^2}{2L}$

Use Table 9-2.

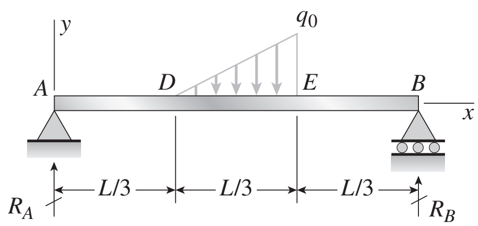
$$q(x) = -R_A\langle x\rangle^{-1} + q\langle x\rangle^{0} - q\langle x-a\rangle^{0} - R_B\langle x-L\rangle^{-1}$$

$$= -(qa/2L)(2L-a)\langle x\rangle^{-1} + q\langle x\rangle^{0} - q\langle x-a\rangle^{0} - (qa^2/2L)\langle x-L\rangle^{-1} \quad \longleftarrow$$

Problem 9.11-9



Solution 9.11-9 **Simple beam**



FROM EQUILIBRIUM: $R_A = \dfrac{2q_0L}{27}$ $\quad R_B = \dfrac{5q_0L}{54}$

USE TABLE 9-2.

$$q(x) = -R_A\langle x\rangle^{-1} + \frac{3q_0}{L}\langle x - \frac{L}{3}\rangle^1 - \frac{3q_0}{L}\langle x - \frac{2L}{3}\rangle^1$$

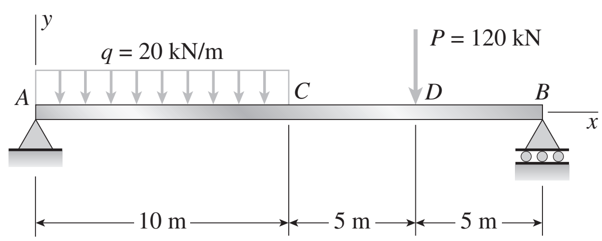
$$-q_0\langle x - \frac{2L}{3}\rangle^0 - R_B\langle x - L\rangle^{-1}$$

$$= -(2q_0L/27)\langle x\rangle^{-1} + (3q_0/L)\langle x - L/3\rangle^1$$

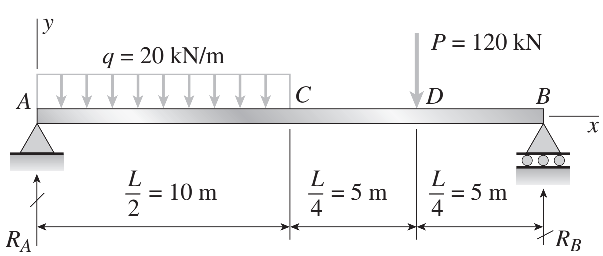
$$-(3q_0/L)\langle x - 2L/3\rangle^1 - q_0\langle x - 2L/3\rangle^0$$

$$-(5q_0L/54)\langle x - L\rangle^{-1} \longleftarrow$$

Problem 9.11-10



Solution 9.11-10 **Simple beam**



FROM EQUILIBRIUM: $R_A = 180$ kN $\quad R_B = 140$ kN

USE TABLE 9-2. Units: kilonewtons, meters

$$q(x) = -R_A\langle x\rangle^{-1} + q\langle x\rangle^0 - q\langle x - L/2\rangle^0$$

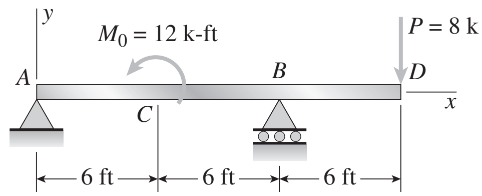
$$+ P\langle x - 3L/4\rangle^{-1} - R_B\langle x - L\rangle^{-1}$$

$$= -180\langle x\rangle^{-1} + 20\langle x\rangle^0 - 20\langle x - 10\rangle^0$$

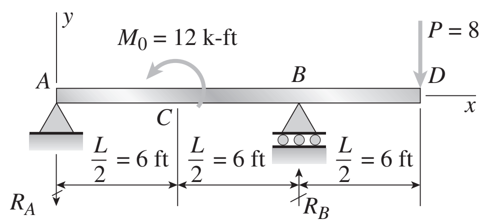
$$+ 120\langle x - 15\rangle^{-1} - 140\langle x - 20\rangle^{-1}$$

(Units: x = meters, q = kN/m)

Problem 9.11-11



Solution 9.11-11 Beam with an overhang



$M_0 = 12$ k-ft $= 144$ k-in.

$\frac{L}{2} = 6$ ft $= 72$ in.

$L = 12$ ft $= 144$ in.

FROM EQUILIBRIUM: $R_A = 3$ k (downward)
$R_B = 11$ k (upward)

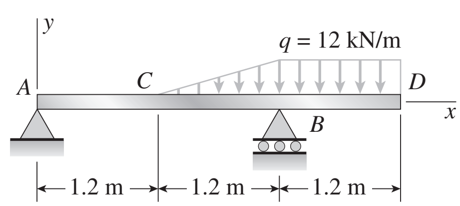
USE TABLE 9-2. Units: kips, inches

$$q(x) = R_A \langle x\rangle^{-1} + M_0 \langle x - L/2\rangle^{-2} - R_B \langle x - L\rangle^{-1} + P \langle x - 3L/2\rangle^{-1}$$

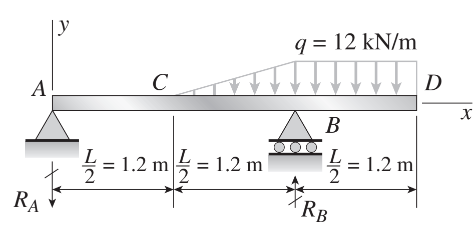
$$= 3 \langle x\rangle^{-1} + 144 \langle x - 72\rangle^{-2} - 11 \langle x - 144 \rangle^{-1} + 8 \langle x - 216 \rangle^{-1} \longleftarrow$$

(Units: x = in., q = k/in.)

Problem 9.11-12



Solution 9.11-12 Beam with an overhang



$q = 12$ kN/m

$\frac{L}{2} = 1.2$ m

$L = 2.4$ m

FROM EQUILIBRIUM: $R_A = 2.4$ kN (downward)
$R_B = 24.0$ kN (upward)

USE TABLE 9-2. Units: kilonewtons, meters

$$q(x) = R_A \langle x\rangle^{-1} + \frac{q}{L/2} \langle x - L/2\rangle^{1} - \frac{q}{L/2} \langle x - L\rangle^{1} - q \langle x - L\rangle^{0} - R_B \langle x - L\rangle^{-1} + q \langle x - L\rangle^{0} - q \langle x - 3L/2\rangle^{0}$$

$$= 2.4 \langle x\rangle^{-1} + 10 \langle x - 1.2\rangle^{1} - 10 \langle x - 2.4\rangle^{1} - 12 \langle x - 2.4\rangle^{0} - 24 \langle x - 2.4\rangle^{-1} + 12 \langle x - 2.4\rangle^{0} - 12 \langle x - 3.6\rangle^{0}$$

$$= 2.4 \langle x\rangle^{-1} + 10 \langle x - 1.2\rangle^{1} - 10 \langle x - 2.4\rangle^{1} - 24 \langle x - 2.4\rangle^{-1} - 12 \langle x - 3.6\rangle^{0} \longleftarrow$$

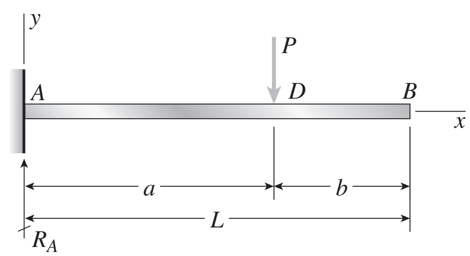
(Units: x = meters, q = kN/m)

Beam Deflections Using Discontinuity Functions

The problems for Section 9.12 are to be solved by using discontinuity functions. All beams have constant flexural rigidity EI. (Obtain the equations for the equivalent distributed loads from the corresponding problems in Section 9.11.)

