Part

3

Formulas and Examples

Each of the following chapters deals with a certain type of structural member or a certain condition of stress. What may be called the common, or typical, case is usually discussed first; special cases, representing peculiarities of form, proportions, or circumstances of loading, are considered subsequently. In the discussion of each case the underlying assumptions are stated, the general behavior of the loaded member is described, and formulas for the stress and deformation are given. The more important of the general equations are numbered consecutively throughout each section to facilitate reference, but, wherever possible, formulas applying to specific cases are tabulated for convenience and economy of space.

In all formulas which contain numerical constants having dimensions, the units are specified.

Most formulas contain only dimensionless constants and can be evaluated in any consistent system of units.

Chapter

7

Tension, Compression, Shear, and Combined Stress

7.1 Bar under Axial Tension (or Compression); Common Case

The bar is straight, of any uniform cross section, of homogeneous material, and (if under compression) short or constrained against lateral buckling. The loads are applied at the ends, centrally, and in such a manner as to avoid nonuniform stress distribution at any section of the part under consideration. The stress does not exceed the proportional limit.

Behavior. Parallel to the load the bar elongates (under tension) or shortens (under compression), the unit longitudinal strain being ε and the total longitudinal deflection in the length l being δ. At right angles to the load the bar contracts (under tension) or expands (under compression); the unit lateral strain ε' is the same in all transverse directions, and the total lateral deflection δ' in any direction is proportional to the lateral dimension d measured in that direction. Both longitudinal and lateral strains are proportional to the applied load. On any right section there is a uniform tensile (or compressive) stress σ; on any oblique section there is a uniform tensile (or compressive) normal stress σ_θ and a uniform shear stress τ_θ. The deformed bar under tension is represented in Fig. 7.1(a), and the stresses in Fig. 7.1(b).

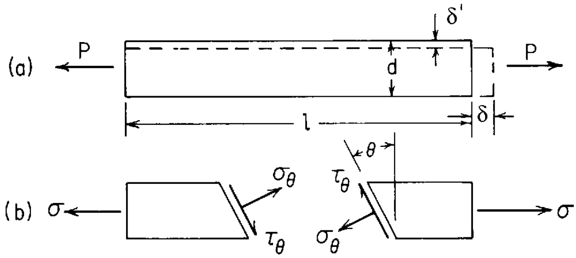

Figure 7.1

Formulas. Let

P = applied load
A = cross-sectional area (before loading)
l = length (before loading)
E = modulus of elasticity
ν = Poisson's ratio

Then

$$\sigma = \frac{P}{A} \tag{7.1-1}$$

$$\sigma_\theta = \frac{P}{A}\cos^2\theta, \qquad \max\ \sigma_\theta = \sigma\ (\text{when } \theta = 0^\circ)$$

$$\tau = \frac{P}{2A}\sin 2\theta, \qquad \max\ \tau_\theta = \frac{1}{2}\sigma(\text{when } \theta = 45 \text{ or } 135^\circ)$$

$$\varepsilon = \frac{\sigma}{E} \tag{7.1-2}$$

$$\delta = l\varepsilon = \frac{Pl}{AE} \tag{7.1-3}$$

$$\varepsilon' = -\nu\varepsilon \tag{7.1-4}$$

$$\delta' = \varepsilon' d \tag{7.1-5}$$

$$\text{Strain energy per unit volume } U = \frac{1}{2}\frac{\sigma^2}{E} \tag{7.1-6}$$

$$\text{Total strain energy } U = \frac{1}{2}\frac{\sigma^2}{E}\,Al = \frac{1}{2}P\delta \tag{7.1-7}$$

For small strain, each unit area of cross section changes by $(-2\nu\varepsilon)$ under load, and each unit of volume changes by $(1 - 2\nu)\varepsilon$ under load.

In some discussions it is convenient to refer to the *stiffness* of a member, which is a measure of the resistance it offers to being deformed. The stiffness of a uniform bar under axial load is shown by Eq. (7.1-3) to be proportional to A and E directly and to l inversely, i.e., proportional AE/l.

EXAMPLE

A cylindrical rod of steel 4 in long and 1.5 in diameter has an axial compressive load of 20,000 lb applied to it. For this steel $\nu = 0.285$ and $E = 30{,}000{,}000$ lb/in^2. Determine (*a*) the unit compressive stress σ; (*b*) the total longitudinal deformation, δ; (*c*) the total transverse deformation δ'; (*d*) the change in volume, ΔV; and (*e*) the total energy, or work done in applying the load.

