Chapter

9

Curved Beams

9.1 Bending in the Plane of the Curve

In a straight beam having either a constant cross section or a cross section which changes gradually along the length of the beam, the neutral surface is defined as the longitudinal surface of zero fiber stress when the member is subjected to pure bending. It contains the neutral axis of every section, and these neutral axes pass through the centroids of the respective sections. In this section on bending in the plane of the curve, the use of the many formulas is restricted to those members for which that axis passing through the centroid of a given section and directed normal to the plane of bending of the member is a principal axis. The one exception to this requirement is for a condition equivalent to the beam being constrained to remain in its original plane of curvature such as by frictionless external guides.

To determine the stresses and deformations in curved beams satisfying the restrictions given above, one first identifies several cross sections and then locates the centroids of each. From these centroidal locations the curved centroidal surface can be defined. For bending in the plane of the curve there will be at each section (1) a force N normal to the cross section and taken to act through the centroid, (2) a shear force V parallel to the cross section in a radial direction, and (3) a bending couple M in the plane of the curve. In addition there will be radial stresses σ_r in the curved beam to establish equilibrium. These internal loadings are shown in Fig. 9.1(a), and the stresses and deformations due to each will be evaluated.

Circumferential normal stresses due to pure bending. When a curved beam is bent in the plane of initial curvature, plane sections remain plane, but because of the different lengths of fibers on the inner and outer portions of the beam, the distribution of unit strain, and therefore stress, is not linear. The neutral axis does not pass through the

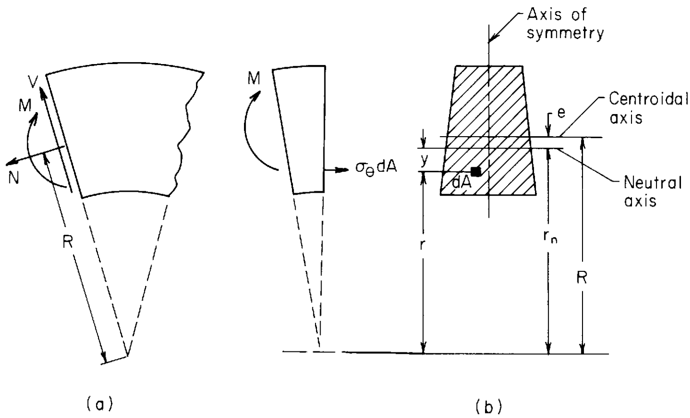


Figure 9.1

centroid of the section and Eqs. (8.1-1) and (8.1-2) do not apply. The error involved in their use is slight as long as the radius of curvature is more than about eight times the depth of the beam. At that curvature the errors in the maximum stresses are in the range of 4 to 5%. The errors created by using the straight-beam formulas become large for sharp curvatures as shown in Table 9.1, which gives formulas and selected numerical data for curved beams of several cross sections and for varying degrees of curvature. In part the formulas and tabulated coefficients are taken from the University of Illinois Circular by Wilson and Quereau (Ref. 1) with modifications suggested by Neugebauer (Ref. 28). For cross sections not included in Table 9.1 and for determining circumferential stresses at locations other than the extreme fibers, one can find formulas in texts on advanced mechanics of materials, for example, Refs. 29 and 36.

The circumferential normal stress σ_θ is given as

$$\sigma_\theta = \frac{My}{Aer} \tag{9.1-1}$$

where M is the applied bending moment, A is the area of the cross section, e is the distance from the centroidal axis to the neutral axis, and y and r locate the radial position of the desired stress from the neutral axis and the center of the curvature, respectively. See Fig. 9.1(b).

$$e = R - r_n = R - \frac{A}{\displaystyle\int_{\text{area}} dA/r} \quad \text{for } \frac{R}{d} < 8 \tag{9.1-2}$$