Problem 9.12-1, 9.12-2, and 9.12-3 Determine the equation of the deflection curve for the cantilever beam ADB shown in the figure. Also, obtain the angle of rotation θ_B and deflection δ_B at the free end. (For the beam of Problem 9.12-3, assume $E = 10 \times 10^3$ ksi and $I = 450$ in.^{4})

Solution 9.12-1 Cantilever beam



FROM PROB: 9.11-1:

$$EIv'''' = -q(x) = P\langle x\rangle^{-1} - Pa\langle x\rangle^{-2} - P\langle x-a\rangle^{-1}$$

INTEGRATE THE EQUATION

$$EIv''' = V = P\langle x\rangle^{0} - Pa\langle x\rangle^{-1} - P\langle x-a\rangle^{0}$$

$$EIv'' = M = P\langle x\rangle^{1} - Pa\langle x\rangle^{0} - P\langle x-a\rangle^{1}$$

Note: $\langle x\rangle^1 = x$ and $\langle x\rangle^0 = 1$

$$EIv' = Px^2/2 - Pax - (P/2)\langle x-a\rangle^2 + C_1$$

B.C. $v'(0) = 0$ $EI(0) = 0 - 0 - 0 + C_1$

$\therefore C_1 = 0$

$$EIv = Px^3/6 - Pax^2/2 - (P/6)\langle x-a\rangle^3 + C_2$$

B.C. $v(0) = 0$ $EI(0) = 0 - 0 - 0 + C_2$

$\therefore C_2 = 0$

FINAL EQUATIONS

$$EIv' = (Px/2)(x - 2a) - (P/2)\langle x-a\rangle^2$$

$$EIv = (Px^2/6)(x - 3a) - (P/6)\langle x-a\rangle^3 \quad \longleftarrow$$

θ_B = CLOCKWISE ROTATION AT END B $(x = L)$

$$EIv'(L) = (PL/2)(L - 2a) - (P/2)\langle L-a\rangle^2$$
$$= (PL/2)(L - 2a) - (P/2)(L - a)^2$$
$$= -Pa^2/2$$

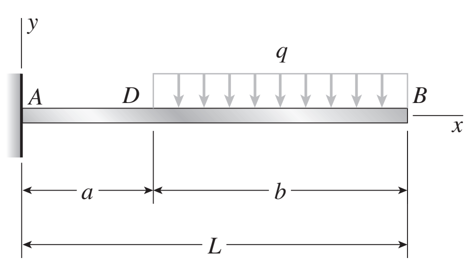
$$\theta_B = -v'(L) = \frac{Pa^2}{2EI} \quad \text{(clockwise)} \quad \longleftarrow$$

δ_B = DOWNWARD DEFLECTION AT END B $(x = L)$

$$EIv(L) = (PL^2/6)(L - 3a) - (P/6)\langle L-a\rangle^3$$
$$= (PL^2/6)(L - 3a) - (P/6)(L - a)^3$$
$$= (Pa^2/6)(-3L + a)$$

$$\delta_B = -v(L) = \frac{Pa^2}{6EI}(3L - a) \quad \text{(downward)} \quad \longleftarrow$$

Solution 9.12-2 Cantilever beam



FROM PROB: 9.11-2:

$$EIv'''' = -q(x) = qb\langle x\rangle^{-1} - (qb/2)(2a + b)\langle x\rangle^{-2} - q\langle x-a\rangle^{0} + q\langle x-L\rangle^{0}$$

Note: $\langle x - L\rangle^0 = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = qb\langle x\rangle^{0} - (qb/2)(2a + b)\langle x\rangle^{-1} - q\langle x-a\rangle^{1}$$

$$EIv'' = M = qb\langle x\rangle^{1} - (qb/2)(2a + b)\langle x\rangle^{0} - q\langle x-a\rangle^{2}/2$$

Note: $\langle x\rangle^1 = x$ and $\langle x\rangle^0 = 1$

$$EIv' = qbx^2/2 - (qb/2)(2a + b)x - (q/6)\langle x-a\rangle^3 + C_1$$

B.C. $v'(0) = 0$ $EI(0) = 0 - 0 - 0 + C_1$

$\therefore C_1 = 0$

$$EIv = qbx^3/6 - (qb/2)(2a + b)(x^2/2) - (q/24)\langle x-a\rangle^4 + C_2$$

B.C. $v(0) = 0$ $EI(0) = 0 - 0 - 0 + C_2$

$\therefore C_2 = 0$

(Continued)

FINAL EQUATIONS

$EIv' = (qbx/2)(x - L - a) - (q/6)\langle x - a\rangle^3$

$EIv = (qbx^2/12)(2x - 3a - 3L)$
$\quad - (q/24)\langle x - a\rangle^4$ ←

θ_B = CLOCKWISE ROTATION AT END B $(x = L)$

$EIv'(L) = (qbL/2)(-a) - (q/6)\langle L - a\rangle^3$
$\quad = -qabL/2 - (q/6)(L - a)^3$
$\quad = -(q/6)(L^3 - a^3)$

$\theta_B = -v'(L) = \dfrac{q}{6EI}(L^3 - a^3)$ (clockwise) ←

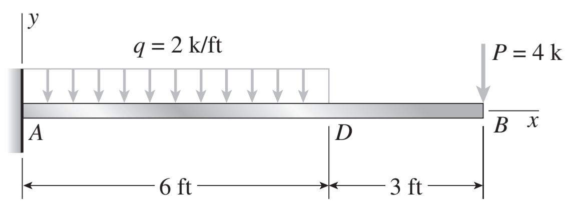
δ_B = DOWNWARD DEFLECTION AT END B $(x = L)$

$EIv(L) = (qbL^2/12)(-3a - L) - (q/24)\langle L - a\rangle^4$
$\quad = (qbL^2/12)(-3a - L) - (q/24)(L - a)^4$
$\quad = -(q/24)(3L^4 - 4a^3L + a^4)$

(After some lengthy algebra)

$\delta_B = -v(L) = \dfrac{q}{24EI}(3L^4 - 4a^3L + a^4)$ (downward) ←

Solution 9.12-3 Cantilever beam



$q = 2\text{ k/ft} = \dfrac{1}{6}\text{ k/in.}$

$a = 72$ in. $\quad b = 36$ in.
$L = 108$ in.
$E = 10 \times 10^3$ ksi. $\quad I = 450$ in.^{4}