Solution

$$(a)\quad \sigma = \frac{P}{A} = \frac{4P}{\pi d^2} = \frac{4(-20{,}000)}{\pi(1.5)^2} = -11{,}320 \text{ lb/in}^2$$

$$(b)\quad \varepsilon = \frac{\sigma}{E} = \frac{-11{,}320}{30{,}000{,}000} = -377(10^{-6})$$

$$\delta = \varepsilon l = (-377)(10^{-6})(4) = -1.509(10^{-3}) \text{ in (''}-\text{'' means shortening)}$$

$$(c)\quad \varepsilon' = -\nu\varepsilon = -0.285(-377)(10^{-6}) = 107.5(10^{-6})$$

$$\delta' = \varepsilon' d = (107.5)(10^{-6})(1.5) = 1.613(10^{-4}) \text{ in (''}+\text{'' means expansion)}$$

$$(d)\quad \Delta V/V = (1-2\nu)\varepsilon = [1-2(0.285)](-377)(10^{-6}) = -162.2(10^{-6})$$

$$\Delta V = -162.2(10^{-6})V = -162.2(10^{-6})\frac{\pi}{4}d^2 l = -162.2(10^{-6})\frac{\pi}{4}(1.5)^2(4)$$
$$= -1.147(10^{-3}) \text{ in}^3 \text{ (''}-\text{'' means decrease)}$$

(*e*) Increase in strain energy,

$$U = \frac{1}{2}P\delta = \frac{1}{2}(-20{,}000)(-1.509)(10^{-3}) = 15.09 \text{ in-lb}$$

7.2 Bar under Tension (or Compression); Special Cases

If the bar is not straight, it is subject to bending; formulas for this case are given in Sec. 12.4.

If the load is applied eccentrically, the bar is subject to bending; formulas for this case are given in Secs. 8.7 and 12.4. If the load is compressive and the bar is long and not laterally constrained, it must be analyzed as a column by the methods of Chapters 12 and 15.

If the stress exceeds the proportional limit, the formulas for stress given in Sec. 7.1 still hold but the deformation and work done in producing it can be determined only from experimental data relating unit strain to unit stress.

If the section is not uniform but changes *gradually*, the stress at any section can be found by dividing the load by the area of that section; the total longitudinal deformation over a length l is given by $\int_0^l \frac{P}{AE} dx$ and the strain energy is given by $\int_0^l \frac{1}{2} \frac{P^2}{AE} dx$, where dx is an infinitesimal length in the longitudinal direction. If the change in section is *abrupt* stress concentration may have to be taken into account, values of K_t being used to find elastic stresses and values of K_r being used to predict the breaking load. Stress concentration may also have to be considered if the end attachments for loading involve pinholes, screw threads, or other stress raisers (see Sec. 3.10 and Chap. 17).

If instead of being applied at the ends of a uniform bar the load is applied at an intermediate point, both ends being held, the *method of consistent deformations* shows that the load is apportioned to the two parts of the bar in inverse proportion to their respective lengths.

If a uniform bar is supported at one end in a vertical position and loaded only by its own weight, the maximum stress occurs at the supported end and is equal to the weight divided by the cross-sectional area. The total elongation is *half* as great and the total strain energy *one-third* as great as if a load equal to the weight were applied at the unsupported end. A bar supported at one end and loaded by its own weight and an axial downward load P (force) applied at the unsupported end will have the same unit stress σ (force per unit area) at all sections if it is tapered so that all sections are similar in form but vary in scale according to the formula

$$y = \frac{\sigma}{w} \log_e \frac{A\sigma}{P} \qquad (7.2\text{-}1)$$

where y is the distance from the free end of the bar to any section, A is the area of that section, and w is the density of the material (force per unit volume).

If a bar is stressed by having both ends rigidly held while a change in temperature is imposed, the resulting stress is found by calculating the longitudinal expansion (or contraction) that the change in temperature would produce if the bar were not held and then calculating the load necessary to shorten (or lengthen) it by that amount (principle of superposition). If the bar is uniform, the unit stress produced is independent of the length of the bar if restraint against buckling is provided. If a bar is stressed by being struck an axial blow at one end, the case is one of *impact* loading, discussed in Sec. 16.3.

EXAMPLES

1. Figure 7.2 represents a uniform bar rigidly held at the ends A and D and axially loaded at the intermediate points B and C. It is required to determine

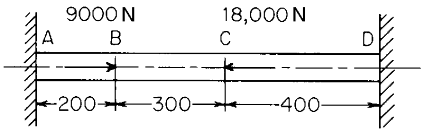

Figure 7.2

the total force in each portion of the bar *AB*, *BC*, *CD*. The loads are in newtons and the lengths in centimeters.

Solution. Each load is divided between the portions of the bar to right and left in inverse proportion to the lengths of these parts (consistent deformations), and the total force sustained by each part is the algebraic sum of the forces imposed by the individual loads (superposition). Of the 9000 N load, therefore, $\frac{7}{9}$, or 7000 N, is carried in tension by segment *AB*, and $\frac{2}{9}$, or 2000 N, is carried in compression by the segment *BD*. Of the 18,000 N load, $\frac{4}{9}$, or 8000 N, is carried in compression by segment *AC*, and $\frac{5}{9}$, or 10,000 N, is carried in tension by segment *CD*. Denoting tension by the plus sign and compression by the minus sign, and adding algebraically, the actual stresses in each segment are found to be

$$\begin{aligned} AB&: \quad 7000 - 8000 = -1000 \text{ N} \\ BC&: \quad -2000 - 8000 = -10{,}000 \text{ N} \\ CD&: \quad -2000 + 10{,}000 = +8000 \text{ N} \end{aligned}$$

The results are quite independent of the diameter of the bar and of *E* provided the bar is completely uniform.