Equations (9.1-1) and (9.1-2) are based on derivations that neglect the contribution of radial normal stress to the circumferential strain. This assumption does not cause appreciable error for curved beams of compact cross section for which the radial normal stresses are small, and it leads to acceptable answers for beams having thin webs where, although the radial stresses are higher, they occur in regions of the cross section where the circumferential bending stresses are small. The use of the equations in Table 9.1 and of Eqs. (9.1-1) and (9.1-2) is limited to values of $R/d > 0.6$ where, for a rectangular cross section, a comparison of this mechanics-of-materials solution [Eq. (9.1-1)] to the solution using the theory of elasticity shows the mechanics of materials solution to indicate stresses approximately 10% too large.

While in theory the curved-beam formula for circumferential bending stress, Eq. (9.1-1), could be used for beams of very large radii of curvature, one should not use the expression for e from Eq. (9.1-2) for cases where R/d, the ratio of the radius of the curvature R to the depth of the cross section, exceeds 8. The calculation for e would have to be done with careful attention to precision on a computer or calculator to get an accurate answer. Instead one should use the following approximate expression for e which becomes very accurate for large values of R/d. See Ref. 29.

$$e \approx \frac{I_c}{RA} \quad \text{for } \frac{R}{d} > 8 \tag{9.1-3}$$

where I_c is the area moment of inertia of the cross section about the centroidal axis. Using this expression for e and letting R approach infinity leads to the usual straight-beam formula for bending stress.

For complex sections where the table or Eq. (9.1-3) are inappropriate, a numerical technique that provides excellent accuracy can be employed. This technique is illustrated on pp. 318–321 of Ref. 36.

In summary, use Eq. (9.1-1) with e from Eq. (9.1-2) for $0.6 < R/d < 8$. Use Eq. (9.1-1) with e from Eq. (9.1-3) for those curved beams for which $R/d > 8$ and where errors of less than 4 to 5% are desired, or use straight-beam formulas if larger errors are acceptable or if $R/d \gg 8$.

Circumferential normal stresses due to hoop tension $N(M = 0)$. The normal force N was chosen to act through the centroid of the cross section, so a constant normal stress N/A would satisfy equilibrium. Solutions carried out for rectangular cross sections using the theory of elasticity show essentially a constant normal stress with higher values on a thin layer of material on the inside of the curved section and lower values on a thin layer of material on the outside of the section. In most engineering applications the stresses due to the moment M are much larger than those due to N, so the assumption of uniform stress due to N is reasonable.

Shear stress due to the radial shear force *V*. Although Eq. (8.1-2) does not apply to curved beams, Eq. (8.1-13), used as for a straight beam, gives the *maximum* shear stress with sufficient accuracy in most instances. Again an analysis for a rectangular cross section carried out using the theory of elasticity shows that the peak shear stress in a curved beam occurs not at the centroidal axis as it does for a straight beam but toward the inside surface of the beam. For a very sharply curved beam, $R/d = 0.7$, the peak shear stress was $2.04V/A$ at a position one-third of the way from the inner surface to the centroid. For a sharply curved beam, $R/d = 1.5$, the peak shear stress was $1.56V/A$ at a position 80% of the way from the inner surface to the centroid. These values can be compared to a peak shear stress of $1.5V/A$ at the centroid for a straight beam of rectangular cross section.

If a mechanics-of-materials solution for the shear stress in a curved beam is desired, the element in Fig. 9.2(*b*) can be used and moments taken about the center of curvature. Using the normal stress distribution $\sigma_\theta = N/A + My/AeR$, one can find the shear stress expression to be

$$\tau_{r\theta} = \frac{V(R-e)}{t_r A e r^2}(RA_r - Q_r) \tag{9.1-4}$$

where t_r is the thickness of the section normal to the plane of curvature at the radial position r and

$$A_r = \int_b^r dA_1 \quad \text{and} \quad Q_r = \int_b^r r_1\, dA_1 \tag{9.1-5}$$

