Chapter

15

Elastic Stability

15.1 General Considerations

Failure through elastic instability has been discussed briefly in Sec. 3.13, where it was pointed out that it may occur when the bending or twisting effect of an applied load is proportional to the deformation it produces. In this chapter, formulas for the critical load or critical unit stress at which such failure occurs are given for a wide variety of members and conditions of loading.

Such formulas can be derived mathematically by integrating the differential equation of the elastic curve or by equating the strain energy of bending to the work done by the applied load in the corresponding displacement of its point of application, the form of the elastic curve being assumed when unknown. Of all possible forms of the curve, that which makes the critical load a minimum is the correct one; but almost any reasonable assumption (consistent with the boundary conditions) can be made without gross error resulting, and for this reason the strain-energy method is especially adapted to the approximate solution of difficult cases. A very thorough discussion of the general problem, with detailed solutions of many specified cases, is given in Timoshenko and Gere (Ref. 1), from which many of the formulas in this chapter are taken. Formulas for many cases are also given in Refs. 35 and 36; in addition Ref. 35 contains many graphs of numerically evaluated coefficients.

At one time, most of the problems involving elastic stability were of academic interest only since engineers were reluctant to use compression members so slender as to fail by buckling at elastic stresses and danger of corrosion interdicted the use of very thin material in exposed structures. The requirements for minimum-weight construction in the fields of aerospace and transportation, however, have given great impetus to the theoretical and experimental investigation of elastic stability and to the use of parts for which it is a governing design consideration.

There are certain definite advantages in lightweight construction, in which stability determines strength. One is that since elastic buckling may occur without damage, part of a structure—such as the skin of an airplane wing or web of a deep beam—may be used safely at loads that cause local buckling, and under these circumstances the resistance afforded by the buckled part is definitely known. Furthermore, members such as Euler columns may be loaded experimentally to their maximum capacity without damage or permanent deformation and subsequently incorporated in a structure.

15.2 Buckling of Bars

In Table 15.1, formulas are given for the critical loads on columns, beams, and shafts. In general, the theoretical values are in good agreement with test results as long as the assumed conditions are reasonably well-satisfied. It is to be noted that even slight changes in the amount of end constraint have a marked effect on the critical loads, and therefore it is important that such constraint be closely estimated. Slight irregularities in form and small accidental eccentricities are less likely to be important in the case of columns than in the case of thin plates. For latticed columns or columns with tie plates, a reduced value of E may be used, calculated as shown in Sec. 12.3. Formulas for the elastic buckling of bars may be applied to conditions under which proportional limit is exceeded if a reduced value of E corresponding to the actual stress is used (Ref. 1), but the procedure requires a stress-strain diagram for the material and, in general, is not practical.

In Table 15.1, cases 1–3, the tabulated buckling coefficients are worked out for various combinations of concentrated and distributed axial loads. Tensile end loads are included so that the effect of axial end restraint under axial loading within the column length can be considered (see the example at the end of this section). Carter and Gere (Ref. 46) present graphs of buckling coefficients for columns with single tapers for various end conditions, cross sections, and degrees of taper. Culver and Preg (Ref. 47) investigate and tabulate buckling coefficients for singly tapered beam-columns in which the effect of torsion, including warping restraint, is considered for the case where the loading is by end moments in the stiffer principal plane.

Kitipornchai and Trahair describe (Ref. 55) the lateral stability of singly tapered cantilever and doubly tapered simple I-beams, including the effect of warping restraint; experimental results are favorably compared with numerical solutions. Morrison (Ref. 57) considers the effect of lateral restraint of the tensile flange of a beam under lateral buckling; example calculations are presented. Massey and McGuire (Ref. 54) present graphs of buckling coefficients for both stepped and tapered cantilever beams; good agreement with experiments is reported. Tables of lateral stability constants for laminated timber beams are presented in Fowler (Ref. 53) along with two design examples.

Clark and Hill (Ref. 52) derive a general expression for the lateral stability of unsymmetrical I-beams with boundary conditions based on both bending and warping supports; tables of coefficients as well as nomographs are presented. Anderson and Trahair (Ref. 56) present tabulated lateral buckling coefficients for uniformly loaded and end-loaded cantilevers and center- and uniformly loaded simply supported beams having unsymmetric I-beam cross sections; favorable comparisons are made with extensive tests on cantilever beams.

The Southwell plot is a graph in which the lateral deflection of a column or any other linearly elastic member undergoing a manner of loading which will produce buckling is plotted versus the lateral deflection divided by the load; the slope of this line gives the critical load. For columns and some frameworks, significant deflections do occur within the range where small-deflection theory is applicable. If the initial imperfections are such that experimental readings of lateral deflection must be taken beyond the small-deflection region, then the Southwell procedure is not adequate. Roorda (Ref. 93) discusses the extension of this procedure into the nonlinear range.

Bimetallic beams. Burgreen and Manitt (Ref. 48) and Burgreen and Regal (Ref. 49) discuss the analysis of bimetallic beams and point out some of the difficulties in predicting the *snap-through instability* of these beams under changes in temperature. The thermal expansion of the support structure is an important design factor.

Rings and arches. Austin (Ref. 50) tabulates in-plane buckling coefficients for circular, parabolic, and catenary arches for pinned and fixed ends as well as for the three-hinged case; he considers cases where the cross section varies with the position in the span as well as the usual case of a uniform cross section. Uniform loads, unsymmetric distributed loads, and concentrated center loads are considered, and the stiffening effect of tying the arch to the girder with columns is also evaluated. (The discussion referenced with the paper gives an extensive bibliography of work on arch stability.)

A thin ring shrunk by cooling and inserted into a circular cavity usually will yield before buckling unless the radius/thickness ratio is very large and the elastic-limit stress is high. Chicurel (Ref. 51) derives approximate solutions to this problem when the effect of friction is considered. He suggests a conservative expression for the *no-friction* condition: $P_o/AE = 2.67(k/r)^{1.2}$, where P_o is the prebuckling hoop compressive force, A is the hoop cross-sectional area, E is the modulus of elasticity, k is the radius of gyration of the cross section, and r is the radius of the ring.

EXAMPLE

A 4-in steel pipe is to be used as a column to carry 8000 lb of transformers centered axially on a platform 20 ft above the foundation. The factor of safety *FS* is to be determined for the following conditions, based on elastic buckling of the column.

(a) The platform is supported only by the pipe fixed at the foundation.

(b) A $3\frac{1}{2}$-in steel pipe is to be slipped into the top of the 4-in pipe a distance of 4 in, welded in place, and extended 10 ft to the ceiling above, where it will extend through a close-fitting hole in a steel plate.

(c) This condition is the same as in (b) except that the $3\frac{1}{2}$-in pipe will be welded solidly into a heavy steel girder passing 10 ft above the platform.

Solution. A 4-in steel pipe has a cross-sectional area of 3.174 in^2 and a bending moment of inertia of 7.233 in^4. For a $3\frac{1}{2}$-in pipe these are 2.68 in^2 and 4.788 in^4, respectively.

(a) This case is a column fixed at the bottom and free at the top with an end load only. In Table 15.1, case la, for $I_2/I_1 = 1.00$ and $P_2/P_1 = 0$, K_1 is given as 0.25. Therefore,

$$P_1' = 0.25\frac{\pi^2 30(10^6)(7.233)}{240^2} = 9295 \text{ lb}$$

$$FS = \frac{9295}{8000} = 1.162$$

(b) This case is a column fixed at the bottom and pinned at the top with a load at a distance of two-thirds the 30-ft length from the bottom: $I_1 = 4.788$ in^4, $I_2 = 7.233$ in^4, and $I_2/I_1 = 1.511$. In Table 15.1, case 2d, for $E_2I_2/E_1I_1 = 1.5$, $P_1/P_2 = 0$, and $a/l = \frac{2}{3}$, K_2 is given as 6.58. Therefore,

$$P_2' = 6.58\frac{\pi^2 30(10^6)(4.788)}{360^2} = 72{,}000 \text{ lb}$$

$$FS = \frac{72{,}000}{8000} = 9$$

(c) This case is a column fixed at both ends and subjected to an upward load on top and a downward load at the platform. The upward load depends to some extent on the stiffness of the girder to which the top is welded, and so we can only bracket the actual critical load. If we assume the girder is infinitely rigid and permits no vertical deflection of the top, the elongation of the upper 10 ft would equal the reduction in length of the lower 20 ft.

Equating these deformations gives

$$\frac{P_1(10)(12)}{2.68(30)(10^6)} = \frac{(P_2 - P_1)(20)(12)}{3.174(30)(10^6)} \qquad \text{or} \qquad P_1 = 0.628P_2$$

From Table 15.1, case 2e, for $E_2I_2/E_1I_1 = 1.5$ and $a/l = \frac{2}{3}$, we find the following values of K_2 for the several values of P_1/P_2:

P_1/P_2	0	0.125	0.250	0.375	0.500
K_2	8.34	9.92	12.09	15.17	19.86

By extrapolation, for $P_1/P_2 = 0.628$, $K_2 = 26.5$.

If we assume the girder provides no vertical load but does prevent rotation of the top, then $K_2 = 8.34$. Therefore, the value of P_2 ranges from 91,200 to 289,900 lb, and the factor of safety lies between 11.4 and 36.2. A reasonable estimate of the rotational and vertical stiffness of the girder will allow a good estimate to be made of the actual factor of safety from the values calculated.