If instead of being *held* at the ends, the bar is prestressed by wedging it between rigid walls under an initial compression of, say, 10,000 N and the loads at *B* and *C* are then applied, the results secured above would represent the *changes* in force the several parts would undergo. The final forces in the bar would therefore be 11,000 N compression in *AB*, 20,000 N compression in *BC*, and 2000 N compression in *CD*. But if the initial compression were less than 8000 N, the bar would break contact with the wall at *D* (no tension possible); there would be no force at all in *CD*, and the forces in *AB* and *BC*, now statically determinate, would be 9000 and 18,000 N compression, respectively.

2. A steel bar 24 in long has the form of a truncated cone, being circular in section with a diameter at one end of 1 in and at the other of 3 in. For this steel, $E = 30{,}000{,}000 \text{ lb/in}^2$ and the coefficient of thermal expansion is $0.0000065/^\circ\text{F}$. This bar is rigidly held at both ends and subjected to a drop in temperature of 50°F. It is required to determine the maximum tensile stress thus caused.

Solution. Using the principle of superposition, the solution is effected in three steps: (*a*) the shortening δ due to the drop in temperature is found, assuming the bar free to contract; (*b*) the force *P* required to produce an elongation equal to δ, that is, to stretch the bar back to its original length, is calculated; (*c*) the maximum tensile stress produced by this force *P* is calculated.

(*a*) $\delta = 50(0.0000065)(24) = 0.00780$ in.

(*b*) Let *d* denote the diameter and *A* the area of any section a distance *x* in from the small end of the bar. Then

$$d = 1 + \frac{x}{12}, \qquad A = \frac{\pi}{4}\left(1 + \frac{x}{12}\right)^2$$

and

$$\delta = \int_0^l \frac{P}{EA}\,dx = \int_0^{24} \frac{4P}{(\pi E)(1 + x/12)^2}\,dx = \frac{4P}{\pi(30)(10^6)}\frac{(-12)}{(1 + x/12)}\Bigg|_0^{24} = 3.395(10^{-7})P$$

Equating this to the thermal contraction of 0.00780 in yields

$$P = 22{,}970 \text{ lb}$$

(*c*) The maximum stress occurs at the smallest section, and is

$$\sigma = \frac{4P}{\pi d_{\min}^2} = \frac{4(22{,}970)}{\pi(1)^2} = 29{,}250 \text{ lb/in}^2$$

The result can be accepted as correct only if the proportional limit of the steel is known to be as great as or greater than the maximum stress and if the concept of a rigid support can be accepted. (See cases 8, 9, and 10 in Table 14.1.)

7.3 Composite Members

A tension or compression member may be made up of parallel elements or parts which jointly carry the applied load. The essential problem is to determine how the load is apportioned among the several parts, and this is easily done by the method of consistent deformations. If the parts are so arranged that all undergo the same total elongation or shortening, then each will carry a portion of the load proportional to its stiffness, i.e., proportional to AE/l if each is a uniform bar and proportional to AE if all these uniform bars are of equal length. It follows that if there are n bars, with section areas $A_1, A_2, \ldots, A_n$, lengths $l_1, l_2, \ldots, l_n$, and moduli E_1, $E_2, \ldots, E_n$, then the loads on the several bars P_1, $P_2, \ldots, P_n$ are given by

$$P_1 = P\frac{\dfrac{A_1E_1}{l_1}}{\dfrac{A_1E_1}{l_1} + \dfrac{A_2E_2}{l_2} + \cdots + \dfrac{A_nE_n}{l_n}} \tag{7.3-1}$$

$$P_2 = P\frac{\dfrac{A_2E_2}{l_2}}{\dfrac{A_1E_1}{l_1} + \dfrac{A_2E_2}{l_2} + \cdots + \dfrac{A_nE_n}{l_n}} \tag{7.3-2}$$

....

A composite member of this kind can be *prestressed*. P_1, P_2, etc., then represent the *increments* of force in each member due to the applied load, and can be found by Eqs. (7.3-1) and (7.3-2), provided all bars can sustain reversal of stress, or provided the applied load is not great enough to cause such reversal in any bar which cannot sustain it. As explained in Sec. 3.12, by proper prestressing, all parts of a composite member can be made to reach their allowable loads, elastic limits, or ultimate strengths simultaneously (Example 2).