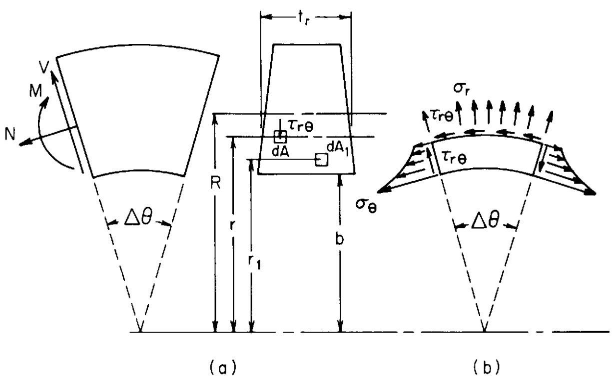


Figure 9.2

Equation (9.1-4) gives conservative answers for the peak values of shear stress in rectangular sections when compared to elasticity solutions. The locations of peak shear stress are the same in both analyses, and the error in magnitude is about 1%.

Radial stresses due to moment *M* and normal force *N*. Owing to the radial components of the fiber stresses, radial stresses are present in a curved beam; these are tensile when the bending moment tends to straighten the beam and compressive under the reverse condition. A mechanics-of-materials solution may be developed by summing radial forces and summing forces perpendicular to the radius using the element in Fig. 9.2.

$$\sigma_r = \frac{R-e}{t_r A e r}\left[(M - NR)\left(\int_b^r \frac{dA_1}{r_1} - \frac{A_r}{R-e}\right) + \frac{N}{r}(RA_r - Q_r)\right] \qquad (9.1\text{-}6)$$

Equation (9.1-6) is as accurate for radial stress as is Eq. (9.1-4) for shear stress when used for a rectangular cross section and compared to an elasticity solution. However, the complexity of Eq. (9.1-6) coupled with the fact that the stresses due to N are generally smaller than those due to M leads to the usual practice of omitting the terms involving N. This leads to the equation for radial stress found in many texts, such as Refs. 29 and 36.

$$\sigma_r = \frac{R-e}{t_r A e r} M\left(\int_b^r \frac{dA_1}{r_1} - \frac{A_r}{R-e}\right) \qquad (9.1\text{-}7)$$

Again care must be taken when using Eqs. (9.1-4), (9.1-6), and (9.1-7) to use an accurate value for e as explained above in the discussion following Eq. (9.1-3).

Radial stress is usually not a major consideration in compact sections for it is smaller than the circumferential stress and is low where the circumferential stresses are large. However, in flanged sections with thin webs the radial stress may be large at the junction of the flange and web, and the circumferential stress is also large at this position. This can lead to excessive shear stress and the possible yielding if the radial and circumferential stresses are of opposite sign. A large compressive radial stress in a thin web may also lead to a buckling of the web. Corrections for curved-beam formulas for sections having thin flanges are discussed in the next paragraph but corrections are also needed if a section has a *thin* web and *very thick* flanges. Under these conditions the individual flanges tend to rotate about their own neutral axes and larger radial and shear stresses are developed. Broughton et al. discuss this configuration in Ref. 31.

EXAMPLES

1. The sharply curved beam with an elliptical cross section shown in Fig. 9.3(*a*) has been used in a machine and has carried satisfactorily a bending moment of $2(10^6)$ N-mm. All dimensions on the figures and in the calculations are given in millimeters. A redesign does not provide as much space for this part, and a decision has been made to salvage the existing stock of this part by machining 10 mm from the inside. The question has been asked as to what maximum moment the modified part can carry without exceeding the peak stress in the original installation.