FROM PROB: 9.11-3 $\quad$ Units: kips, inches

$EIv'''' = -q(x) = 16\langle x\rangle^{-1} - 864\langle x\rangle^{-2} - (1/6)\langle x\rangle^0$
$\quad + (1/6)\langle x - 72\rangle^0 - 4\langle x - 108\rangle^{-1}$

Note: $\langle x - 108\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$EIv''' = V = 16\langle x\rangle^0 - 864\langle x\rangle^{-1} - (1/6)\langle x\rangle^1$
$\quad + (1/6)\langle x - 72\rangle^1$

Note: $\langle x\rangle^0 = 1$ and $\langle x\rangle^1 = x$

$EIv'' = M = 16x - 864\langle x\rangle^0 - x^2/12 + (1/12)\langle x - 72\rangle^2$

$EIv' = 8x^2 - 864\langle x\rangle^1 - x^3/36$
$\quad + (1/36)\langle x - 72\rangle^3 + C_1$

Note: $\langle x\rangle^1 = x$

B.C. $v'(0) = 0 \quad EI(0) = 0 - 0 - 0 + 0 + C_1$
$\therefore C_1 = 0$

$EIv = 8x^3/3 - 432x^2 - x^4/144$
$\quad + (1/144)\langle x - 72\rangle^4 + C_2$

B.C. $v(0) = 0 \quad EI(0) = 0 - 0 - 0 + 0 + C_2$
$\therefore C_2 = 0$

FINAL EQUATIONS

$EIv' = (x/36)(-x^2 + 288x - 31{,}104) + (1/36)\langle x - 72\rangle^3$

$EIv = (x^2/144)(-x^2 + 384x - 62{,}208)$
$\quad + (1/144)\langle x - 72\rangle^4$ ←

Units: E = ksi, $\quad I$ = in.^{4}, $\quad v'$ = radians,
$\quad v$ = in., $\quad x$ = in.

θ_B = CLOCKWISE ROTATION AT END B $(x = L = 108$ in.)

$\theta_B = -v'(L) = -v'(108)$

$$\theta_B = -\frac{108}{36EI}[-(108)(108) + 288(108) - 31{,}104] - \left(\frac{1}{36EI}\right)(108 - 72)^3$$

$$= \frac{108}{36EI}(11{,}664) - \frac{1}{36EI}(46{,}656) = \frac{1}{EI}(33{,}696)$$

$EI = (10 \times 10^3\text{ ksi})(450\text{ in.}^4) = 4.5 \times 10^6\text{ k-in.}^2$

$$\theta_B = \frac{33{,}696}{4.5 \times 10^6}$$
$= 0.007488$ radians (clockwise) ←

δ_B = DOWNWARD DEFLECTION AT END B

$(x = L = 108$ in.)

$\delta_B = -v(L) = -v(108)$

$$\delta_B = -\frac{(108)^2}{144EI}[-(108)(108) + 384(108) - 62{,}208] - \frac{1}{144EI}(108 - 72)^4$$

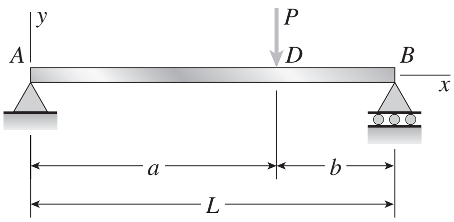
$$= \frac{(108)^2}{144EI}(32{,}400) - \frac{1}{144EI}(1{,}679{,}616)$$

$$= \frac{2{,}612{,}736}{EI} = \frac{2{,}612{,}736}{4.5 \times 10^6}$$

$= 0.5806$ in. (downward) ←

Problem 9.12-4, 9.12-5, and 9.12-6 Determine the equation of the deflection curve for the simple beam AB shown in the figure. Also, obtain the angle of rotation θ_A at the left-hand support and the deflection δ_D at point D.

Solution 9.12-4 Simple beam



FROM PROB: 9.11-4:

$$EIv'''' = -q(x) = (Pb/L)\langle x\rangle^{-1} - P\langle x-a\rangle^{-1} + (Pa/L)\langle x-L\rangle^{-1}$$

Note: $\langle x-L\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = (Pb/L)\langle x\rangle^{0} - P\langle x-a\rangle^{0}$$

$$EIv'' = M = (Pb/L)\langle x\rangle^{1} - P\langle x-a\rangle^{1}$$

$$EIv' = (Pb/2L)\langle x\rangle^{2} - (P/2)\langle x-a\rangle^{2} + C_1$$

$$EIv = (Pb/6L)\langle x\rangle^{3} - (P/6)\langle x-a\rangle^{3} + C_1x + C_2$$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$

B.C. $v(0) = 0$ $\quad EI(0) = 0 - 0 + 0 + C_2 \quad \therefore C_2 = 0$

B.C. $v(L) = 0$

$$EI(0) = PL^2b/6 - (P/6)\langle L-a\rangle^3 + C_1L$$

$$= PbL^2/6 - (P/6)(b^3) + C_1L$$

$$\therefore C_1 = -\frac{PbL}{6} + \frac{Pb^3}{6L} = -\frac{Pb}{6L}(L^2 - b^2)$$

FINAL EQUATIONS

$$EIv' = Pbx^2/2L - (P/2)\langle x-a\rangle^2 - \frac{Pb}{6L}(L^2 - b^2)$$

$$= (Pb/6L)(3x^2 + b^2 - L^2) - (P/2)\langle x-a\rangle^2$$

$$EIv = (Pb/6L)(x)^3 - (P/6)\langle x-a\rangle^3 - (Pbx/6L)(L^2 - b^2)$$

$$= (Pbx/6L)(x^2 + b^2 - L^2) - (P/6)\langle x-a\rangle^3 \quad \longleftarrow$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A $(x = 0)$

$$EIv'(0) = (Pb/6L)(b^2 - L^2) + (P/2)(0)$$

$$\theta_A = -v'(0) = (Pb/6L)(L^2 - b^2)(1/EI)$$

$$\theta_A = \frac{Pb}{6LEI}(L^2 - b^2) = \frac{Pb}{6LEI}(L-b)(L+b)$$

$$= \frac{Pab}{6LEI}(L+b) \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = (Pba/6L)(a^2 + b^2 - L^2) - (P/6)(0)$$

$$= -(Pab/6L)(L^2 - b^2 - a^2)$$

$$\delta_D = -v(a) = \frac{Pab}{6LEI}(L^2 - b^2 - a^2) = \frac{Pa^2b^2}{3LEI} \quad \longleftarrow$$

Solution 9.12-5 Simple beam

FROM PROB: 9.11-5:

$$EIv'''' = -q(x) = (M_0/L)\langle x\rangle^{-1} - M_0\langle x-a\rangle^{-2} - (M_0/L)\langle x-L\rangle^{-1}$$