EXAMPLES

1. A ring is suspended by three vertical bars, A, B, and C of unequal lengths. The upper ends of the bars are held at different levels, so that as assembled none of the bars is stressed. A is 4 ft long, has a section area of 0.3 in^2, and is of steel for which $E = 30{,}000{,}000$ lb/in^2; B is 3 ft long and has a section area of 0.2 in^2, and is of copper for which $E = 17{,}000{,}000$ lb/in^2; C is 2 ft long, has a section area of 0.4 in^2, and is of aluminum for which $E = 10{,}000{,}000$ lb/in^2. A load of 10,000 lb is hung on the ring. It is required to determine how much of this load is carried by each bar.

Solution. Denoting by P_A, P_B, and P_C the loads carried by A, B, and C, respectively, and expressing the moduli of elasticity in millions of pounds per square inch and the lengths in feet, we substitute in Eq. (7.3-1) and find

$$P_A = 10{,}000\left[\frac{\dfrac{(0.3)(30)}{4}}{\dfrac{(0.3)(30)}{4}+\dfrac{(0.2)(17)}{3}+\dfrac{(0.4)(10)}{2}}\right] = 4180 \text{ lb}$$

Similarly

$$P_B = 2100 \text{ lb} \quad \text{and} \quad P_C = 3720 \text{ lb}$$

2. A composite member is formed by passing a steel rod through an aluminum tube of the same length and fastening the two parts together at both ends. The fastening is accomplished by adjustable nuts, which make it possible to assemble the rod and tube so that one is under initial tension and the other is under an equal initial compression. For the steel rod the section area is 1.5 in^2, the modulus of elasticity 30,000,000 lb/in^2 and the allowable stress 15,000 lb/in^2. For the aluminum tube the section area is 2 in^2, the modulus of elasticity 10,000,000 lb/in^2 and the allowable stress 10,000 lb/in^2. It is desired to prestress the composite member so that under a tensile load both parts will reach their allowable stresses simultaneously.

Solution. When the allowable stresses are reached, the force in the steel rod will be $1.5(15{,}000) = 22{,}500$ lb, the force in the aluminum tube will be $2(10{,}000) = 20{,}000$ lb, and the total load on the member will be $22{,}500 + 20{,}000 \; ; = 42{,}500$ lb. Let P_i denote the initial tension or compression in the members, and, as before, let tension be considered positive and compression negative. Then, since Eq. (7.3-1) gives the *increment* in force, we have for the aluminum tube

$$P_i + 42{,}500 \frac{(2)(10)}{(2)(10) + (1.5)(30)} = 20{,}000$$

or

$$P_i = +6920 \text{ lb} \qquad \text{(initial tension)}$$

For the steel rod, we have

$$P_i + 42{,}500 \frac{(1.5)(30)}{(2)(10) + (1.5)(30)} = 22{,}500$$

or

$$P_i = -6920 \text{ lb} \qquad \text{(initial compression)}$$

If the member were not prestressed, the unit stress in the steel would always be just three times as great as that in the aluminum because it would sustain the same unit deformation and its modulus of elasticity is three times as great. Therefore, when the steel reached its allowable stress of 15,000 lb/in^2, the aluminum would be stressed to only 5000 lb/in^2 and the allowable load on the composite member would be only 32,500 lb instead of 42,500 lb.

7.4 Trusses

A conventional truss is essentially an assemblage of straight uniform bars that are subjected to axial tension or compression when the truss is loaded at the joints. The deflection of any joint of a truss is easily found by the *method of unit loads* (Sec. 4.5). Let p_1, p_2, p_3, etc., denote the forces produced in the several members by an *assumed unit load* acting in the direction x at the joint whose deflection is to be found, and let δ_1, δ_2, δ_3, etc., denote the longitudinal deformations produced in the several members by the *actual applied loads*. The deflection Δ_x in the direction x of the joint in question is given by

$$\Delta_x = p_1\delta_1 + p_2\delta_2 + p_3\delta_3 + \cdots = \sum_{i=1}^{n} p_i\delta_i \tag{7.4-1}$$

The deflection in the direction y, at right angles to x, can be found similarly by assuming the unit load to act in the y direction; the resultant deflection is then determined by combining the x and y deflections. Attention must be given to the *signs* of p and δ, p is positive if a member is subjected to tension and negative if under compression, and δ is positive if it represents an elongation and negative if it represents a shortening. A positive value for $\sum p\delta$ means that the deflection is in the direction of the assumed unit load, and a negative value means that it is in the opposite direction. (This procedure is illustrated in Example 1 below.)