Solution. First compute the maximum stress in the original section by using case 6 of Table 9.1. $R = 100, c = 50, R/c = 2, A = \pi(50)(20) = 3142$, $e/c = 0.5[2 - (2^2 - 1)^{1/2}] = 0.1340$, $e = 6.70$, and $r_n = 100 - 6.7 = 93.3$. Using these values the stress σ_i can be found as

$$\sigma_i = \frac{My}{Aer} = \frac{2(10^6)(93.3 - 50)}{3142(6.7)(50)} = 82.3\,\text{N/mm}^2$$

Alternatively one can find σ_i from $\sigma_i = k_i Mc/I_x$, where k_i is found to be 1.616 in the table of values from case 6

$$\sigma_i = \frac{(1.616)(2)(10^6)(50)}{\pi(20)(50)^3/4} = 82.3\,\text{N/mm}^2$$

Next consider the same section with 10 mm machined from the inner edge as shown in Fig. 9.3(*b*). Use case 9 of Table 9.1 with the initial calculations based on the equivalent modified circular section shown in Fig. 9.3(*c*). For this configuration $\alpha = \cos^{-1}(-40/50) = 2.498$ rad (143.1°), $\sin\alpha = 0.6$, $\cos\alpha = -0.8$, $R_x = 100$, $a = 50$, $a/c = 1.179$, $c = 42.418$, $R = 102.418$, and $R/c = 2.415$. In this problem $R_x > a$, so by using the appropriate expression from case 9 one obtains $e/c = 0.131$ and $e = 5.548$. R, c, and e have the same values for the machined ellipse, Fig. 9.3(*b*), and from case 18 of Table A.1 the area is found to be $A = 20(50)(\alpha - \sin\alpha\cos\alpha) = 2978$. Now the maximum stress on the inner surface can be found and set equal to $82.3\,\text{N/mm}^2$.

$$\sigma_i = 82.3 = \frac{My}{Aer} = \frac{M(102.42 - 5.548 - 60)}{2978(5.548)(60)}$$

$$82.3 = 37.19(10^6)M, \quad M = 2.21(10^6)\,\text{N-mm}$$

One might not expect this increase in M unless consideration is given to the machining away of a stress concentration. Be careful, however, to note that,

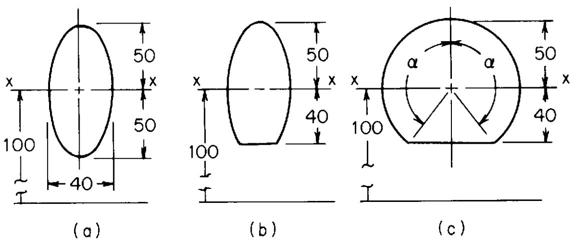


Figure 9.3

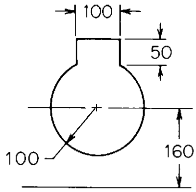


Figure 9.4

although removing the material reduced the peak stress in this case, the part will undergo greater deflections under the same moment that it carried before.

2. A curved beam with a cross section shown in Fig. 9.4 is subjected to a bending moment of 10^7 N-mm in the plane of the curve and to a normal load of 80,000 N in tension. The center of the circular portion of the cross section has a radius of curvature of 160 mm. All dimensions are given and used in the formulas in millimeters. The circumferential stresses in the inner and outer fibers are desired.

Solution. This section can be modeled as a summation of three sections: (1) a solid circular section, (2) a negative (materials removed) segment of a circle, and (3) a solid rectangular section. The section properties are evaluated in the order listed above and the results summed for the composite section.

Section 1. Use case 6 of Table 9.1. $R = 160$, $b = 200$, $c = 100$, $R/c = 1.6$, $\int dA/r = 200[1.6 - (1.6^2 - 1)^{1/2}] = 220.54$, and $A = \pi(100^2) = 31{,}416$.

Section 2. Use case 9 of Table 9.1. $\alpha = \pi/6 \quad (30^\circ)$, $R_x = 160$, $a = 100$, $R_x/a = 1.6$, $a/c = 18.55$, $c = 5.391$, $R = 252.0$, $\int dA/r = 3.595$, and from case 20 of Table A.1, $A = 905.9$.