15.3 Buckling of Flat and Curved Plates

In Table 15.2, formulas are given for the critical loads and critical stresses on plates and thin-walled members. Because of the greater likelihood of serious geometrical irregularities and their greater relative effect, the critical stresses actually developed by such members usually fall short of the theoretical values by a wider margin than in the case of bars. The discrepancy is generally greater for pure compression (thin tubes under longitudinal compression or external pressure) than for tension and compression combined (thin tubes under torsion or flat plates under edge shear), and increases with the thinness of the material. The critical stress or load indicated by any one of the theoretical formulas should therefore be regarded as an upper limit, approached more or less closely according to the closeness with which the actual shape of the member approximates the geometrical form assumed. In Table 15.2, the approximate discrepancy to be expected between theory and experiment is indicated wherever the data available have made this possible.

Most of the theoretical analyses of the stability of plates and shells require a numerical evaluation of the resulting equations. Considering the variety of shapes and combinations of shapes as well as the multiplicity of boundary conditions and loading combinations, it is not possible in the limited space available to present anything like a comprehensive coverage of plate and shell buckling. As an alternative, Table 15.2 contains many of the simpler loadings and shapes. The following paragraphs and the References contain some, but by no means all, of the more easily acquired sources giving results in tabular or graphic form that can be applied directly to specific problems. See also Refs. 101–104, and 109–111.

Rectangular plates. Stability coefficients for *orthotropic* rectangular plates with several combinations of boundary conditions and several ratios of the bending stiffnesses parallel to the sides of the plate are tabulated in Shuleshko (Ref. 60); these solutions were obtained by reducing the problem of plate buckling to that of an isotropic bar that is in a state of vibration and under tension. Srinivas and Rao (Ref. 63) evaluate the effect of shear deformation on the stability of simply supported rectangular plates under edge loads parallel to one side; the effect becomes noticeable for $h/b > 0.05$ and is greatest when the loading is parallel to the short side.

Skew plates. Ashton (Ref. 61) and Durvasula (Ref. 64) consider the buckling of skew (parallelogram) plates under combinations of edge compression, edge tension, and edge shear. Since the loadings evaluated are generally parallel to orthogonal axes and not to both sets of the plate edges, we would not expect to find the particular case desired represented in the tables of coefficients; the general trend of results is informative.

Circular plates. Vijayakumar and Joga Rao (Ref. 58) describe a technique for solving for the radial buckling loads on a *polar orthotropic annular plate*. They give graphs of stability coefficients for a wide range of rigidity ratios and for the several combinations of free, simply supported, and fixed inner and outer edges for the radius ratio (outer to inner) $2:1$. Two loadings are presented: outer edge only under uniform compression and inner and outer edges under equal uniform compression.

Amon and Widera (Ref. 59) present graphs showing the effect of an edge beam on the stability of a circular plate of uniform thickness.

Sandwich plates. There is a great amount of literature on the subject of sandwich construction. References 38 and 100 and the publications listed in Ref. 39 provide initial sources of information.

15.4 Buckling of Shells

Baker, Kovalevsky, and Rish (Ref. 97) discuss the stability of unstiffened orthotropic composite, stiffened, and sandwich shells. They represent data based on theory and experiment which permit the designer to choose a loading or pressure with a 90% probability of no stability failure; the work is extensively referenced. For similar collected data see Refs. 41 and 42.

Stein (Ref. 95) discusses some comparisons of theory with experimentation in shell buckling. Rabinovich (Ref. 96) describes in some detail the work in structural mechanics, including shell stability, in the U.S.S.R. from 1917 to 1957.

In recent years, there have been increasing development and application of the finite-element method for the numerical solution of shell problems. Navaratna, Pian, and Witmer (Ref. 94) describe a finite-element method of solving axisymmetric shell problems where the element considered is either a conical frustum or a frustum with a curved meridian; examples are presented of cylinders with uniform or tapered walls under axial load, a truncated hemisphere under axial tension, and a conical shell under torsion. Bushnell (Ref. 99) presents a very general finite-element program for shell analysis and Perrone (Ref. 98) gives a compendium of such programs. See also Refs. 101 to 108.

Cylindrical and conical shells. In general, experiments to determine the axial loads required to buckle cylindrical shells yield results that are between one-half and three-fourths of the classical buckling loads predicted by theory. The primary causes of these discrepancies are the deviations from a true cylindrical form in most manufactured vessels and the inability to accurately define the boundary conditions. Hoff (Refs. 67 and 68) shows that removing the in-plane shear stress at the boundary of a simply supported cylindrical shell under axial compression can reduce the theoretical buckling load by a factor of 2 from that predicted by the more usual boundary conditions associated with a simply supported edge. Baruch, Harari, and Singer (Ref. 84) find similar low-buckling loads for simply supported conical shells under axial load but for a different modification of the boundary support. Tani and Yamaki (Ref. 83) carry out further work on this problem, including the effect of clamped edges.

The random nature of manufacturing deviations leads to the use of the statistical approach, as mentioned previously (Ref. 97) and as Hausrath and Dittoe have done for conical shells (Ref. 77). Weingarten, Morgan, and Seide (Ref. 80) have developed empirical expressions for lower bounds of stability coefficients for cylindrical and conical shells under axial compression with references for the many data they present.

McComb, Zender, and Mikulas (Ref. 44) discuss the effects of internal pressure on the bending stability of very thin-walled cylindrical shells. Internal pressure has a stabilizing effect on axially and/or torsionally loaded cylindrical and conical shells. This subject is discussed in several references: Seide (Ref. 75), Weingarten (Ref. 76), and Weingarten, Morgan, and Seide (Ref. 82) for conical and cylindrical shells; Ref. 97 contains much information on this subject as well.

Axisymmetric snap-buckling of conical shells is discussed by Newman and Reiss (Ref. 73), which leads to the concept of the Belleville spring for the case of shallow shells. (See also Sec. 11.8.)

External pressure as a cause of buckling is examined by Singer (Ref. 72) for cones and by Newman and Reiss (Ref. 73) and Yao and Jenkins (Ref. 69) for elliptic cylinders. External pressure caused by pretensioned filament winding on cylinders is analyzed by Mikulas and Stein (Ref. 66); they point out that material compressibility in the thickness direction is important in this problem.

The combination of external pressure and axial loads on cylindrical and conical shells is very thoroughly examined and referenced by Radkowski (Ref. 79) and Weingarten and Seide (Ref. 81). The combined loading on orthotropic and stiffened conical shells is discussed by Singer (Ref. 74).

Attempts to manufacture nearly perfect shells in order to test the theoretical results have led to the construction of thin-walled shells by electroforming; Sendelbeck and Singer (Ref. 85) and Arbocz and Babcock (Ref. 91) describe the results of such tests.

A very thorough survey of buckling theory and experimentation for conical shells of constant thickness is presented by Seide (Ref. 78).

Spherical shells. Experimental work is described by Loo and Evan-Iwanowski on the effect of a concentrated load at the apex of a spherical cap (Ref. 90) and the effect of multiple concentrated loads (Ref. 89). Carlson, Sendelbeck, and Hoff (Ref. 70) report on the experimental study of buckling of electroformed complete spherical shells; they report experimental critical pressures of up to 86% of those predicted by theory and the correlation of flaws with lower test pressures.

Burns (Ref. 92) describes tests of static and dynamic buckling of thin spherical caps due to external pressure; both elastic and plastic buckling are considered and evaluated in these tests. Wu and Cheng (Ref. 71) discuss in detail the buckling due to circumferential hoop compression which is developed when a truncated spherical shell is subjected to an axisymmetric tensile load.

Toroidal shells. Stein and McElman (Ref. 86) derive nonlinear equations of equilibrium and buckling equations for segments of toroidal shells; segments that are symmetric with the equator are considered for both inner and outer diameters, as well as segments centered at the crown. Sobel and Flügge (Ref. 87) tabulate and graph the minimum buckling external pressures on full toroidal shells. Almroth, Sobel, and Hunter (Ref. 88) compare favorably the theory in Ref. 87 with experiments they performed.

Corrugated tubes or bellows. An instability can develop when a corrugated tube or bellows is subjected to an internal pressure with the ends partially or totally restrained against axial displacement. (This instability can also occur in very long cylindrical vessels under similar restraints.) For a discussion and an example of this effect, see Sec. 13.5.