A statically indeterminate truss can be solved by the *method of least work* (Sec. 4.5). To do this, it is necessary to write down the expression for the total strain energy in the structure, which, being simply the sum of the strain energies of the constituent bars, is given by

$$\frac{1}{2}P_1\delta_1 + \frac{1}{2}P_2\delta_2 + \frac{1}{2}P_3\delta_3 + \cdots = \sum_{i=1}^{n}\frac{1}{2}P_i\delta_i = \sum_{i=1}^{n}\frac{1}{2}\left(\frac{P^2 l}{AE}\right)_i \qquad (7.4\text{-}2)$$

Here P_1, P_2, etc., denote the forces in the individual members due to the applied loads and δ has the same meaning as above. It is necessary to express each force P_i as the sum of the two forces; one of these is the force the applied loads would produce with the redundant member removed, and the other is the force due to the unknown force (say, F) exerted by this redundant member on the rest of the structure. The total strain energy is thus expressed as a function of the known applied forces and F, the force in the redundant member. The partial derivative with respect to F of this expression for strain energy is then set equal to zero and solved for F. If there are two or more redundant members, the expression for strain energy with all the redundant forces, F_1, F_2, etc., represented is differentiated once with respect to each. The equations thus obtained are then solved simultaneously for the unknown forces. (The procedure is illustrated in Example 2.)

EXAMPLES

1. The truss shown in Fig. 7.3 is composed of tubular steel members, for which $E = 30{,}000{,}000$ lb/in^2. The section areas of the members are given in the table below. It is required to determine Δ_x and Δ_y, the horizontal and vertical components of the displacement of joint A produced by the indicated loading.

Solution. The method of unit loads is used. The force P in each member due to the applied loads is found, and the resulting elongation or shortening δ is calculated. The force p_x in each member due to a load of 1 lb acting to the right at A, and the force p_y in each member due to a load of 1 lb acting down at A are calculated. By Eq. (7.4-1), $\sum p_x\delta$, then gives the horizontal and $\sum p_y\delta$ gives the vertical displacement or deflection of A. Tensile forces and elongations are

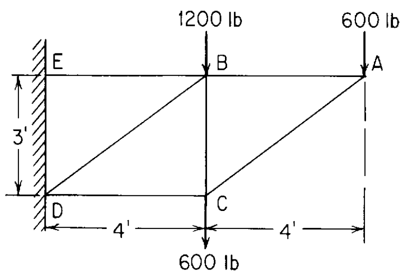

Figure 7.3

denoted by +, compressive forces and shortenings by −. The work is conveniently tabulated as follows:

Member	Area, A_i, in^2	Length, l_i, in	P_i, lb	$\delta_i = \left(\frac{Pl}{AE}\right)_i$, in	$(p_x)_i$	$(p_x\delta)_i$, in (a)	$(p_y)_i$	$(p_y\delta)_i$, in (b)
(1) *AB*	0.07862	48	800	0.01628	1.000	0.01628	1.333	0.02171
(2) *AC*	0.07862	60	−1000	−0.02544	0	0	−1.667	0.04240
(3) *BC*	0.1464	36	1200	0.00984	0	0	1.000	0.00984
(4) *BE*	0.4142	48	4000	0.01545	1.000	0.01545	2.667	0.04120
(5) *BD*	0.3318	60	−4000	−0.02411	0	0	−1.667	0.04018
(6) *CD*	0.07862	48	−800	−0.01628	0	0	−1.333	0.02171
						$\Delta_x = 0.03173$ in		$\Delta_y = 0.17704$ in

Δ_x and Δ_y are both found to be positive, which means that the displacements are in the directions of the assumed unit loads—to the right and down. Had either been found to be negative, it would have meant that the displacement was in a direction opposite to that of the corresponding unit load.

2. Assume a diagonal member, running from *A* to *D* and having a section area 0.3318 in^2 and length 8.544 ft, is to be added to the truss of Example 1; the structure is now statically indeterminate. It is required to determine the force in each member of the altered truss due to the loads shown.

Solution. We use the method of least work. The truss has one redundant member; any member except *BE* may be regarded as redundant, since if any one were removed, the remaining structure would be stable and statically determinate. We select *AD* to be regarded as redundant, denote the unknown force in *AD* by *F*, and assume *F* to be tension. We find the force in each member assuming *AD* to be removed, then find the force in each member due to a pull *F* exerted at *A* by *AD*, and then add these forces, thus getting an expression for the force in each member of the actual truss in terms of *F*. The expression for the strain energy can then be written out, differentiated with respect to *F*, equated to zero, and solved for *F*. *F* being known, the force in each member of the truss is easily found. The computations are conveniently tabulated as follows:

	Forces in members†			
Member	Due to applied loads without AD (a) (lb)	Due to pull, *F*, exerted by *AD* (b)	Total forces, P_i. Superposition of (a) and (b) (c)	Actual total values with $F = -1050$ lb in (c) (d) (lb)
(1) *AB*	800	−0.470 *F*	800 − 0.470 *F*	1290
(2) *AC*	−1000	−0.584 *F*	−1000 − 0.584 *F*	−390
(3) BC	1200	0.351 *F*	1200 + 0.351 *F*	830
(4) *BE*	4000	0	4000	4000
(5) *BD*	−4000	−0.584 *F*	−4000 − 0.584 *F*	−3390
(6) *CD*	−800	−0.470 *F*	−800 − 0.470 *F*	−306
(7) *AD*	0	*F*	*F*	−1050