Note: $\langle x-L\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

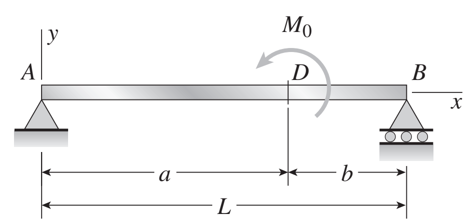
$$EIv''' = V = (M_0/L)\langle x\rangle^{0} - M_0\langle x-a\rangle^{-1}$$

$$EIv'' = M = (M_0/L)\langle x\rangle^{1} - M_0\langle x-a\rangle^{0}$$

$$EIv' = (M_0/2L)\langle x\rangle^{2} - M_0\langle x-a\rangle^{1} + C_1$$

$$EIv = (M_0/6L)\langle x\rangle^{3} - (M_0/2)\langle x-a\rangle^{2} + C_1x + C_2$$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$



B.C. $v(0) = 0$ $\quad EI(0) = 0 - 0 + 0 + C_2$

$\therefore C_2 = 0$

B.C. $v(L) = 0$

$$EI(0) = M_0L^2/6 - (M_0/2)\langle L-a\rangle^2 + C_1L$$

$$= M_0L^2/6 - (M_0/2)(L-a)^2 + C_1L$$

$$\therefore C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

(Continued)

FINAL EQUATIONS

$$EIv' = (M_0/2L)x^2 - M_0\langle x-a\rangle^1 + (M_0/6L)(2L^2 - 6aL + 3a^2)$$
$$= (M_0/6L)(3x^2 - 6aL + 3a^2 + 2L^2) - M_0\langle x-a\rangle^1$$

$$EIv = (M_0/6L)(x)^3 - (M_0/2)\langle x-a\rangle^2 + (M_0x/6L)(2L^2 - 6aL + 3a^2)$$
$$= (M_0x/6L)(x^2 - 6aL + 3a^2 + 2L^2) - (M_0/2)\langle x-a\rangle^2 \quad \longleftarrow$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A $(x = 0)$

$$EIv'(0) = (M_0/6L)(-6aL + 3a^2 + 2L^2) - (M_0/2)(0)$$

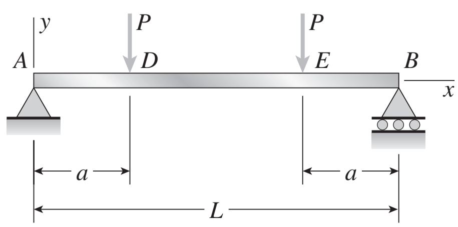
$$\theta_A = -v'(0) = \frac{M_0}{6LEI}(6aL - 3a^2 - 2L^2) \quad \text{(clockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = (M_0/6L)(a^3) - (M_0/2)(0) + (M_0a/6L)(2L^2 - 6aL + 3a^2)$$
$$= \frac{M_0a}{6L}(a^2 + 2L^2 - 6aL + 3a^2)$$
$$= \frac{M_0a}{6L}(L-a)(2)(L-2a)$$
$$= \frac{M_0ab}{3L}(L-2a)$$

$$\delta_D = -v(a) = \frac{M_0ab}{3LEI}(2a - L) \quad \text{(downward)} \quad \longleftarrow$$

Solution 9.12-6 Simple beam



FROM PROB: 9.11-6:

$$EIv'''' = -q(x) = P\langle x\rangle^{-1} - P\langle x-a\rangle^{-1} - P\langle x-L+a\rangle^{-1} + P\langle x-L\rangle^{-1}$$

Note: $\langle x-L\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = P\langle x\rangle^0 - P\langle x-a\rangle^0 - P\langle x-L+a\rangle^0$$
$$EIv'' = M = P\langle x\rangle^1 - P\langle x-a\rangle^1 - P\langle x-L+a\rangle^1$$
$$EIv' = (P/2)\langle x\rangle^2 - (P/2)\langle x-a\rangle^2 - (P/2)\langle x-L+a\rangle^2 + C_1$$

B.C. (symmetry) $EIv'(L/2) = 0$

$$0 = (P/2)(L/2)^2 - (P/2)(L/2 - a)^2 - (P/2)(0) + C_1$$

$$\therefore\ C_1 = -\frac{Pa}{2}(L-a)$$

$$EIv' = (P/2)\langle x\rangle^2 - (P/2)\langle x-a\rangle^2 - (P/2)\langle x-L+a\rangle^2 - (Pa/2)(L-a)$$

$$EIv = (P/6)\langle x\rangle^3 - (P/6)\langle x-a\rangle^3 - (P/6)\langle x-L+a\rangle^3 - (Pa/2)(L-a)\,x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 - 0 - 0 + C_2$

$\therefore C_2 = 0$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^3 = x^3$

FINAL EQUATIONS

$$EIv' = Px^2/2 - (P/2)\langle x-a\rangle^2 - (P/2)\langle x-L+a\rangle^2 - (Pa/2)(L-a)$$
$$= (P/2)(x^2 - aL + a^2) - (P/2)\langle x-a\rangle^2 - (P/2)\langle x-L+a\rangle^2$$

$$EIv = Px^3/6 - (P/6)\langle x-a\rangle^3 - (P/6)\langle x-L+a\rangle^3 - (3Pax/6)(L-\text{a})$$
$$= (Px/6)(x^2 - 3aL + 3a^2) - (P/6)\langle x-a\rangle^3 - (P/6)\langle x-L+a\rangle^3 \quad \longleftarrow$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A $(x = 0)$

$$EIv'(0) = (Pa/2)(-L+a) - (P/2)(0) - (P/2)(0)$$
$$= (Pa/2)(-L+a)$$

$$\theta_A = -v'(0) = \frac{Pa}{2EI}(L-a) \quad \text{(clockwise)} \quad \longleftarrow$$

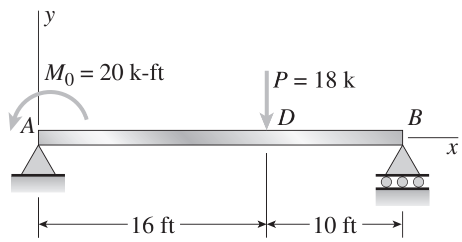
δ_D = DOWNWARD DEFLECTION AT POINT D $(x = a)$

$$EIv(a) = (Pa/6)(4a^2 - 3aL) - (P/6)(0) - (P/6)\langle -L+2a\rangle^3$$
$$= (Pa/6)(4a^2 - 3aL) - (P/6)(0)$$
$$= (Pa^2/6)(4a - 3L)$$

$$\delta_D = -v(a) = \frac{Pa^2}{6EI}(3L - 4a) \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.12-7 Determine the equation of the deflection curve for the simple beam *ADB* shown in the figure. Also, obtain the angle of rotation θ_A at the left-hand support and the deflection δ_D at point *D*. Assume $E = 30 \times 10^6$ psi and $I = 720$ in.^{4}

Solution 9.12-7 Simple beam



$M_0 = 20$ k-ft $= 240$ k-in.
$P = 18$ k
$a = 16$ ft $= 192$ in.
$b = 10$ ft $= 120$ in.
$L = a + b = 312$ in.
$E = 30 \times 10^3$ ksi
$I = 720$ in.^{4}