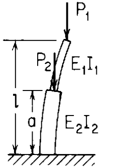
15.5 Tables

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams

NOTATION: P' = critical load (force); p' = critical unit load (force per unit length); T' = critical torque (force-length); M' = critical bending moment (force-length); E = modulus of elasticity (force per unit area); and I = moment of inertia of cross section about central axis perpendicular to plane of buckling

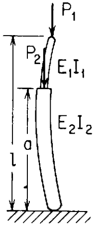
Reference number, form of bar, and manner of loading and support

1a. Stepped straight bar under end load P_1 and intermediate load P_2; upper end free, lower end fixed $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	0.250	0.250	0.250	0.250	0.250	0.279	0.312	0.342	0.364	0.373	0.296	0.354	0.419	0.471	0.496
0.5	0.249	0.243	0.228	0.208	0.187	0.279	0.306	0.317	0.306	0.279	0.296	0.350	0.393	0.399	0.372
1.0	0.248	0.237	0.210	0.177	0.148	0.278	0.299	0.295	0.261	0.223	0.296	0.345	0.370	0.345	0.296
2.0	0.246	0.222	0.178	0.136	0.105	0.277	0.286	0.256	0.203	0.158	0.295	0.335	0.326	0.267	0.210
4.0	0.242	0.195	0.134	0.092	0.066	0.274	0.261	0.197	0.138	0.099	0.294	0.314	0.257	0.184	0.132
8.0	0.234	0.153	0.088	0.056	0.038	0.269	0.216	0.132	0.084	0.057	0.290	0.266	0.174	0.112	0.076

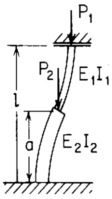
1b. Stepped straight bar under end load P_1 and intermediate load P_2; both ends pinned $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	1.000	1.000	1.000	1.000	1.000	1.010	1.065	1.180	1.357	1.479	1.014	1.098	1.297	1.633	1.940
0.5	0.863	0.806	0.797	0.789	0.740	0.876	0.872	0.967	1.091	1.098	0.884	0.908	1.069	1.339	1.452
1.0	0.753	0.672	0.663	0.646	0.584	0.769	0.736	0.814	0.908	0.870	0.776	0.769	0.908	1.126	1.153
2.0	0.594	0.501	0.493	0.473	0.410	0.612	0.557	0.615	0.676	0.613	0.621	0.587	0.694	0.850	0.814
4.0	0.412	0.331	0.325	0.307	0.256	0.429	0.373	0.412	0.442	0.383	0.438	0.397	0.470	0.566	0.511
8.0	0.254	0.197	0.193	0.180	0.147	0.267	0.225	0.248	0.261	0.220	0.272	0.240	0.284	0.336	0.292

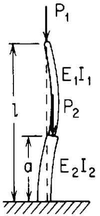
TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

1c. Stepped straight bar under end load P_1 and intermediate load P_2; upper end guided, lower end fixed $P_1' = K_1 \frac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	1.000	1.000	1.000	1.000	1.000	1.113	1.208	1.237	1.241	1.309	1.184	1.367	1.452	1.461	1.565
0.5	0.986	0.904	0.792	0.711	0.672	1.105	1.117	1.000	0.897	0.885	1.177	1.288	1.192	1.063	1.063
1.0	0.972	0.817	0.650	0.549	0.507	1.094	1.026	0.830	0.697	0.669	1.171	1.206	1.000	0.832	0.805
2.0	0.937	0.671	0.472	0.377	0.339	1.073	0.872	0.612	0.482	0.449	1.156	1.047	0.745	0.578	0.542
4.0	0.865	0.480	0.304	0.231	0.204	1.024	0.642	0.397	0.297	0.270	1.126	0.794	0.486	0.358	0.327
8.0	0.714	0.299	0.176	0.130	0.114	0.910	0.406	0.232	0.169	0.151	1.042	0.511	0.284	0.203	0.182

1d. Stepped straight bar under end load P_1 and intermediate load P_2; upper end pinned, lower end fixed $P_1' = K_1 \frac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below

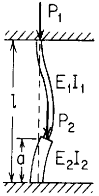


E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	2.046	2.046	2.046	2.046	2.046	2.241	2.289	2.338	2.602	2.976	2.369	2.503	2.550	2.983	3.838
0.5	1.994	1.814	1.711	1.700	1.590	2.208	2.071	1.991	2.217	2.344	2.344	2.286	2.196	2.570	3.066
1.0	1.938	1.613	1.464	1.450	1.290	2.167	1.869	1.727	1.915	1.918	2.313	2.088	1.915	2.250	2.525
2.0	1.820	1.300	1.130	1.111	0.933	2.076	1.535	1.355	1.506	1.390	2.250	1.742	1.518	1.796	1.844
4.0	1.570	0.918	0.773	0.753	0.594	1.874	1.107	0.941	1.042	0.891	2.097	1.277	1.065	1.270	1.184
8.0	1.147	0.569	0.469	0.454	0.343	1.459	0.697	0.582	0.643	0.514	1.727	0.812	0.664	0.796	0.686

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

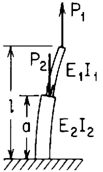
Reference number, form of bar, and manner of loading and support

1e. Stepped straight bar under end load P_1 and intermediate load P_2; both ends fixed $P_1' = K_1 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_1 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l \ P_2/P_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0.0	4.000	4.000	4.000	4.000	4.000	4.389	4.456	4.757	5.359	5.462	4.657	4.836	5.230	6.477	6.838
0.5	3.795	3.298	3.193	3.052	2.749	4.235	3.756	3.873	4.194	3.795	4.545	4.133	4.301	5.208	4.787
1.0	3.572	2.779	2.647	2.443	2.094	4.065	3.211	3.254	3.411	2.900	4.418	3.568	3.648	4.297	3.671
2.0	3.119	2.091	1.971	1.734	1.414	3.679	2.459	2.459	2.452	1.968	4.109	2.766	2.782	3.136	2.496
4.0	2.365	1.388	1.297	1.088	0.857	2.921	1.659	1.649	1.555	1.195	3.411	1.882	1.885	2.008	1.523
8.0	1.528	0.826	0.769	0.623	0.479	1.943	1.000	0.992	0.893	0.671	2.334	1.138	1.141	1.158	0.854

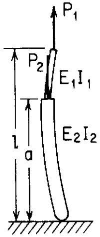
2a. Stepped straight bar under end load P_1 and intermediate load P_2; upper end free, lower end fixed $P_2' = K_2 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l \ P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	9.00	2.25	1.00	0.56	0.36	13.50	3.38	1.50	0.84	0.54	18.00	4.50	2.00	1.13	0.72
0.125	15.55	3.75	1.48	0.74	0.44	21.87	5.36	2.19	1.11	0.65	27.98	6.92	2.89	1.48	0.87
0.250	21.33	5.30	2.19	1.03	0.55	29.51	7.36	3.13	1.53	0.82	37.30	9.31	4.02	2.02	1.10
0.375	29.02	7.25	3.13	1.52	0.74	39.89	9.97	4.37	2.21	1.10	50.10	12.52	5.52	2.86	1.46
0.500	40.50	10.12	4.46	2.31	1.08	55.66	13.92	6.16	3.28	1.60	69.73	17.43	7.73	4.18	2.12

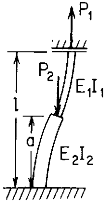
TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

2b. Stepped straight bar under tensile end load P_1 and intermediate load P_2; both ends pinned $P_2' = K_2 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	2.60	1.94	1.89	1.73	1.36	2.77	2.24	2.47	2.54	2.04	2.86	2.41	2.89	3.30	2.72
0.125	3.51	2.49	2.43	2.14	1.62	3.81	2.93	3.26	3.18	2.43	3.98	3.21	3.89	4.19	3.24
0.250	5.03	3.41	3.32	2.77	1.99	5.63	4.15	4.64	4.15	2.99	5.99	4.65	5.75	5.52	3.98
0.375	7.71	5.16	4.96	3.76	2.55	8.98	6.61	7.26	5.63	3.82	9.80	7.67	9.45	7.50	5.09
0.500	12.87	9.13	8.00	5.36	3.48	15.72	12.55	12.00	7.96	5.18	17.71	15.45	16.00	10.54	6.87

2c. Stepped straight bar under tensile end load P_1 and intermediate load P_2; upper end guided, lower end fixed $P_2' = K_2 \dfrac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below

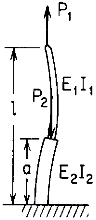


E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	10.40	3.08	1.67	1.19	1.03	14.92	4.23	2.21	1.55	1.37	19.43	5.37	2.73	1.88	1.65
0.125	15.57	4.03	2.03	1.40	1.18	21.87	5.57	2.71	1.82	1.57	27.98	7.07	3.36	2.21	1.90
0.250	21.33	5.37	2.54	1.67	1.38	29.52	7.40	3.42	2.20	1.84	37.32	9.34	4.26	2.68	2.24
0.375	29.02	7.26	3.31	2.08	1.67	39.90	9.97	4.50	2.76	2.24	50.13	12.53	5.61	3.39	2.73
0.500	40.51	10.12	4.53	2.72	2.10	55.69	13.91	6.21	3.66	2.84	69.76	17.43	7.76	4.52	3.47

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

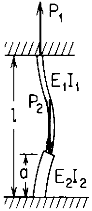
Reference number, form of bar, and manner of loading and support

2d. Stepped straight bar under tensile end load P_1 and intermediate load P_2; upper end pinned, lower end fixed $P_2' = K_2 \frac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l \ P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	13.96	5.87	4.80	4.53	3.24	18.66	7.33	6.04	6.58	4.86	23.26	8.64	6.98	8.40	6.48
0.125	20.21	7.93	6.50	5.84	3.91	27.12	10.12	8.43	8.71	5.86	33.71	12.06	9.92	11.51	7.81
0.250	28.58	11.35	9.64	7.68	4.87	38.32	14.82	13.13	11.50	7.27	47.36	17.85	15.96	15.30	9.65
0.375	41.15	17.64	15.82	10.15	6.26	55.43	23.67	23.44	14.96	9.30	68.40	28.88	30.65	19.66	12.28
0.500	62.90	31.73	23.78	13.58	8.42	85.64	44.28	34.97	19.80	12.40	106.27	55.33	45.81	25.83	16.27