Section 3. Use case 1 of Table 9.1. $R = 160 + 100\cos 30^\circ + 25 = 271.6$, $b = 100$, $c = 25$, $R/c = 10.864$, $A = 5000$, $\int dA/r = 100\ \ln(11.864/9.864) = 18.462$.

For the composite section; $A = 31{,}416 - 905.9 + 5000 = 35{,}510$, $R = [31{,}416(160) - 905.9(252) + 5000(272.6)]/35{,}510 = 173.37$, $c = 113.37$, $\int dA/r = 220.54 - 3.595 + 18.462 = 235.4$, $r_n = A/(\int dA/r) = 35{,}510/235.4 = 150.85$, $e = R - r_n = 22.52$.

Using these data the stresses on the inside and outside are found to be

$$\sigma_i = \frac{My}{Aer} + \frac{N}{A} = \frac{10^7(150.85 - 60)}{35{,}510(22.52)(60)} + \frac{80{,}000}{35{,}510}$$

$$= 18.93 + 2.25 = 21.18\,\text{N/mm}^2$$

$$\sigma_o = \frac{10^7(150.85 - 296.6)}{35{,}510(22.52)(296.6)} + \frac{80{,}000}{35{,}510}$$

$$= -6.14 + 2.25 = -3.89\,\text{N/mm}^2$$

Curved beams with wide flanges. In reinforcing rings for large pipes, airplane fuselages, and ship hulls, the combination of a curved sheet and attached web or stiffener forms a curved beam with wide flanges.

Formulas for the effective width of a flange in such a curved beam are given in Ref. 9 and are as follows.

When the flange is indefinitely wide (e.g., the inner flange of a pipe-stiffener ring), the effective width is

$$b' = 1.56\sqrt{Rt}$$

where b' is the total width assumed effective, R is the mean radius of curvature of the flange, and t is the thickness of the flange.

When the flange has a definite unsupported width b (gross width less web thickness), the ratio of effective to actual width b'/b is a function of qb, where

$$q = \sqrt[4]{\frac{3(1-\nu^2)}{R^2t^2}}$$

Corresponding values of qb and b'/b are as follows:

qb	1	2	3	4	5	6	7	8	9	10	11
b'/b	0.980	0.850	0.610	0.470	0.380	0.328	0.273	0.244	0.217	0.200	0.182

For the curved beam each flange should be considered as replaced by one of corresponding effective width b', and all calculations for direct, bending, and shear stresses, including corrections for curvature, should be based on this transformed section.

Bleich (Ref. 10) has shown that under a *straightening* moment where the curvature is decreased, the radial components of the fiber stresses in the flanges bend both flanges radially away from the web, thus producing tension in the fillet between flange and web in a direction normal to both the circumferential and radial normal stresses discussed in the previous section. Similarly, a moment which increases the curvature causes both flanges to bend radially toward the web and produce compressive stresses in the fillet between flange and web. The *nominal* values of these transverse bending stresses σ' in the fillet, without any correction for the stress concentration at the fillet, are given by $|\sigma'| = |\beta\sigma_m|$, where σ_m is the circumferential bending stress at the *midthickness* of the flange. This is less than the maximum value found in Table 9.1 and can be calculated by using Eq. (9.1-1). See the first example problem. The value of the coefficient β depends upon the ratio c^2/Rt, where c is the actual unsupported projecting width of the flange to either side of the web and R and t have the same meaning they did in the expressions for b' and q. Values of β may be found from the following table; they were taken from Ref. 10, where values of b' are also tabulated.

$c^2/Rt = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.8
$\beta = 0$	0.297	0.580	0.836	1.056	1.238	1.382	1.577
$c^2/Rt = 1$	1.2	1.4	1.5	2	3	4	5
$\beta = 1.677$	1.721	1.732	1.732	1.707	1.671	1.680	1.700

Derivations of expressions for b'/b and for β are also found in Ref. 29. Small differences in the values given in various references are due to compensations for secondary effects. The values given here are conservative.