FROM PROB. 9.11-7: Units: kips, inches

$$EIv'''' = -q(x) = 7.692\langle x\rangle^{-1} - 240\langle x\rangle^{-2} - 18\langle x - 192\rangle^{-1} + 10.308\langle x - 312\rangle^{-1}$$

Note: $\langle x - 312\rangle^{-1} = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = V = 7.692\langle x\rangle^0 - 240\langle x\rangle^{-1} - 18\langle x - 192\rangle^0$$

$$EIv'' = M = 7.692\langle x\rangle^1 - 240\langle x\rangle^0 - 18\langle x - 192\rangle^1$$

$$EIv' = (7.692/2)\langle x\rangle^2 - 240\langle x\rangle^1 - (18/2)\langle x - 192\rangle^2 + C_1$$

Note: $\langle x\rangle^2 = x^2$ and $\langle x\rangle^1 = x$

$$EIv' = 3.846\,x^2 - 240\,x - 9\langle x - 192\rangle^2 + C_1$$

$$EIv = 1.282\,x^3 - 120\,x^2 - 3\langle x - 192\rangle^3 + C_1x + C_2$$

B.C. $EIv\,(0) = 0 \qquad 0 = 0 - 0 - 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv\,(312) = 0$

$0 = 1.282(312)^3 - 120(312)^2 - 3\langle 120\rangle^3 + C_1(312)$

Note: $\langle 120\rangle^3 = (120)^3$

$0 = 22{,}071 \times 10^3 + C_1(312) \quad \therefore C_1 = -70{,}740$

FINAL EQUATIONS

(Note: x = in., E = ksi, I = in.^{4}, v' = rad, v = in.)

$$EIv' = 3.846x^2 - 240x - 9\langle x - 192\rangle^2 - 70{,}740$$

$$EIv = 1.282x^3 - 120x^2 - 3\langle x - 192\rangle^3 - 70{,}740x \quad \longleftarrow$$

θ_A = CLOCKWISE ROTATION AT SUPPORT A ($x = 0$)

$$EIv'(0) = -9\langle -192\rangle^2 - 70{,}740 = -70{,}740$$

$$\theta_A = -v'(0) = \frac{70{,}740}{EI} = \frac{70{,}740}{(30 \times 10^3)(720)}$$

$= 0.00327$ rad (clockwise) $\longleftarrow$

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 192$)

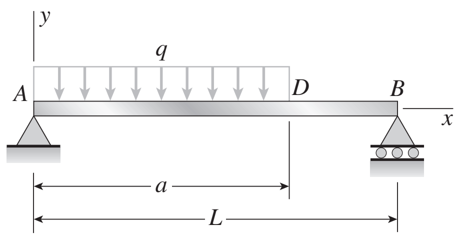
$$EIv(192) = 1.282(192)^3 - 120(192)^2 - 70{,}740(192) = -8.932 \times 10^6$$

$$\delta_D = -v(192) = \frac{8.932 \times 10^6}{EI} = \frac{8.932 \times 10^6}{(30 \times 10^3)(720)}$$

$= 0.414$ in. (downward) $\longleftarrow$

Problems 9.12-8, 9.12-9, and 9.12-10 Obtain the equation of the deflection curve for the simple beam *AB* (see figure). Also, determine the angle of rotation θ_B at the right-hand support and the deflection δ_D at point *D*. (For the beam of Problem 9.12-10, assume $E = 200$ GPa and $I = 2.60 \times 10^9$ mm^4.)

Solution 9.12-8 Simple beam



FROM PROB. 9.11-8:

$$EIv'''' = -q(x) = (qa/2L)(2L - a)\langle x\rangle^{-1} - q\langle x\rangle^0 + q\langle x - a\rangle^0 + (qa^2/2L)\langle x - L\rangle^{-1}$$

Note: $\langle x - L\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = (qa/2L)(2L - a)\langle x\rangle^0 - q\langle x\rangle^1 + q\langle x - a\rangle^1$$

$$EIv'' = M = (qa/2L)(2L - a)\langle x\rangle^1 - (q/2)\langle x\rangle^2 + (q/2)\langle x - a\rangle^2$$

$$EIv' = (qa/4L)(2L - a)\langle x\rangle^2 - (q/6)\langle x\rangle^3 + (q/6)\langle x - a\rangle^3 + C_1$$

$$EIv = (qa/12L)(2L - a)\langle x\rangle^3 - (q/24)\langle x\rangle^4 + (q/24)\langle x - a\rangle^4 + C_1x + C_2$$

Note: $\langle x\rangle^2 = x^2$, $\langle x\rangle^3 = x^3$, and $\langle x\rangle^4 = x^4$

B.C. $EIv(0) = 0$ $\quad 0 = 0 - 0 + (q/24)(0) + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(L) = 0$

$$0 = (qaL^2/12)(2L - a) - qL^4/24 + (q/24)(L - a)^4 + C_1L$$

After lengthy algebra,

$$C_1 = -\frac{qa^2}{24L}(2L - a)^2$$

FINAL EQUATIONS

$$EIv' = (qax^2/4L)(2L - a) - qx^3/6 + (q/6)\langle x - a\rangle^3 - (qa^2/24L)(2L - a)^2$$

$$EIv = (qax^3/12L)(2L - a) - qx^4/24 + (q/24)\langle x - a\rangle^4 - (qa^2x/24L)(2L - a)^2$$

$$= qx[-a^2(2L - a)^2 + 2a(2L - a)x^2 - Lx^3]/24L + q\langle x - a\rangle^4/24 \quad \longleftarrow$$

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT *B* ($x = L$)

$$EIv'(L) = (qaL/4)(2L - a) - qL^3/6 + (q/6)(L - a)^3 - (qa^2/24L)(2L - a)^2$$

After lengthy algebra,

$$EIv'(L) = (qa^2/24L)(2L^2 - a^2)$$

$$\theta_B = v'(L) = \frac{qa^2}{24LEI}(2L^2 - a^2) \quad \text{(counterclockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT *D* ($x = a$)

$$EIv(a) = qa[-a^2(2L - a)^2 + 2a^3(2L - a) - a^3L]/24L + q(0)$$

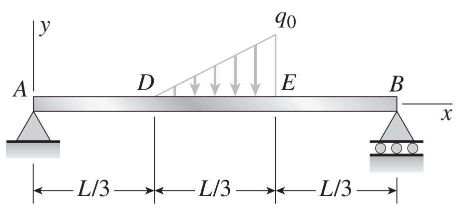
$$= (qa^3/24L)[-(2L - a)^2 + 2a(2L - a) - aL]$$

$$= (qa^3/24L)(-4L^2 + 7aL - 3a^2)$$

$$\delta_D = -v(a) = \frac{qa^3}{24LEI}(4L^2 - 7aL + 3a^2)$$

$$= \frac{qa^3}{24LEI}(4L - 3a)(L - a) \text{(downward)} \quad \longleftarrow$$

Solution 9.12-9 Simple beam



FROM PROB. 9.11-9:

$$EIv'''' = -q(x) = (2q_0L/27)\langle x\rangle^{-1} - (3q_0/L)\langle x - L/3\rangle^1 + (3q_0/L)\langle x - 2L/3\rangle^1 + q_0\langle x - 2L/3\rangle^0 + (5q_0L/54)\langle x - L\rangle^{-1}$$