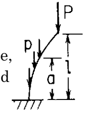
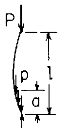
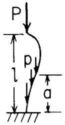
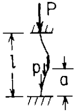
2e. Stepped straight bar under tensile end load P_1 and intermediate load P_2; both ends fixed $P_2' = K_2 \frac{\pi^2 E_1 I_1}{l^2}$ where K_2 is tabulated below



E_2I_2/E_1I_1	1.000					1.500					2.000				
a/l \ P_1/P_2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
0	16.19	8.11	7.54	5.79	4.34	21.06	9.93	9.89	8.34	6.09	25.75	11.44	11.55	10.87	7.78
0.125	21.83	10.37	9.62	6.86	5.00	28.74	12.93	13.03	9.92	7.05	35.28	15.06	15.55	12.96	9.01
0.250	30.02	14.09	12.86	8.34	5.91	39.81	17.99	18.36	12.09	8.35	48.88	21.25	22.98	15.79	10.69
0.375	42.72	20.99	17.62	10.47	7.19	57.14	27.66	26.02	15.17	10.20	70.23	33.29	34.36	19.79	13.11
0.500	64.94	36.57	24.02	13.70	9.16	86.23	50.39	35.09	19.86	13.07	102.53	61.71	45.87	25.86	16.85

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

3a. Uniform straight bar under end load P and a uniformly distributed load p over a lower portion of the length; several end conditions. $(pa)' = K\frac{\pi^2 EI}{l^2}$ where K is tabulated below (a negative value for P/pa means the end load is tensile)

End conditions	Upper end free, lower end fixed 				Both ends pinned 				Upper end pinned, lower end fixed 				Both ends fixed 			
a/l \ P/pa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
−0.25		11.31	5.18	2.38	9.03	5.32	4.25	3.30		27.9	17.4	11.3		31.3	19.4	13.4
0.00	12.74	3.185	1.413	0.795	3.52	2.53	2.22	1.88	22.2	9.46	7.13	5.32	25.3	13.0	9.78	7.56
0.25	0.974	0.825	0.614	0.449	1.97	1.59	1.46	1.30	6.83	4.70	3.98	3.30	11.2	7.50	6.25	5.20
0.50	0.494	0.454	0.383	0.311	1.34	1.15	1.08	0.98	3.76	3.03	2.71	2.37	6.75	5.18	4.54	3.94
1.00	0.249	0.238	0.218	0.192	0.81	0.73	0.70	0.66	1.97	1.75	1.64	1.51	3.69	3.17	2.91	2.65

3b. Uniform straight bar under end load P and a uniformly distributed load p over an upper portion of the length; several end conditions. $(pa)' = K\frac{\pi^2 EI}{l^2}$ where K is tabulated below (a negative value for P/pa means the end load is tensile)

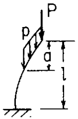
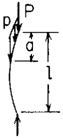
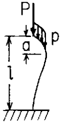
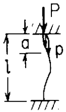
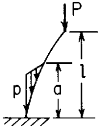
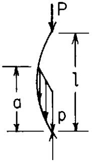
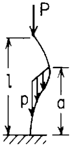
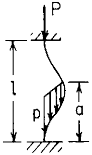
End conditions	Upper end free, lower end fixed 			Both ends pinned 			Upper end pinned, lower end fixed 			Both ends fixed 		
a/l \ P/pa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
−0.25	0.481	0.745	1.282	1.808	2.272	2.581	4.338	5.937	7.385	5.829	7.502	9.213
0.00	0.327	0.440	0.600	1.261	1.479	1.611	2.904	3.586	4.160	4.284	5.174	5.970
0.25	0.247	0.308	0.380	0.963	1.088	1.159	2.164	2.529	2.815	3.384	3.931	4.383
0.50	0.198	0.236	0.276	0.778	0.859	0.903	1.720	1.943	2.111	2.796	3.164	3.453
1.00	0.142	0.161	0.179	0.561	0.603	0.624	1.215	1.323	1.400	2.073	2.273	2.419

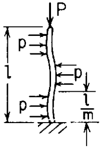
TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

Reference number, form of bar, and manner of loading and support

3c. Stepped straight bar under end load P and a distributed load of maximum value p at the bottom linearly decreasing to zero at a distance a from the bottom. $(pa)' = K\frac{\pi^2 EI}{l^2}$ where K is tabulated below (a negative value for P/pa means the end load is tensile)

End conditions	Upper end free, lower end fixed 				Both ends pinned 				Upper end pinned, lower end fixed 				Both ends fixed 			
a/l \ P/pa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
−0.250						58.9	41.1	30.4								
−0.125			26.7	15.5	31.9	15.7	12.0	9.41			62.1	43.8		113.0	70.2	48.7
0.000	52.4	13.1	5.80	3.26	9.66	6.31	5.32	4.72		30.3	20.6	16.1		38.9	27.8	21.9
0.125	1.98	1.85	1.58	1.29	4.65	3.66	3.29	3.03	15.2	11.7	9.73	8.50	27.3	18.9	15.6	13.4
0.250	0.995	0.961	0.887	0.787	2.98	2.54	2.35	2.22	7.90	6.92	6.18	5.66	14.9	12.1	10.6	9.53
0.500	0.499	0.490	0.471	0.441	1.72	1.56	1.49	1.43	4.02	3.77	3.54	3.36	7.73	6.95	6.43	6.00
1.000	0.250	0.248	0.243	0.235	0.93	0.88	0.86	0.84	2.03	1.96	1.90	1.85	3.93	3.73	3.57	3.44

4. Uniform straight bar under end load P; both ends hinged and bar elastically supported by lateral pressure p proportional to deflection ($p = ky$, where $k =$ lateral force per unit length per unit of deflection)



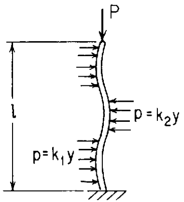
$$P' = \frac{\pi^2 EI}{l^2}\left(m^2 + \frac{kl^4}{m^2\pi^4 EI}\right)$$

where m represents the number of half-waves into which the bar buckles and is equal to the lowest integer greater than

$$\frac{1}{2}\left(\sqrt{1 + \frac{4l^2}{\pi^2}\sqrt{\frac{k}{EI}}} - 1\right)$$

(Ref. 1)

5. Uniform straight bar under end load P; both ends hinged and bar elastically supported by lateral pressure p proportional to deflection but where the constant of proportionality depends upon the direction of the deflection ($p = k_1 y$ for deflection toward the softer foundation; $p = k_2 y$ for deflection toward the harder foundation); these are also called unattached foundations

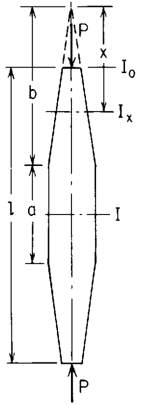


$$P' = \frac{\pi^2 EI}{l^2}\left(m^2 + \frac{k_2 l^4}{m^2 \pi^4 EI}\phi^\alpha\right)$$ where $\phi = \frac{k_1}{k_2}$ and α depends upon m as given below

m	α
1	1
2	$1 + \phi(0.23 - 0.017 l^2 \sqrt{k_2/EI})$
3	$0.75 - 0.56\phi$

This is an empirical expression which closely fits numerical solutions found in Ref. 45 and is valid only over the range $0 \leqslant l^2\sqrt{k_2/EI} \leqslant 120$. Solutions for P' are carried out for values of $m = 1$, 2, and 3, and the lowest one governs

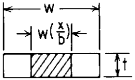
6. Straight bar, middle portion uniform, end portions tapered and alike; end load; I = moment of inertia of cross section of middle portion; I_0 = moment of inertia of end cross sections; I_x = moment of inertia of section x



(For singly tapered columns see Ref. 46.)

6a. $I_x = I\frac{x}{b}$

for example, rectangular section tapering uniformly in width



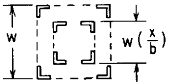
$P' = \frac{KEI}{l^2}$ where K depends on $\frac{I_0}{I}$ and $\frac{a}{l}$ and may be found from the following table:

I_0/I / a/l	K for ends hinged							K for ends fixed			
	0	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	5.78	5.87	6.48	7.01	7.86	8.61	9.27	20.36	26.16	31.04	35.40
0.2	7.04	7.11	7.58	7.99	8.59	9.12	9.53	22.36	27.80	32.20	36.00
0.4	8.35	8.40	8.63	8.90	9.19	9.55	9.68	23.42	28.96	32.92	36.36
0.6	9.36	9.40	9.46	9.73	9.70	9.76	9.82	25.44	30.20	33.80	36.84
0.8	9.80	9.80	9.82	9.82	9.83	9.85	9.86	29.00	33.08	35.80	37.84

(Ref. 5)

6b. $I_x = I\left(\frac{x}{b}\right)^2$

for example, section of four slender members latticed together



$P' = \frac{KEI}{l^2}$ where K may be found from the following table:

I_0/I / a/l	K for ends hinged							K for ends fixed			
	0	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	1.00	3.45	5.40	6.37	7.61	8.51	9.24	18.94	25.54	30.79	35.35
0.2	1.56	4.73	6.67	7.49	8.42	9.04	9.50	21.25	27.35	32.02	35.97
0.4	2.78	6.58	8.08	8.61	9.15	9.48	9.70	22.91	28.52	32.77	36.34
0.6	6.25	8.62	9.25	9.44	9.63	9.74	9.82	24.29	29.69	33.63	36.80
0.8	9.57	9.71	9.79	9.81	9.84	9.85	9.86	27.67	32.59	35.64	37.81