† + for tension and − for compression.

$$U = \sum_{i=1}^{7} \frac{1}{2}\left(\frac{P^2 l}{AE}\right)_i = \frac{1}{2E}\left[\frac{(800-0.470F)^2(48)}{0.07862} + \frac{(-1000-0.584F)^2(60)}{0.07862} + \frac{(1200+0.351F)^2(36)}{0.1464} + \frac{(4000)^2(48)}{0.4142} + \frac{(-4000-0.584F)^2(60)}{0.3318} + \frac{(-800-0.470F)^2(48)}{0.07862} + \frac{F^2(102.5)}{0.3318}\right]$$

Setting the partial derivative of U relative to F to zero,

$$\frac{\partial U}{\partial F} = \frac{1}{2E}\left[\frac{2(800-0.470F)(-0.470)(48)}{0.07862} + \frac{2(-1000-0.584F)(-0.584)(60)}{0.07862} + \cdots\right] = 0$$

and solving for F gives $F = -1050$ lb.

The negative sign here simply means that AD is in compression. A positive value of F would have indicated tension. Substituting the value of F into the terms of column (c) yield the actual total forces in each member as tabulated in column (d).

7.5 Body under Pure Shear Stress

A condition of pure shear may be produced by any one of the methods of loading shown in Fig. 7.4. In Fig. 7.4(a), a rectangular block of length a, height b, and uniform thickness t is shown loaded by forces P_1 and P_2, uniformly distributed over the surfaces to which they are applied and satisfying the equilibrium equation $P_1 b = P_2 a$. There are equal shear stresses on all vertical and horizontal planes, so that any contained cube oriented like $ABCD$ has on each of four faces the shear stress $\tau = P_1/at = P_2/bt$ and no other stress.

In Fig. 7.4(b) a rectangular block is shown under equal and opposite biaxial stresses σ_t and σ_c. There are equal shear stresses on all planes inclined at 45° to the top and bottom faces, so that a contained cube

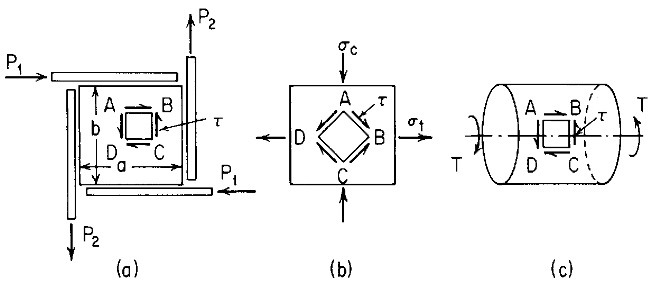

Figure 7.4

oriented like *ABCD* has on each of four faces the shear stress $\tau = \sigma_t = \sigma_c$ and no other stress.

In Fig. 7.4(*c*), a circular shaft is shown under a twisting moment T; a cube of infinitesimal dimensions, a distance z from the axis and oriented like *ABCD* has on each of four faces an essentially uniform shear stress $\tau = Tz/J$ (Sec. 10.1) and no other stress.

In whatever way the loading is accomplished, the result is to impose on an elementary cube of the loaded body the condition of stress represented in Fig. 7.5, that is, shearing stress alone on each of four faces, these stresses being equal and so directed as to satisfy the equilibrium condition $T_x = 0$ (Sec. 4.1).

The stresses, σ_θ and τ_θ on a transformed surface rotated counterclockwise through the angle θ can be determined from the transformation equations given by Eqs. (2.3-17). They are given by

$$\sigma_\theta = \tau \sin 2\theta, \qquad \tau_\theta = \tau \cos 2\theta \tag{7.5-1}$$

where $(\sigma_\theta)_{\max,\min} = \pm\tau$ at $\theta = \pm 45^\circ$.

The strains produced by pure shear are shown in Fig. 7.5(*b*), where the cube *ABCD* is deformed into a rhombohedron *A′B′C′D′*. The unit shear strain, γ, referred to as the *engineering shear strain*, is reduction of angles $\angle ABC$ and $\angle ADC$, and the increase in angles $\angle DAB$ and $\angle BCD$ in radians. Letting G denote the modulus of rigidity, the shear strain is related to the shear stress as

$$\gamma = \frac{\tau}{G} \tag{7.5-2}$$

Assuming a linear material, the strain energy per unit volume for pure shear, u_s, within the elastic range is given by

$$u_s = \frac{1}{2}\frac{\tau^2}{G} \tag{7.5-3}$$

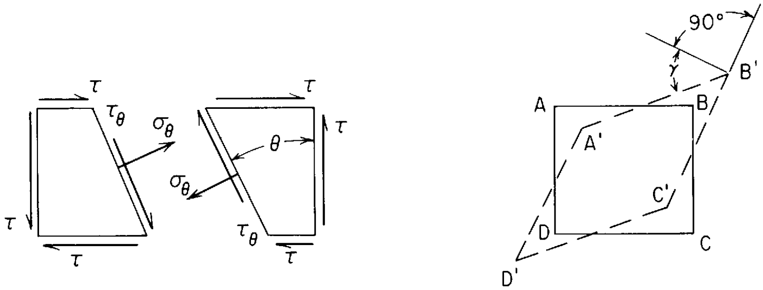

Figure 7.5 (*a*) Shear stress and transformation. (*b*) Shear strain.