Note: $\langle x - L\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = (2q_0L/27)\langle x\rangle^0 - (3q_0/2L)\langle x - L/3\rangle^2 + (3q_0/2L)\langle x - 2L/3\rangle^2 + q_0\langle x - 2L/3\rangle^1$$

Note: $\langle x\rangle^0 = 1$

$$EIv'' = M = (2q_0L/27)x - (q_0/2L)\langle x - L/3\rangle^3 + (q_0/2L)\langle x - 2L/3\rangle^3 + (q_0/2)\langle x - 2L/3\rangle^2$$

$$EIv' = (q_0L/27)x^2 - (q_0/8L)\langle x - L/3\rangle^4 + (q_0/8L)\langle x - 2L/3\rangle^4 + (q_0/6)\langle x - 2L/3\rangle^3 + C_1$$

$$EIv = (q_0L/81)x^3 - (q_0/40L)\langle x - L/3\rangle^5 + (q_0/40L)\langle x - 2L/3\rangle^5 + (q_0/24)\langle x - 2L/3\rangle^4 + C_1x + C_2$$

B.C. $EIv(0) = 0 \qquad 0 = 0 - 0 + 0 + 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(L) = 0$

$$0 = q_0L^4/81 - (q_0/40L)(2L/3)^5 + (q_0/40L)(L/3)^5 + (q_0/24)(L/3)^4 + C_1L$$

$$0 = \frac{47q_0L^4}{4860} + C_1L \qquad \therefore C_1 = -\frac{47q_0L^3}{4860}$$

FINAL EQUATIONS

$$EIv' = (q_0L/27)x^2 - (q_0/8L)\langle x - L/3\rangle^4 + (q_0/8L)\langle x - 2L/3\rangle^4 + (q_0/6)\langle x - 2L/3\rangle^3 - 47q_0L^3/4860$$

$$EIv = (q_0L/81)x^3 - (q_0/40L)\langle x - L/3\rangle^5 + (q_0/40L)\langle x - 2L/3\rangle^5 + (q_0/24)\langle x - 2L/3\rangle^4 - 47q_0L^3x/4860 \quad \longleftarrow$$

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT B $(x = L)$

$$EIv'(L) = q_0L^3/27 - (q_0/8L)(2L/3)^4 + (q_0/8L)(L/3)^4 + (q_0/6)(L/3)^3 - 47q_0L^3/4860 = 101q_0L^3/9720$$

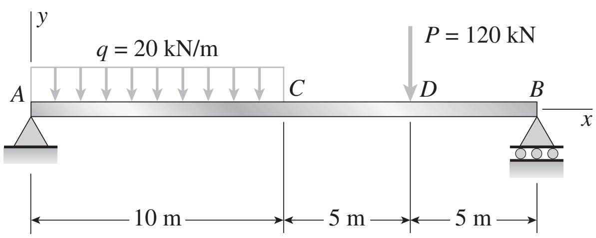
$$\theta_B = v'(L) = \frac{101q_0L^3}{9720EI} \quad \text{(counterclockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D $(x = L/3)$

$$EIv(L/3) = (q_0L/81)(L/3)^3 - (q_0/40L)(0) + (q_0/40L)(0) + (q_0/24)(0) - 47q_0L^3(L/3)/4860 = -121q_0L^4/43{,}740$$

$$\delta_D = -v\left(\frac{L}{3}\right) = \frac{121q_0L^4}{43{,}740EI} \quad \text{(downward)} \quad \longleftarrow$$

Solution 9.12-10 Simple beam



$q = 20$ kN/m
$P = 120$ kN
$\frac{L}{2} = 10$ m
$L = 20$ m
$E = 200$ GPa
$I = 2.60 \times 10^{-3}$ m^4

FROM PROB. 9.11-10: Units: kilonewtons, meters

$$EIv'''' = -q(x) = 180\langle x\rangle^{-1} - 20\langle x\rangle^0 + 20\langle x-10\rangle^0 - 120\langle x-15\rangle^{-1} + 140\langle x-20\rangle^{-1}$$

Note: $\langle x - 20\rangle^{-1} = 0$ and may be dropped from the equation

INTEGRATE THE EQUATION

$$EIv''' = V = 180\langle x\rangle^0 - 20\langle x\rangle^1 + 20\langle x-10\rangle^1 - 120\langle x-15\rangle^0$$

Note: $\langle x\rangle^0 = 1$ and $\langle x\rangle^1 = x$

$$EIv'' = M = 180x - 20(x^2/2) + (20/2)\langle x-10\rangle^2 - 120\langle x-15\rangle^1$$

$$EIv' = 180(x^2/2) - 20(x^3/6) + (10/3)\langle x-10\rangle^3 - 60\langle x-15\rangle^2 + C_1$$

$$EIv = 30x^3 - (5/6)x^4 + (5/6)\langle x-10\rangle^4 - 20\langle x-15\rangle^3 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 - 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(20) = 0$

$$0 = 30(20)^3 - (5/6)(20)^4 + (5/6)(10)^4 - 20(5)^3 + C_1(20)$$

$$0 = 112{,}500 + 20C_1 \quad \therefore C_1 = -5625$$

FINAL EQUATIONS

$$EIv' = 90x^2 - (10/3)x^3 + (10/3)\langle x-10\rangle^3 - 60\langle x-15\rangle^2 - 5625$$

$$EIv = 30x^3 - (5/6)x^4 + (5/6)\langle x-10\rangle^4 - 20\langle x-15\rangle^3 - 5625x \quad \longleftarrow$$

(x = meters, v = meters, v' = radians, E = kilopascals, I = meters^4)

θ_B = COUNTERCLOCKWISE ROTATION AT SUPPORT B ($x = 20$)

$$EIv'(20) = 90(20)^2 - (10/3)(20)^3 + (10/3)(10)^3 - 60(5)^2 - 5625 = 5541.67$$

$$\theta_B = v'(20) = \frac{5541.67}{EI} = \frac{5541.67}{(200 \times 10^6 \text{ kPa})(2.60 \times 10^{-3} \text{ m})}$$

$$= 0.01066 \text{ rad} \quad \text{(counterclockwise)} \quad \longleftarrow$$

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 15$)

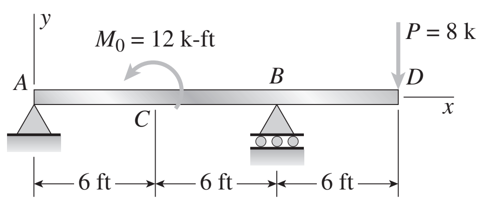
$$EIv(15) = 30(15)^3 - (5/6)(15)^4 + (5/6)(5)^4 - 20(0) - 5625(15) = -24{,}791.7$$

$$\delta_D = -v(15) = \frac{24{,}791.7}{EI} = \frac{24{,}791.7}{(200 \times 10^6 \text{ kPa})(2.60 \times 10^{-3} \text{ m})}$$

$$= 0.04768 \text{ m} = 47.68 \text{ mm} \quad \text{(downward)} \quad \longleftarrow$$

Problem 9.12-11 A beam *ACBD* with simple supports at *A* and *B* and an overhang *BD* is shown in the figure. (a) Obtain the equation of the deflection curve for the beam. (b) Calculate the deflections δ_C and δ_D at points *C* and *D*, respectively. (Assume $E = 30 \times 10^6$ psi and $I = 280$ in.^{4})

Solution 9.12-11 **Beam with an overhang**



$M_0 = 144$ k-in.