In a similar way, the radial components of the circumferential normal stresses distort thin tubular cross sections of curved beams. This distortion affects both the stresses and deformations and is discussed in the next section.

U-shaped members. A U-shaped member having a semicircular inner boundary and a rectangular outer boundary is sometimes used as a punch or riveter frame. Such a member can usually be analyzed as a curved beam having a concentric outer boundary, but when the back thickness is large, a more accurate analysis may be necessary. In Ref. 11 are presented the results of a photoelastic stress analysis of such members in which the effects of variations in the several dimensions were determined. See case 23, Table 17.1

Deflections. If a sharply curved beam is only a small portion of a larger structure, the contribution to deflection made by the curved portion can best be calculated by using the stresses at the inner and outer surfaces to calculate strains and the strains then used to determine the rotations of the plane sections. If the structure is made up primarily of a sharply curved beam or a combination of such beams, then refer to the next section.

9.2 Deflection of Curved Beams

Deflections of curved beams can generally be found most easily by applying an energy method such as Castigliano's second theorem. One such expression is given by Eq. (8.1-7). The proper expression to use for the complementary energy depends upon the degree of curvature in the beam.

Deflection of curved beams of large radius. If for a curved beam the radius of curvature is large enough such that Eqs. (8.1-1) and (8.1-2) are acceptable, i.e., the radius of curvature is greater than 10 times the depth, then the stress distribution across the depth of the beam is very nearly linear and the complementary energy of flexure is given with sufficient accuracy by Eq. (8.1-3). If, in addition, the angular span is great enough such that deformations due to axial stress from the normal force N and the shear stresses due to transverse shear V can be neglected, deflections can be obtained by applying Eqs. (8.1-3) and (8.1-7) and rotations by Eq. (8.1-8). The following example shows how this is done.

EXAMPLE

Figure 9.5 represents a slender uniform bar curved to form the quadrant of a circle; it is fixed at the lower end and at the upper end is loaded by a vertical force V, a horizontal force H, and a couple M_0. It is desired to find the vertical deflection δ_y, the horizontal deflection δ_x, and the rotation θ of the upper end.

Solution. According to Castigliano's second theorem, $\delta_y = \partial U/\partial V$, $\delta_x = \partial U/\partial H$, and $\theta = \partial U/\partial M_0$. Denoting the angular position of any section by x, it is evident that the moment there is $M = VR\sin x + HR(1 - \cos x) + M_0$. Disregarding shear and axial stress, and replacing ds by $R\,dx$, we have [Eq. (8.1-3)]

$$U = U_f = \int_0^{\pi/2} \frac{[VR\sin x + HR(1-\cos x) + M_0]^2 R\,dx}{2EI}$$

Instead of integrating this and then carrying out the partial differentiations, we will differentiate first and then integrate, and for convenience suppress the constant term EI until all computations are completed. Thus

$$\begin{aligned}
\delta_y &= \frac{\partial U}{\partial V} \\
&= \int_0^{\pi/2} [VR\sin x + HR(1-\cos x) + M_0](R\sin x)R\,dx \\
&= VR^3(\tfrac{1}{2}x - \tfrac{1}{2}\sin x\cos x) - HR^3(\cos x + \tfrac{1}{2}\sin^2 x) - M_0R^2\cos x\Big|_0^{\pi/2} \\
&= \frac{(\pi/4)VR^3 + \frac{1}{2}HR^3 + M_0R^2}{EI}
\end{aligned}$$