The relations between τ, σ, and the strains represented in Fig. 7.5(b) make it possible to express G in terms of E and Poisson's ratio, ν, for a linear, homogeneous, isotropic material. The relationship is

$$G = \frac{E}{2(1+\nu)} \tag{7.5-4}$$

From known values of E (determined by a tensile test) and G (determined by a torsion test) it is thus possible to calculate ν.

7.6 Cases of Direct Shear Loading

By *direct shear loading* is meant any case in which a member is acted on by equal, parallel, and opposite forces so nearly colinear that the material between them is subjected primarily to shear stress, with negligible bending. Examples of this are provided by rivets, bolts, and pins, shaft splines and keys, screw threads, short lugs, etc. These are not really cases of pure shear; the actual stress distribution is complex and usually indeterminate because of the influence of fit and other factors. In designing such parts, however, it is usually assumed that the shear is uniformly distributed on the critical section, and since working stresses are selected with due allowance for the approximate nature of this assumption, the practice is usually permissible. In *beams* subject to transverse shear, this assumption cannot be made as a rule.

Shear and other stresses in rivets, pins, keys, etc., are discussed more fully in Chap. 14, shear stresses in beams in Chap. 8, and shear stresses in torsion members in Chap. 10.

7.7 Combined Stress

Under certain circumstances of loading, a body is subjected to a combination of tensile and compressive stresses (usually designated as *biaxial* or *triaxial stress*) or to a combination of tensile, compressive, and shear stresses (usually designated as *combined stress*). For example, the material at the inner surface of a thick cylindrical pressure vessel is subjected to triaxial stress (radial compression, longitudinal tension, and circumferential tension), and a shaft simultaneously bent and twisted is subjected to combined stress (longitudinal tension or compression, and torsional shear).

In most instances the normal and shear stresses on each of three mutually perpendicular planes are due to flexure, axial loading, torsion, beam shear, or some combination of these which separately can be calculated readily by the appropriate formulas. Normal stresses arising from different load conditions acting on the same plane can be combined simply by algebraic addition considering tensile stresses positive and compressive stresses negative. Similarly, shear stresses can be combined by algebraic addition following a consistent sign convention. Further analysis of the combined states of normal and shear stresses must be performed using the transformation techniques outlined in Sec. 2.3. The principal stresses, the maximum shear stress, and the normal and shear stresses on any given plane can be found by the equations given in Sec. 2.3.

The strains produced by any combination of stresses not exceeding the proportional limit can also be found using Hooke's law for each stress and then combined by superposition. Consideration of the strains caused by equal triaxial stresses leads to an expression for the bulk modulus of elasticity given by

$$K = \frac{E}{3(1-2\nu)} \tag{7.7-1}$$

EXAMPLES

1. A rectangular block 12 in long, 4 in high, and 2 in thick is subjected to a longitudinal tensile stress $\sigma_x = 12{,}000$ lb/in^2, a vertical compressive stress $\sigma_y = 15{,}000$ lb/in^2, and a lateral compressive stress $\sigma_z = 9000$ lb/in^2. The material is steel, for which $E = 30{,}000{,}000$ lb/in^2 and $\nu = 0.30$. It is required to find the total change in length.

Solution. The longitudinal deformation is found by superposition: The unit strain due to each stress is computed separately by Eqs. (7.1-2) and (7.1-4); these results are added to give the resultant longitudinal unit strain, which is multiplied by the length to give the total elongation. Denoting unit longitudinal strain by ε_x and total longitudinal deflection by δ_x, we have

$$\begin{aligned}\varepsilon_x &= \frac{12{,}000}{E} - \nu\frac{-15{,}000}{E} - \nu\frac{-9000}{E}\\ &= 0.000400 + 0.000150 + 0.000090 = +0.00064\\ \delta_x &= 12(0.00064) = 0.00768 \text{ in}\end{aligned}$$

The lateral dimensions have nothing to do with the result since the lateral stresses, not the lateral loads, are given.

2. A piece of "standard extra-strong" pipe, 2 in nominal diameter, is simultaneously subjected to an internal pressure of $p = 2000$ lb/in^2 and to a twisting moment of $T = 5000$ in-lb caused by tightening a cap screwed on at one end. Determine the maximum tensile stress and the maximum shear stress thus produced in the pipe.