$\frac{L}{2} = 72$ in.

$L = L_{AB} = 144$ in.

$\frac{3L}{2} = 216$ in.

$E = 30 \times 10^3$ ksi

$I = 280$ in.^{4}

From Prob. 9.11-11: Units: kips, inches

$$EIv'''' = -q(x) = -3\langle x\rangle^{-1} - 144\langle x-72\rangle^{-2} + 11\langle x-144\rangle^{-1} - 8\langle x-216\rangle^{-1}$$

Note: $\langle x - 216\rangle^{-1} = 0$ and may be dropped from the equation.

Integrate the equation

$$EIv''' = V = -3\langle x\rangle^0 - 144\langle x-72\rangle^{-1} + 11\langle x-144\rangle^0$$

$$EIv'' = M = -3\langle x\rangle^1 - 144\langle x-72\rangle^0 + 11\langle x-144\rangle^1$$

$$EIv' = -(3/2)\langle x\rangle^2 - 144\langle x-72\rangle^1 + (11/2)\langle x-144\rangle^2 + C_1$$

$$EIv = -(1/2)\langle x\rangle^3 - (144/2)\langle x-72\rangle^2 + (11/6)\langle x-144\rangle^3 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(144) = 0 \quad 0 = -(1/2)(144)^3 - (72)(72)^2 + (11/6)(0) + C_1(144)$

$0 = -1{,}866{,}240 + 144\,C_1$

$\therefore C_1 = 12{,}960$

Final equations

$$EIv' = -3x^2/2 - 144\langle x-72\rangle^1 + (11/2)\langle x-144\rangle^2 + 12{,}960$$

$$EIv = -x^3/2 - 72\langle x-72\rangle^2 + (11/6)\langle x-144\rangle^3 + 12{,}960\,x \longleftarrow$$

(x = in., v = in., v' = rad, $E = 30 \times 10^3$ ksi, $I = 280$ in.^{4})

δ_C = upward deflection at point *C* ($x = 72$)

$$EIv(15) = -(72)^3/2 - 72(0) + (11/6)(0) + 12{,}960(72) = 746{,}496$$

$$\delta_C = v(15) = \frac{746{,}496}{EI} = \frac{746{,}496}{(30 \times 10^3)(280)} = 0.08887 \text{ in.} \quad \text{(upward)} \longleftarrow$$

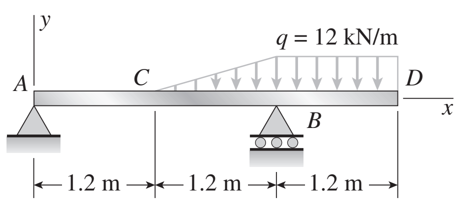
δ_D = downward deflection at point *D* ($x = 216$)

$$EIv(216) = -(216)^3/2 - 72(144)^2 + (11/6)(72)^3 + 12{,}960(216) = -3{,}048{,}192$$

$$\delta_D = -v(216) = \frac{3{,}048{,}192}{EI} = \frac{3{,}048{,}192}{(30 \times 10^3)(280)} = 0.3629 \text{ in.} \quad \text{(downward)} \longleftarrow$$

Problem 9.12-12 The overhanging beam *ACBD* shown in the figure is simply supported at *A* and *B*. Obtain the equation of the deflection curve and the deflections δ_C and δ_D at points *C* and *D*, respectively. (Assume $E = 200$ GPa and $I = 15 \times 10^6\ \text{mm}^4$.)

Solution 9.12-12 Beam with an overhang



$q = 12$ kN/m

$\dfrac{L}{2} = 1.2$ m

$L = L_{AB} = 2.4$ m

$E = 200$ GPa

$I = 15 \times 10^{-6}\ \text{m}^4$

FROM PROB. 9.11-12: Units: kilometers, meters

$$EIv'''' = -q(x) = -2.4\,\langle x\rangle^{-1} - 10\langle x - 1.2\rangle^1 + 10\,\langle x - 2.4\rangle^1 + 24\,\langle x - 2.4\rangle^{-1} + 12\,\langle x - 3.6\rangle^0$$

Note: $\langle x - 3.6\rangle^0 = 0$ and may be dropped from the equation.

INTEGRATE THE EQUATION

$$EIv''' = v = -2.4\,\langle x\rangle^0 - (10/2)\,\langle x - 1.2\rangle^2 + (10/2)\,\langle x - 2.4\rangle^2 + 24\langle x - 2.4\rangle^0$$

$$EIv'' = M = -2.4\,\langle x\rangle^1 - (5/3)\,\langle x - 1.2\rangle^3 + (5/3)\,\langle x - 2.4\rangle^3 + 24\,\langle x - 2.4\rangle^1$$

Note: $\langle x\rangle^1 = x$

$$EIv' = -1.2x^2 - (5/12)\,\langle x - 1.2\rangle^4 + (5/12)\,\langle x - 2.4\rangle^4 + 12\,\langle x - 2.4\rangle^2 + C_1$$

$$EIv = -0.4x^3 - (1/12)\,\langle x - 1.2\rangle^5 + (1/12)\,\langle x - 2.4\rangle^5 + 4\,\langle x - 2.4\rangle^3 + C_1x + C_2$$

B.C. $EIv(0) = 0 \quad 0 = 0 - 0 + 0 + 0 + C_1(0) + C_2$

$\therefore C_2 = 0$

B.C. $EIv(2.4) = 0$

$$0 = -0.4(2.4)^3 - (1/12)(1.2)^5 + (1/12)(0) + 4(0) + 2.4\,C_1$$

$$0 = -5.73696 + 2.4\,C_1$$

$\therefore C_1 = 2.3904$

FINAL EQUATIONS

$$EIv' = -1.2x^2 - (5/12)\,\langle x - 1.2\rangle^4 + (5/12)\,\langle x - 2.4\rangle^4 + 12\,\langle x - 2.4\rangle^2 + 2.3904$$

$$EIv = -0.4x^3 - (1/12)\,\langle x - 1.2\rangle^5 + (1/12)\,\langle x - 2.4\rangle^5 + 4\,\langle x - 2.4\rangle^3 + 2.3904x$$

(x = meters, v = meters, v' = radians, $E = 200 \times 10^6$ kPa, $I = 15 \times 10^{-6}\ \text{m}^4$)

δ_C = UPWARD DEFLECTION AT POINT C ($x = 1.2$)

$$EIv(1.2) = -0.4(1.2)^3 - (1/12)\,(0) + (1/12)\,(0) + 4\,(0) + 2.3904\,(1.2) = 2.17728$$

$$\delta_C = v(1.2) = \frac{2.17728}{EI} = \frac{2.17728}{(200 \times 10^6)(15 \times 10^{-6})}$$