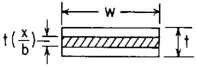
(Ref. 5)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

Reference number, form of bar, and manner of loading and support

6c. $I_x = I\left(\frac{x}{b}\right)^3$

for example, rectangular section tapering uniformly in thickness



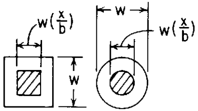
$P' = \frac{KEI}{l^2}$ where K may be found from the following table:

I_0/I	K for ends hinged						K for ends fixed			
a/l	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	2.55	5.01	6.14	7.52	8.50	9.23	18.48	25.32	30.72	35.32
0.2	3.65	6.32	7.31	8.38	9.02	9.50	20.88	27.20	31.96	35.96
0.4	5.42	7.84	8.49	9.10	9.46	9.69	22.64	28.40	32.72	36.32
0.6	7.99	9.14	9.39	9.62	9.74	9.81	23.96	29.52	33.56	36.80
0.8	9.63	9.77	9.81	9.84	9.85	9.86	27.24	32.44	35.60	37.80

(Ref. 5)

6d. $I_x = I\left(\frac{x}{b}\right)^4$

for example, end portions pyramidal or conical



$P' = \frac{KEI}{l^2}$ where K may be found from the following table:

I_0/I	K for ends hinged						K for ends fixed			
a/l	0.01	0.10	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
0	2.15	4.81	6.02	7.48	8.47	9.23	18.23	25.23	30.68	35.33
0.2	3.13	6.11	7.20	8.33	9.01	9.49	20.71	27.13	31.94	35.96
0.4	4.84	7.68	8.42	9.10	9.45	9.69	22.49	28.33	32.69	36.32
0.6	7.53	9.08	9.38	9.62	9.74	9.81	23.80	29.46	33.54	36.78
0.8	9.56	9.77	9.80	9.84	9.85	9.86	27.03	32.35	35.56	37.80

(Ref. 5)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

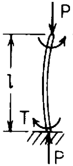
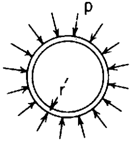
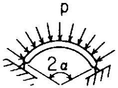
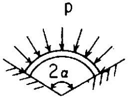
7. Uniform straight bar under end loads P and end twisting couples T; cross section of bar has same I for all central axes; both ends hinged  	Critical combination of P and T is given by $\frac{T^2}{4(EI)^2}+\frac{P}{EI}=\frac{\pi^2}{l^2}$ (Ref. 1) If $P=0$, the formula gives critical twisting moment T' which, acting alone, would cause buckling If for a given value of T the formula gives a negative value for P, $T>T'$ and P represents tensile load required to prevent buckling For thin circular tube of diameter D and thickness t under torsion only, critical shear stress $\tau=\frac{\pi ED}{l(1-\nu)}\left(1-\frac{t}{D}+\frac{1}{3}\frac{t^2}{D^2}\right)$ for helical buckling only (not for shell-type buckling in the thin wall) (Ref. 2)
8. Uniform circular ring under uniform radial pressure p lb/in; mean radius of ring r  	$p'=\frac{3EI}{r^3}$ (Ref. 1)
9. Uniform circular arch under uniform radial pressure p lb/in; mean radius r; ends hinged  	$p'=\frac{EI}{r^3}\left(\frac{\pi^2}{\alpha^2}-1\right)$ (Ref. 1) (For symmetrical arch of any form under central concentrated loading, see Ref. 40; for parabolic and catenary arches, see Ref. 50)
10. Uniform circular arch under uniform radial pressure p lb/in; mean radius r; ends fixed  	$p'=\frac{EI}{r^3}(k^2-1)$ (Ref. 1) where k depends on α and is found by trial from the equation: $k\tan\alpha\cot k\alpha=1$ or from the following table: $\begin{array}{ccccccccc} \alpha & 15^\circ & 30^\circ & 45^\circ & 60^\circ & 75^\circ & 90^\circ & 120^\circ & 180^\circ \\ \hline k & 17.2 & 8.62 & 5.80 & 4.37 & 3.50 & 3.00 & 2.36 & 2.00 \end{array}$ (For parabolic and catenary arches, see Ref. 50)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

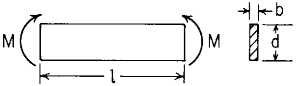
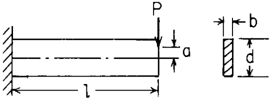
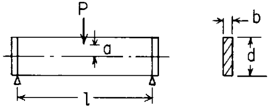
	Reference number, form of bar, and manner of loading and support
11. Straight uniform beam of narrow rectangular section under pure bending 	For ends held vertical but not fixed in horizontal plane: $M' = \dfrac{\pi b^3 d\sqrt{EG\left(1-0.63\frac{b}{d}\right)}}{6l}$ For ends held vertical and fixed in horizontal plane $M' = \dfrac{2\pi b^3 d\sqrt{EG\left(1-0.63\frac{b}{d}\right)}}{6l}$ (Refs. 1, 3, 4)
12. Straight uniform cantilever beam of narrow rectangular section under end load applied at a point a distance a above (a positive) or below (a negative) centroid of section 	$P' = \dfrac{0.669b^3 d\sqrt{\left(1-0.63\frac{b}{d}\right)EG}}{l^2}\left[1-\dfrac{a}{2l}\sqrt{\dfrac{E}{G\left(1-0.63\frac{b}{d}\right)}}\right]$ For a load W uniformly distributed along the beam the critical load, $W' = 3P'$ (approximately) (For tapered and stepped beams, see Ref. 54) (Refs. 1, 3, 4)
13. Straight uniform beam of narrow rectangular section under center load applied at a point a distance a above (a positive) or below (a negative) centroid of section: ends of beam simply supported and constrained against twisting 	$P' = \dfrac{2.82b^3 d\sqrt{\left(1-0.63\frac{b}{d}\right)EG}}{l^2}\left[1-\dfrac{1.74a}{l}\sqrt{\dfrac{E}{G\left(1-0.63\frac{b}{d}\right)}}\right]$ For a uniformly distributed load, the critical load $W' = 1.67P'$ (approximately) If P is applied at an intermediate point, a distance C from one end, its critical value is practically the same as for central loading if $0.4l < C < 0.5l$: if $C < 0.4l$, the critical load is given approximately by multiplying the P' for central loading by $0.36 + 0.28\frac{l}{C}$ If the ends of the beam are fixed and the load P is applied at the centroid of the middle cross section, $P' = \dfrac{4.43b^3 d}{l^2}\sqrt{\left(1-0.63\frac{b}{d}\right)EG}$ (Refs. 1, 3, 4)

TABLE 15.1 Formulas for elastic stability of bars, rings, and beams (*Continued*)

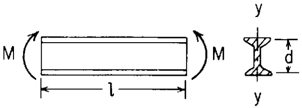
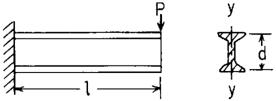
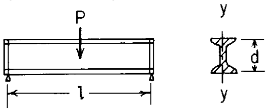
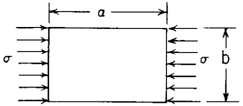
14. Straight uniform I-beam under pure bending; d = depth center to center of flange; ends constrained against twisting  	$M' = \dfrac{\pi\sqrt{EI_y KG}}{l}\sqrt{1+\pi^2\dfrac{I_f Ed^2}{2KGl^2}}$ where I_y is the moment of inertia of the cross section about its vertical axis of symmetry, I_f is the moment of inertia of one flange about this axis, and KG is the torsional rigidity of the section (see Table 10.1, case 26) (For tapered I-beams, see Ref. 47.) (Refs. 1, 3)
15. Straight uniform cantilever beam of I-section under end load applied at centroid of cross section; d = depth center to center of flanges  	$P' = \dfrac{m\sqrt{EI_y KG}}{l^2}$ where m is approximately equal to $4.01 + 11.7\sqrt{\dfrac{I_f Ed^2}{2KGl^2}}$ and I_y, I_f, and KG have the same significance as in case 14 (For unsymmetric I-beams, see Refs. 52 and 56; for tapered I-beams, see Ref. 55.) (Refs. 1, 3)
16. Straight uniform I-beam loaded at centroid of middle section; ends simply supported and constrained against twisting  	$P' = \dfrac{m\sqrt{EI_y KG}}{l^2}$ where m is approximately equal to $16.93 + 45\left(\dfrac{I_f Ed^2}{2KGl^2}\right)^{0.8}$ and I_y, I_f, and KG have same significance as in case 14. (For unsymmetric I-beams, see Refs. 52 and 56; for tapered I-beams, see Ref. 55.) (Refs. 1, 3)

TABLE 15.2 Formulas for elastic stability of plates and shells

NOTATION: E = modulus of elasticity; ν = Poisson's ratio; and t = thickness for all plates and shells. All angles are in radians. Compression is positive; tension is negative. For the plates, the smaller width should be greater than 10 times the thickness unless otherwise specified.