Solution. The calculations will be made, first, for a point at the outer surface and, second, for a point at the inner surface. The dimensions of the pipe and properties of the cross section are as follows: inner radius $r_i = 0.9695$ in, outer radius $r_o = 1.1875$ in, cross-sectional area of bore $A_b = 2.955$ in^2, cross-sectional area of pipe wall $A_w = 1.475$ in^2, and polar moment of inertial $J = 1.735$ in^4.

We take axis x along the axis of the pipe, axis y tangent to the cross section, and axis z radial in the plane of the cross section. For a point at the outer surface of the pipe, σ_x is the longitudinal tensile stress due to pressure and σ_y is the circumferential (hoop) stress due to pressure, the radial stress $\sigma_z = 0$ (since the pressure is zero on the outer surface of the pipe), and τ_{xy} is the shear stress due to torsion. Equation (7.1-1) can be used for σ_x, where $P = pA_b$ and $A = A_w$. To calculate σ_y, we use the formula for stress in thick cylinders (Table 13.5, case 1b). Finally, for τ_{xy}, we use the formula for torsional stress (Eq. (10.1-2). Thus,

$$\sigma_x = \frac{pA_b}{A_w} = \frac{(2000)(2.955)}{1.475} = 4007 \text{ lb/in}^2$$

$$\sigma_y = p\frac{r_i^2(r_o^2 + r_o^2)}{r_o^2(r_o^2 - r_i^2)} = 2000\frac{(0.9695^2)(1.1875^2 + 1.1875^2)}{(1.1875^2)(1.1875^2 - 0.9695^2)} = 7996 \text{ lb/in}^2$$

$$\tau_{xy} = \frac{Tr_o}{J} = \frac{(5000)(1.1875)}{1.735} = 3422 \text{ lb/in}^2$$

This is a case of plane stress where Eq. (2.3-23) applies. The principal stresses are thus

$$\sigma_p = \tfrac{1}{2}[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}]$$
$$= \tfrac{1}{2}[(4007 + 7996) \pm \sqrt{(4007 - 7996)^2 + 4(3422^2)}] = 9962, \quad 2041 \text{ lb/in}^2$$

Thus, $\sigma_{\max} = 9962 \text{ lb/in}^2$.

In order to determine the maximum shear stress, we order the *three* principal stresses such that $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. For plane stress, the out-of-plane principal stresses are zero. Thus, $\sigma_1 = 9962 \text{ lb/in}^2$, $\sigma_2 = 2041 \text{ lb/in}^2$, and $\sigma_3 = 0$. From Eq. (2.3-25), the maximum shear stress is

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}(9962 - 0) = 4981 \text{ lb/in}^2$$

For a point on the inner surface, the stress conditions are three-dimensional since a radial stress due to the internal pressure is present. The longitudinal stress is the same; however, the circumferential stress and torsional shear stress change. For the inner surface,

$$\sigma_x = 4007 \text{ lb/in}^2$$

$$\sigma_y = p\frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 2000\frac{1.1875^2 + 0.9695^2}{1.1875^2 - 0.9695^2} = 9996 \text{ lb/in}^2$$

$$\sigma_z = -p = -2000 \text{ lb/in}^2$$

$$\tau_{xy} = \frac{Tr_i}{J} = \frac{(5000)(0.9695)}{1.735} = 2794 \text{ lb/in}^2$$

$$\tau_{yz} = \tau_{zx} = 0$$

The principal stresses are found using Eq. (2.3-20):†

$$\sigma_p^3 - (4007 + 9996 - 2000)\sigma_p^2 + [(4007)(9996) + (9996)(-2000) + (-2000)(4007) - 2794^2 - 0 - 0]\sigma_p - [(4007)(9996)(-2000) + 2(2794)(0)(0) - (4007)(0^2) - (9996)(0^2) - (-2000)(2794^2)] = 0$$

or

$$\sigma_p^3 - 12.003(10^3)\sigma_p^2 + 4.2415(10^6)\sigma_p + 64.495(10^9) = 0$$

Solving this gives $\sigma_p = 11{,}100$, 2906, and -2000 lb/in^2, which are the principal stresses σ_1, σ_2, and σ_3, respectively. Obviously, the maximum tensile stress is 11,100 lb/in^2. Again, the maximum shear stress comes from Eq. (2.3-25), and is $\frac{1}{2}[11{,}100 - (-2000)] = 6550$ lb/in^2.

Note that for this problem, if the pipe is a ductile material, and one were looking at failure due to shear stress (see Sec. 3.7), the stress conditions for the pipe are more severe at the inner surface compared with the outer surface.

† *Note:* Since $\tau_{yz} = \tau_{zx} = 0$, σ_z is one of the principal stresses and the other two can be found from the plane stress equations. Consequently, the other two principal stresses are in the xy plane.