$= 0.00072576$ m $= 0.7258$ mm (upward) ←

δ_D = DOWNWARD DEFLECTION AT POINT D ($x = 3.6$)

$$EIv(3.6) = -0.4(3.6)^3 - (1/12)\,(2.4)^5 + (1/12)\,(1.2)^5 + 4\,(1.2)^3 + 2.3904\,(3.6) = -9.57312$$

$$\delta_D = -v(3.6) = \frac{9.57312}{EI} = \frac{9.57312}{(200 \times 10^6)(15 \times 10^{-6})}$$

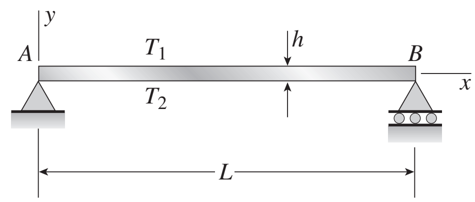
$= 0.00319104$ m $= 3.191$ mm (downward) ←

Temperature Effects

The beams described in the problems for Section 9.13 have constant flexural rigidity EI. In every problem, the temperature varies linearly between the top and bottom of the beam.

Problem 9.13-1 A simple beam AB of length L and height h undergoes a temperature change such that the bottom of the beam is at temperature T_2 and the top of the beam is at temperature T_1 (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_A at the left-hand support, and the deflection δ_{max} at the midpoint.



Solution 9.13-1 **Simple beam with temperature differential**

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)x}{h} + C_1$

B.C. 1 (Symmetry) $v'\left(\dfrac{L}{2}\right) = 0$

$\therefore C_1 = -\dfrac{\alpha L(T_2 - T_1)}{2h}$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h} - \dfrac{\alpha L(T_2 - T_1)x}{2h} + C_2$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

$v = -\dfrac{\alpha(T_2 - T_1)(x)(L - x)}{2h}$ ←

(positive v is upward deflection)

$v' = -\dfrac{\alpha(T_2 - T_1)(L - 2x)}{2h}$

$\theta_A = -v'(0) = \dfrac{\alpha L(T_2 - T_1)}{2h}$ ←

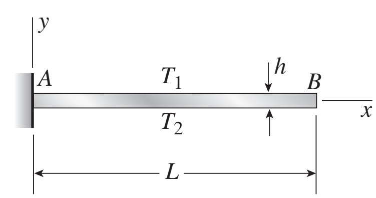
(positive θ_A is clockwise rotation)

$\delta_{max} = -v\left(\dfrac{L}{2}\right) = \dfrac{\alpha L^2(T_2 - T_1)}{8h}$ ←

(positive δ_{max} is downward deflection)

Problem 9.13-2 A cantilever beam AB of length L and height h (see figure) is subjected to a temperature change such that the temperature at the top is T_1 and at the bottom is T_2.

Determine the equation of the deflection curve of the beam, the angle of rotation θ_B at end B, and the deflection δ_B at end B.



Solution 9.13-2 **Cantilever beam with temperature differential**

Eq. (9-147): $v'' = \dfrac{d^2v}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{h}$

$v' = \dfrac{dv}{dx} = \dfrac{\alpha(T_2 - T_1)}{h}x + C_1$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$v' = \dfrac{\alpha(T_2 - T_1)}{h}x$

$v = \dfrac{\alpha(T_2 - T_1)}{h}\left(\dfrac{x^2}{2}\right) + C_2$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

$v = \dfrac{\alpha(T_2 - T_1)x^2}{2h}$ ←

(positive v is upward deflection)

$\theta_B = v'(L) = \dfrac{\alpha L(T_2 - T_1)}{h}$ ←

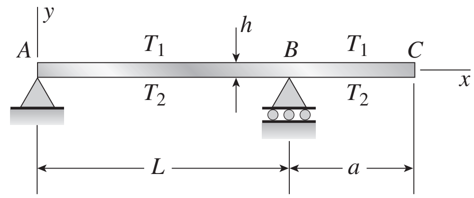
(positive θ_B is counterclockwise rotation)

$\delta_B = v(L) = \dfrac{\alpha L^2(T_2 - T_1)}{2h}$ ←

(positive δ_B is upward deflection)

Problem 9.13-3 An overhanging beam ABC of height h is heated to a temperature T_1 on the top and T_2 on the bottom (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_C at end C, and the deflection δ_C at end C.



Solution 9.13-3 Overhanging beam with temperature differential

Eq. (9-147): $v'' = \frac{d^2v}{dx^2} = \frac{\alpha(T_2 - T_1)}{h}$

(This equation is valid for the entire length of the beam.)

$$v' = \frac{\alpha(T_2 - T_1)x}{h} + C_1$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} + C_1x + C_2$$

B.C. 1 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $v(L) = 0 \quad \therefore C_1 = -\frac{\alpha(T_2 - T_1)L}{2h}$

$$v = \frac{\alpha(T_2 - T_1)}{2h}(x^2 - Lx) \quad \leftarrow$$

(positive v is upward deflection)

$$v' = \frac{\alpha(T_2 - T_1)}{2h}(2x - L)$$

$$\theta_C = v'(L + a) = \frac{\alpha(T_2 - T_1)}{2h}(L + 2a) \quad \leftarrow$$

(positive θ_C is counterclockwise rotation)

$$\delta_C = v(L + a) = \frac{\alpha(T_2 - T_1)\,(L + a)\,(a)}{2h} \quad \leftarrow$$

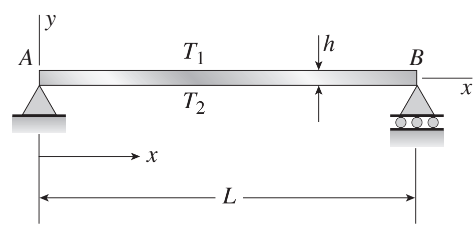
(positive δ_C is upward deflection)

Problem 9.13-4 A simple beam AB of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A; that is,

$$T_2 - T_1 = T_0x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance.

Determine the maximum deflection δ_{max} of the beam.



Solution 9.13-4 Simple beam with temperature differential proportional to distance x

$T_2 - T_1 = T_0x$

Eq. (9-147): $v'' = \frac{d^2v}{dx^2} = \frac{\alpha(T_2 - T_1)}{h} = \frac{\alpha T_0 x}{h}$

$$v' = \frac{dv}{dx} = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1x + C_2$$

B.C. 1 $v\,(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $v\,(L) = 0 \quad \therefore C_1 = -\frac{\alpha T_0 L^2}{6h}$

$$v = -\frac{\alpha T_0 x}{6h}(L^2 - x^2)$$

(positive v is upward deflection)

$$v' = -\frac{\alpha T_0}{6h}(L^2 - 3x^2)$$

(positive v' is upward to the right)

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x.

$$L^2 - 3x^2 = 0 \qquad x_1 = \frac{L}{\sqrt{3}}$$

$$v_{max} = v(x_1) = -\frac{\alpha T_0 L^3}{9\sqrt{3}h}$$

$$\delta_{max} = -v_{max} = \frac{\alpha T_0 L^3}{9\sqrt{3}h} \quad \leftarrow$$

(positive δ_{max} is downward)