<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ′, unit shear stress τ′, load P′, bending moment M′, or unit external pressure q′ at which elastic buckling occurs</th></tr>
<tr><td rowspan="5">1. Rectangular plate under equal uniform compression on two opposite edges b
 </td><td>1a. All edges simply supported</td><td>

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

Here K depends on ratio $\frac{a}{b}$ and may be found from the following table:

<table>
<tr><th>a/b</th><th>0.2</th><th>0.3</th><th>0.4</th><th>0.6</th><th>0.8</th><th>1.0</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2.0</th><th>2.2</th><th>2.4</th><th>2.7</th><th>3.0</th><th>∞</th></tr>
<tr><td>K</td><td>22.2</td><td>10.9</td><td>6.92</td><td>4.23</td><td>3.45</td><td>3.29</td><td>3.40</td><td>3.68</td><td>3.45</td><td>3.32</td><td>3.29</td><td>3.32</td><td>3.40</td><td>3.32</td><td>3.29</td><td>3.29</td></tr>
</table>

(For unequal end compressions, see Ref. 33) (Refs. 1, 6)</td></tr>
<tr><td>1b. All edges clamped</td><td>

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

<table>
<tr><th>a/b</th><th>1</th><th>2</th><th>3</th><th>∞</th></tr>
<tr><td>K</td><td>7.7</td><td>6.7</td><td>6.4</td><td>5.73</td></tr>
</table>

(Refs. 1, 6, 7)</td></tr>
<tr><td>1c. Edges b simply supported, edges a clamped</td><td>

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

<table>
<tr><th>a/b</th><th>0.4</th><th>0.5</th><th>0.6</th><th>0.7</th><th>0.8</th><th>1.0</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2.1</th><th>∞</th></tr>
<tr><td>K</td><td>7.76</td><td>6.32</td><td>5.80</td><td>5.76</td><td>6.00</td><td>6.32</td><td>5.80</td><td>5.76</td><td>6.00</td><td>5.80</td><td>5.76</td><td>5.73</td></tr>
</table>

(Refs. 1, 6)</td></tr>
<tr><td>1d. Edges b simply supported, one edge a simply supported, other edge a free</td><td>

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

<table>
<tr><th>a/b</th><th>0.5</th><th>1.0</th><th>1.2</th><th>1.4</th><th>1.6</th><th>1.8</th><th>2.0</th><th>2.5</th><th>3.0</th><th>4.0</th><th>5.0</th></tr>
<tr><td>K</td><td>3.62</td><td>1.18</td><td>0.934</td><td>0.784</td><td>0.687</td><td>0.622</td><td>0.574</td><td>0.502</td><td>0.464</td><td>0.425</td><td>0.416</td></tr>
</table>

(Ref. 1)</td></tr>
<tr><td>1e. Edges b simply supported, one edge a clamped, other edge a free</td><td>

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

<table>
<tr><th>a/b</th><th>1</th><th>1.1</th><th>1.2</th><th>1.3</th><th>1.4</th><th>1.5</th><th>1.6</th><th>1.7</th><th>1.8</th><th>1.9</th><th>2.0</th><th>2.2</th><th>2.4</th></tr>
<tr><td>K</td><td>1.40</td><td>1.28</td><td>1.21</td><td>1.16</td><td>1.12</td><td>1.10</td><td>1.09</td><td>1.09</td><td>1.10</td><td>1.12</td><td>1.14</td><td>1.19</td><td>1.21</td></tr>
</table>

(Ref. 1)</td></tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

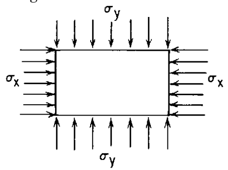
1f. Edges b clamped, edges a simply supported

$$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

$\frac{a}{b}$	0.6	0.8	1.0	1.2	1.4	1.6	1.7	1.8	2.0	2.5	3.0
K	11.0	7.18	5.54	4.80	4.48	4.39	4.39	4.26	3.99	3.72	3.63

(Ref. 1)

2. Rectangular plate under uniform compression (or tension) σ_x on edges b and uniform compression (or tension) σ_y on edges a



2a. All edges simply supported

$$\sigma'_x\frac{m^2}{a^2}+\sigma'_y\frac{n^2}{b^2}=0.823\frac{E}{1-\nu^2}t^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

Here m and n signify the number of half-waves in the buckled plate in the x and y directions, respectively. To find σ'_y for a given σ_x, take $m=1$, $n=1$ if $C\left(1-4\frac{a^4}{b^4}\right)<\sigma_x<C\left(5+2\frac{a^2}{b^2}\right)$, where $C=\frac{0.823Et^2}{(1-\nu^2)a^2}$.

If σ_x is too large to satisfy this inequality, take $n=1$ and m to satisfy:

$C\left(2m^2-2m+1+2\frac{a^2}{b^2}\right)<\sigma_x<C\left(2m^2+2m+1+2\frac{a^2}{b^2}\right)$. If σ_x is too small to satisy the first inequality, take $m=1$ and n to satisfy:

$$C\left[1-n^2(n-1)^2\frac{a^4}{b^4}\right]>\sigma_x>C\left[1-n^2(n+1)^2\frac{a^4}{b^4}\right]$$

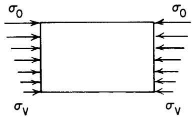
(Refs. 1, 6)

2b. All edges clamped

$$\sigma'_x+\frac{a^2}{b^2}\sigma'_y=1.1\frac{Et^2a^2}{1-\nu^2}\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right)$$

(This equation is approximate and is most accurate when the plate is nearly square and σ_x and σ_y nearly equal (Ref. 1)

3. Rectangular plate under linearly varying stress on edges b (bending or bending combined with tension or compression)



3a. All edges simply supported

$$\sigma'_o = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$$

Here K depends on $\frac{a}{b}$ and on $\alpha=\frac{\sigma_o}{\sigma_o-\sigma_v}$ and may be found from the following table:

	$\frac{a}{b}=0.4$	0.5	0.6	0.667	0.75	0.8	0.9	1.0	1.5
$\alpha=0.5$	$K=23.9$	21.1	19.8	19.7	19.8	20.1	21.1	21.1	19.8
0.75	15.4		10.6		9.5	9.2		9.1	9.5
1.00	12.4		8.0		6.9	6.7		6.4	6.9
1.25	10.95		6.8		5.8	5.7		5.4	5.8
1.50	8.9		5.3		5.0	4.9		4.8	5.0
∞ (pure compression)	6.92		4.23			3.45		3.29	3.57

(Refs. 1, 6)

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

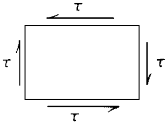
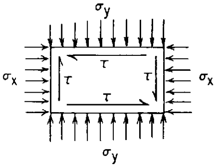
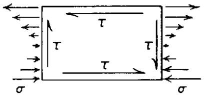
<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs</th></tr>
<tr><td rowspan="2">4. Rectangular plate under uniform shear on all edges

 </td><td>4a. All edges simply supported</td><td>$\tau' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table><tr><td>$\frac{a}{b}$</td><td>1.0</td><td>1.2</td><td>1.4</td><td>1.5</td><td>1.6</td><td>1.8</td><td>2.0</td><td>2.5</td><td>3.0</td><td>∞</td></tr><tr><td>K</td><td>7.75</td><td>6.58</td><td>6.00</td><td>5.84</td><td>5.76</td><td>5.59</td><td>5.43</td><td>5.18</td><td>5.02</td><td>4.40</td></tr></table>(Refs. 1, 6, 8, 22)</td></tr>
<tr><td>4b. All edges clamped</td><td>$\tau' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
<table><tr><td>$\frac{a}{b}$</td><td>1</td><td>2</td><td>∞</td></tr><tr><td>K</td><td>12.7</td><td>9.5</td><td>7.38</td></tr></table>Test results indicate a value for K of about 4.1 for very large values of $\frac{a}{b}$ (Ref. 9)
(For continuous panels, see Ref. 30)</td></tr>
<tr><td rowspan="2">5. Rectangular plate under uniform shear on all edges; compression (or tension) σ_x on edges b; compression (or tension) σ_y on edges a; a/b very large

 </td><td>5a. All edges simply supported</td><td>$\tau' = \sqrt{C^2\left(2\sqrt{1-\frac{\sigma_y}{C}}+2-\frac{\sigma_x}{C}\right)\left(2\sqrt{1-\frac{\sigma_y}{C}}+6-\frac{\sigma_x}{C}\right)}$
where $C = \frac{0.823}{1-\nu^2}\left(\frac{t}{b}\right)^2 E$ (Refs. 1, 6, 23, and 31)</td></tr>
<tr><td>5b. All edges clamped</td><td>$\tau' = \sqrt{C^2\left(2.31\sqrt{4-\frac{\sigma_y}{C}}+\frac{4}{3}-\frac{\sigma_x}{C}\right)\left(2.31\sqrt{4-\frac{\sigma_y}{C}}+8-\frac{\sigma_x}{C}\right)}$
where $C = \frac{0.823}{1-\nu^2}\left(\frac{t}{b}\right)^2 E$
(σ_x and σ_y are negative when tensile) (Ref. 6)</td></tr>
<tr><td>6. Rectangular plate under uniform shear on all edges and bending stresses on edges b

 </td><td>6a. All edges simply supported</td><td>$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$
Here K depends on $\frac{\tau}{\tau'}$ (ratio of actual shear stress to shear stress that, acting alone, would be critical) and on $\frac{a}{b}$. K varies less than 10% for values $\frac{a}{b}$ from 0.5 to 1, and for $\frac{a}{b}=1$ is approximately as follows:
<table><tr><td>$\frac{\tau}{\tau'}$</td><td>0</td><td>0.2</td><td>0.3</td><td>0.4</td><td>0.5</td><td>0.6</td><td>0.7</td><td>0.8</td><td>0.9</td><td>1.0</td></tr><tr><td>K</td><td>21.1</td><td>20.4</td><td>19.6</td><td>18.5</td><td>17.7</td><td>16.0</td><td>14.0</td><td>11.9</td><td>8.20</td><td>0</td></tr></table>(Refs. 1, 10)</td></tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

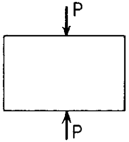
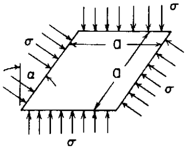
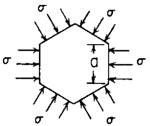
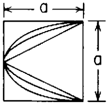
<table>
<tr>
<td rowspan="2">7. Rectangular plate under concentrated center loads on two opposite edges
 </td>
<td>7a. All edges simply supported</td>
<td>$P' = \frac{\pi}{3}\frac{Et^3}{(1-\nu^2)b} \quad \left(\text{for } \frac{a}{b} > 2\right)$ (Ref. 1)</td>
</tr>
<tr>
<td>7b. Edges b simply supported, edges a clamped</td>
<td>$P' = \frac{2\pi}{3}\frac{Et^3}{(1-\nu^2)b} \quad \left(\text{for } \frac{a}{b} > 2\right)$ (Ref. 1)</td>
</tr>
<tr>
<td>8. Rhombic plate under uniform compression on all edges
 </td>
<td>8a. All edges simply supported</td>
<td>$\sigma' = K\frac{Et^2}{a^2(1-\nu^2)}$
<table>
<tr><td>α</td><td>0°</td><td>9°</td><td>18°</td><td>27°</td><td>36°</td><td>45°</td></tr>
<tr><td>K</td><td>1.645</td><td>1.678</td><td>1.783</td><td>1.983</td><td>2.338</td><td>2.898</td></tr>
</table>
Ref. 65)</td>
</tr>
<tr>
<td>9. Polygon plate under uniform compression on all edges

N = number of sides</td>
<td>9a. All edges simply supported</td>
<td>$\sigma' = K\frac{Et^2}{a^2(1-\nu^2)}$
<table>
<tr><td>N</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td>K</td><td>4.393</td><td>1.645</td><td>0.916</td><td>0.597</td><td>0.422</td><td>0.312</td></tr>
</table>
(Ref. 65)</td>
</tr>
<tr>
<td rowspan="2">10. Parabolic and semielliptic plates under uniform compression on all edges
 </td>
<td>10a. All edges simply supported</td>
<td rowspan="2">$\sigma' = K\frac{Et^2}{a^2(1-\nu^2)}$
where K is tabulated below for the several shapes and boundary conditions for $\nu = \frac{1}{3}$:
<table>
<tr><td></td><td>Square</td><td>Semiellipse</td><td>Parabola</td><td>Triangle</td></tr>
<tr><td>Simply supported</td><td>1.65</td><td>1.86</td><td>2.50</td><td>3.82</td></tr>
<tr><td>Fixed</td><td>4.36</td><td>5.57</td><td>7.22</td><td>10.60</td></tr>
</table>
(Ref. 62)</td>
</tr>
<tr>
<td>10b. All edges fixed</td>
</tr>
</table>

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

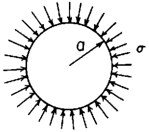
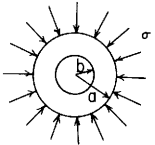
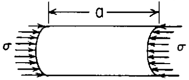
Form of plate or shell and manner of loading	Manner of support	Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs
11. Isotropic circular plate under uniform radial edge compression  	11a. Edges simply supported	$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$ $\begin{array}{c\|c\|c\|c\|c\|c} \nu & 0 & 0.1 & 0.2 & 0.3 & 0.4 \\ \hline K & 0.282 & 0.306 & 0.328 & 0.350 & 0.370 \end{array}$ (Ref. 1)
	11b. Edges clamped	$\sigma' = 1.22\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$ (Ref. 1) For elliptical plate with major semiaxis a, minor semiaxis b, $\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{b}\right)^2$, where K has values as follows: $\begin{array}{c\|cccccc} \frac{a}{b} & 1.0 & 1.1 & 1.2 & 1.3 & 2.0 & 5.0 \\ \hline K & 1.22 & 1.13 & 1.06 & 1.01 & 0.92 & 0.94 \end{array}$ (Ref. 21)
12. Circular plate with concentric hole under uniform radial compression on outer edge  $\frac{a}{t} > 10$	12a. Outer edge simply supported, inner edge free	$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$ Here K depends on $\frac{b}{a}$ and is given approximately by following table: $\begin{array}{c\|cccccccccc} \frac{b}{a} & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ \hline K & 0.35 & 0.33 & 0.30 & 0.27 & 0.23 & 0.21 & 0.19 & 0.18 & 0.17 & 0.16 \end{array}$ (Ref. 1)
	12b. Outer edge clamped, inner edge free	$\sigma' = K\frac{E}{1-\nu^2}\left(\frac{t}{a}\right)^2$ Here K depends on $\frac{b}{a}$ and is given approximately by following table: $\begin{array}{c\|cccccc} \frac{b}{a} & 0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\ \hline K & 1.22 & 1.17 & 1.11 & 1.21 & 1.48 & 2.07 \end{array}$ (Ref. 1)
13. Curved panel under uniform compression on curved edges b (b = width of panel measured on arc; r = radius of curvature)  $\frac{b}{t} > 10$	13a. All edges simply supported	$\sigma' = \frac{1}{6}\frac{E}{1-\nu^2}\left[\sqrt{12(1-\nu^2)\left(\frac{t}{r}\right)^2 + \left(\frac{\pi t}{b}\right)^4} + \left(\frac{\pi t}{b}\right)^2\right]$ (*Note:* With $a > b$, the solution does not depend upon a) or $\sigma' = 0.6E\frac{t}{r}$ if $\frac{b}{r}$ (central angle of curve) is less than $\frac{1}{2}$ and b and a are nearly equal (Refs. 1 and 6) (For compression combined with shear, see Refs. 28 and 34.)

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

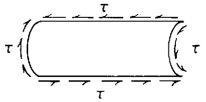
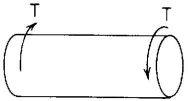
14. Curved panel under uniform shear on all edges  $b/t > 10$. See case 13 for b and r.	14a. All edges simply supported	$\tau' = 0.1E\frac{t}{r} + 5E\left(\frac{t}{b}\right)^2$ (Refs. 6, 27, 29)
	14b. All edges clamped	$\tau' = 0.1E\frac{t}{r} + 7.5E\left(\frac{t}{b}\right)^2$ (Ref. 6) Tests show $\tau' = 0.075E\frac{t}{r}$ for panels curved to form quadrant of a circle (Ref. 11) (See also Refs. 27, 29)
15. Thin-walled circular tube under uniform longitudinal compression (radius of tube = r)  $\frac{r}{t} > 10$	15a. Ends not constrained	$\sigma' = \frac{1}{\sqrt{3}}\frac{E}{\sqrt{1-\nu^2}}\frac{t}{r}$ (Refs. 6, 12, 13, 24) Most accurate for very long tubes, but applicable if length is several times as great as $1.72\sqrt{rt}$, which is the length of a half-wave of buckling. Tests indicate an actual buckling strength of 40–60% of this theoretical value, or $\sigma' = 0.3Et/r$ approximately
16. Thin-walled circular tube under a transverse bending moment M (radius of tube = r)  $\frac{r}{t} > 10$	16a. No constraint	$M' = K\frac{E}{1-\nu^2}rt^2$ Here the theoretical value of K for pure bending and long tubes is 0.99. The average value of K determined by tests is 1.14, and the minimum value is 0.72. Except for very short tubes, length effect is negligible and a small transverse shear produces no appreciable reduction in M'. A very short cylinder under transverse (beam) shear may fail by buckling at neutral axis when shear stress there reaches a value of about $1.25\tau'$ for case 17a (Refs. 6, 14, 15)
17. Thin-walled circular tube under a twisting moment T that produces a uniform circumferential shear stress: $\tau = \frac{T}{2\pi r^2 t}$ (length of tube = l; radius of tube = r)  $\frac{r}{t} > 10$	17a. Ends hinged, i.e., wall free to change angle with cross section, but circular section maintained	$\tau' = \frac{E}{1-\nu^2}\left(\frac{t}{l}\right)^2(1.27 + \sqrt{9.64 + 0.466H^{1.5}})$ where $H = \sqrt{1-\nu^2}\frac{l^2}{tr}$ Tests indicate that the actual buckling stress is 60–75% of this theoretical value, with the majority of the data points nearer 75% (Refs. 6, 16, 18, 25)
	17b. Ends clamped, i.e., wall held perpendicular to cross section and circular section maintained	$\tau' \frac{E}{1-\nu^2}\left(\frac{t}{l}\right)^2(-2.39 + \sqrt{96.9 + 0.605H^{1.5}})$ where H is given in part 17a. The statement in part a regarding actual buckling stress applies here as well (Refs. 6, 16, 18, 25)

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

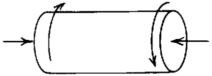
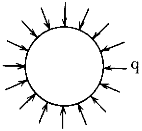
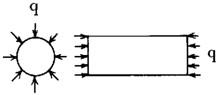
Form of plate or shell and manner of loading	Manner of support	Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs
18. Thin-walled circular tube under uniform longitudinal compression σ and uniform circumferential shear τ due to torsion (case 15 combined with case 17)  $\frac{r}{t} > 10$	18a. Edges hinged as in case 17a. 18b. Edges clamped as in case 17b.	The equation $1 - \frac{\sigma'}{\sigma'_o} = \left(\frac{\tau'}{\tau'_o}\right)^n$ holds, where σ' and τ' are the critical compressive and shear stresses for the combined loading, σ'_o is the critical compressive stress for the cylinder under compression alone (case 15), and τ'_o is the critical shear stress for the cylinder under torsion alone (case 17a or 17b according to end conditions). Tests indicate that n is approximately 3. If σ is tensile, then σ' should be considered negative. (Ref. 6) (See also Ref. 26. For square tube, see Ref. 32)
19. Thin tube under uniform lateral external pressure (radius of tube $= r$)  $\frac{r}{t} > 10$	19a. Very long tube with free ends; length l	$q' = \frac{1}{4}\frac{E}{1-\nu^2}\frac{t^3}{r^3}$ Applicable when $l > 4.90r\sqrt{\frac{r}{t}}$ (Ref. 19)
	19b. Short tube, of length l, ends held circular, but not otherwise constrained, or long tube held circular at intervals l	$q' = 0.807\frac{Et^2}{lr}\sqrt[4]{\left(\frac{1}{1-\nu^2}\right)^3\frac{t^2}{r^2}}$ approximate formula (Ref. 19)
20. Thin tube with closed ends under uniform external pressure, lateral and longitudinal (length of tube $= l$; radius of tube $= r$)  $\frac{r}{t} > 10$	20a. Ends held circular	$q' = \frac{E\frac{t}{r}}{1+\frac{1}{2}\left(\frac{\pi r}{nl}\right)^2}\left\{\frac{1}{n^2\left[1+\left(\frac{nl}{\pi r}\right)^2\right]^2} + \frac{n^2t^2}{12r^2(1-\nu^2)}\left[1+\left(\frac{\pi r}{nl}\right)^2\right]^2\right\}$ (Refs. 19, 20) where $n =$ number of lobes formed by the tube in buckling. To determine q' for tubes of a given t/r, plot a group of curves, one curve for each integral value of n of 2 or more, with l/r as ordinates and q' as abscissa; that curve of the group which gives the least value of q' is then used to find the q' corresponding to a given l/r. If $60 < \left(\frac{l}{r}\right)^2\left(\frac{r}{t}\right) < 2.5\left(\frac{r}{t}\right)^2$, the critical pressure can be approximated by $q' = \frac{0.92E}{\left(\frac{l}{r}\right)\left(\frac{r}{t}\right)^{2.5}}$ (Ref. 81) For other approximations see ref. 109 Values of experimentally determined critical pressures range 20% above and below the theoretical values given by the expressions above. A recommended probable minimum critical pressure is 0.80q'.

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

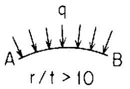
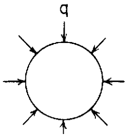
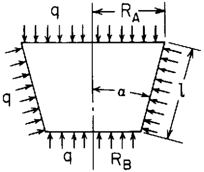
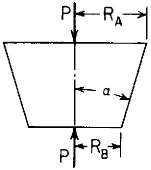
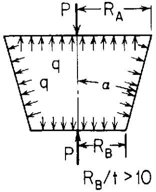
21. Curved panel under uniform radial pressure (radius of curvature r, central angle 2α, when $2\alpha = \text{arc } AB/r$)  	21a. Curved edges free, straight edges at A and B simply supported (i.e., hinged)	$q' = \frac{Et^3\left(\frac{\pi^2}{\alpha^2} - 1\right)}{12r^3(1-\nu^2)}$ (Ref. 1)
	21b. Curved edges free, straight edges at A and B clamped	Here k is found from the equation $k \tan\alpha \cot k\alpha = 1$ and has the following values: $q' = \frac{Et^3(k^2-1)}{12r^3(1-\nu^2)}$ $\begin{array}{c\|ccccccc} \alpha & 15^\circ & 30^\circ & 60^\circ & 90^\circ & 120^\circ & 150^\circ & 180^\circ \\ \hline k & 17.2 & 8.62 & 4.37 & 3.0 & 2.36 & 2.07 & 2.0 \end{array}$ (Ref. 1)
22. Thin sphere under uniform external pressure (radius of sphere $= r$)  r/t > 10	22a. No constraint	$q' = \frac{2Et^2}{r^2\sqrt{3(1-\nu^2)}}$ (for ideal case) (Refs. 1, 37) $q' = \frac{0.365Et^2}{r^2}$ (probable actual minimum q') For spherical cap, half-central angle ϕ between 20 and 60°, R/t between 400 and 2000, $q' = [1 - 0.00875(\phi^\circ - 20^\circ)]\left(1 - 0.000175\frac{R}{t}\right)(0.3E)\left(\frac{t}{R}\right)^2$ (Empirical formula, Ref. 43)
23. Thin truncated conical shell with closed ends under external pressure (both lateral and longitudinal pressure)  $R_B/t > 10$	23a. Ends held circular	q' can be found from the formula of case 20a if the slant length of the cone is substituted for the length of the cylinder and if the average radius of curvature of the wall of the cone normal to the meridian $(R_A + R_B)/(2\cos\alpha)$ is substituted for the radius of the cylinder. The same recommendation of a probable minimum critical pressure of $0.8q'$ is made from the examination of experimental data for cones. (Refs. 78, 81)
24. Thin truncated conical shell under axial load  $R_B/t > 10$	24a. Ends held circular	$P' = \frac{2\pi Et^2\cos^2\alpha}{\sqrt{3(1-\nu^2)}}$ (theoretical) Tests indicate an actual buckling strength of from 40 to 60% of the above theoretical value, or $P' = 0.3(2\pi Et^2\cos^2\alpha)$ approximately. (Ref. 78) In Ref. 77 it is stated that $P' = 0.277(2\pi Et^2\cos^2\alpha)$ will give 95% confidence in at least 90% of the cones carrying more than this critical load. This is based on 170 tests.

TABLE 15.2 Formulas for elastic stability of plates and shells (*Continued*)

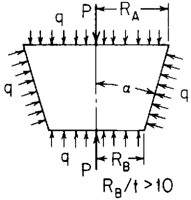
<table>
<tr><th>Form of plate or shell and manner of loading</th><th>Manner of support</th><th>Formulas for critical unit compressive stress σ', unit shear stress τ', load P', bending moment M', or unit external pressure q' at which elastic buckling occurs</th></tr>
<tr><td>25. Thin truncated conical shell under combined axial load and internal pressure


</td><td>25a. Ends held circular</td><td>$P' - q\pi R_B^2 = K_A 2\pi E t^2 \cos^2\alpha$

The probable minimum values of K_A are tabulated for several values of $K_P = \frac{q}{E}\left(\frac{R_B}{t\cos\alpha}\right)^2$.
$k_B = 2\left[\frac{12(1-\nu^2)R_B^2}{t^2\tan^2\alpha\sin^2\alpha}\right]^{1/4}$

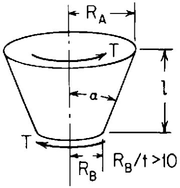
<table>
<tr><th>K_p</th><th>0.00</th><th>0.25</th><th>0.50</th><th>1.00</th><th>1.50</th><th>2.00</th><th>3.00</th></tr>
<tr><td>for $k_B \leqslant 150$</td><td>0.30</td><td>0.52</td><td>0.60</td><td>0.68</td><td>0.73</td><td>0.76</td><td>0.80</td></tr>
<tr><td>for $k_B > 150$</td><td>0.20</td><td>0.36</td><td>0.48</td><td>0.60</td><td>0.64</td><td>0.66</td><td>0.69</td></tr>
</table>

(Ref. 78))</td></tr>
<tr><td>26. Thin truncated conical shell under combined axial load and external pressure


</td><td>26a. Ends held circular</td><td>The following conservative interaction formula may be used for design. It is applicable equally to theoretical values or to minimum probable values of critical load and pressure.

$$\frac{P'}{P'_{\text{case 24}}} + \frac{q'}{q'_{\text{case 23}}} = 1$$

This expression can be used for cylinders if the angle α is set equal to zero or use is made of cases 15 and 20. For small values of $P'/P'_{\text{case 24}}$ the external pressure required to collapse the shell is greater than that required to initiate buckling. See Ref. 78.</td></tr>
<tr><td>27. Thin truncated conical shell under torsion


</td><td>27a. Ends held circular</td><td>Let $T = \tau' 2\pi r_e^2 t$ and for τ' use the formulas for thin-walled circular tubes, case 17, substituting for the radius r of the tube the equivalent radius r_e, where $r_e = R_B\cos\alpha\left\{1+\left[\frac{1}{2}\left(1+\frac{R_A}{R_B}\right)\right]^{1/2} - \left[\frac{1}{2}\left(1+\frac{R_A}{R_B}\right)\right]^{-1/2}\right\}$. l and t remain the axial length and wall thickness, respectively.

(Ref.17)</td></tr>
</table>

15.6 References

1. Timoshenko, S. P., and J. M. Gere: "Theory of Elastic Stability," 2nd ed., McGraw-Hill, 1961.
2. Schwerin, E.: Die Torsionstabilität des dünnwandigen Rohres, *Z. angew. Math. Mech.*, vol. 5, no. 3, p. 235, 1925.
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