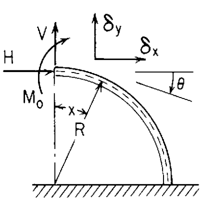


Figure 9.5

$$\begin{aligned}
\delta_x &= \frac{\partial U}{\partial H} \\
&= \int_0^{\pi/2} [VR\sin x + HR(1-\cos x) + M_0]R(1-\cos x)R\,dx \\
&= VR^3(-\cos x - \tfrac{1}{2}\sin^2 x) + HR^3(\tfrac{3}{2}x - 2\sin x + \tfrac{1}{2}\sin x\cos x) + M_0R^2(x-\sin x)\Big|_0^{\pi/2} \\
&= \frac{\tfrac{1}{2}VR^3 + (\tfrac{3}{4}\pi - 2)HR^3 + (\pi/2 - 1)M_0R^2}{EI} \\
\theta &= \frac{\partial U}{\partial M_0} \\
&= \int_0^{\pi/2} [VR\sin x + HR(1-\cos x) + M_0]R\,dx \\
&= -VR^2\cos x + HR^2(x - \sin x) + M_0Rx\Big|_0^{\pi/2} \\
&= \frac{VR^2 + (\pi/2 - 1)HR^2 + (\pi/2)M_0R}{EI}
\end{aligned}$$

The deflection produced by any one load or any combination of two loads is found by setting the other load or loads equal to zero; thus, V alone would produce $\delta_x = \frac{1}{2}VR^3/EI$, and M alone would produce $\delta_y = M_0R^2/EI$. In this example all results are positive, indicating that δ_x is in the direction of H, δ_y in the direction of V, and θ in the direction of M_0.

Distortion of tubular sections. In curved beams of thin tubular section, the distortion of the cross section produced by the radial components of the fiber stresses reduces both the strength and stiffness. If the beam curvature is not so sharp as to make Eqs. (8.1-1) and (8.1-4) inapplicable, the effect of this distortion of the section can be taken into account as follows.

In calculating deflection of curved beams of hollow circular section, replace I by KI, where

$$K = 1 - \frac{9}{10 + 12(tR/a^2)^2}$$

(Here $R =$ the radius of curvature of the beam axis, $a =$ the outer radius of tube section, and $t =$ the thickness of tube wall.) In calculating the maximum bending stress in curved beams of hollow circular section, use the formulas

$$\sigma_{\max} = \frac{Ma}{I}\frac{2}{3K\sqrt{3\beta}} \qquad \text{at } y = \frac{a}{\sqrt{3\beta}} \qquad \text{if } \frac{tR}{a^2} < 1.472$$

or

$$\sigma_{\max} = \frac{Ma}{I}\frac{1-\beta}{K} \qquad \text{at } y = a \qquad \text{if } \frac{tR}{a^2} > 1.472$$

where

$$\beta = \frac{6}{5 + 6(tR/a^2)^2}$$

and y is measured from the neutral axis. Torsional stresses and deflections are unchanged.

In calculating deflection or stress in curved beams of hollow square section and uniform wall thickness, replace I by

$$\frac{1 + 0.0270n}{1 + 0.0656n} I$$

where $n = b^4/R^2t^2$. (Here $R =$ the radius of curvature of the beam axis, $b =$ the length of the side of the square section, and $t =$ the thickness of the section wall.)

The preceding formulas for circular sections are from von Kármán (Ref. 4); the formulas for square sections are from Timoshenko (Ref. 5), who also gives formulas for rectangular sections.

Extensive analyses have been made for thin-walled pipe elbows with sharp curvatures for which the equations given above do not apply directly. Loadings may be *in-plane*, *out-of-plane*, or in various combinations (Ref. 8). Internal pressure increases and external pressure decreases pipe-bend stiffness. To determine ultimate load capacities of pipe bends or similar thin shells, elastic-plastic analyses, studies of the several modes of instability, and the stabilizing effects of flanges and the piping attached to the elbows are some of the many subjects presented in published works. Bushnell (Ref. 7) included an extensive list of references. Using numerical results from computer codes, graphs of stress indices and flexibility factors provide design data (Refs. 7, 19, and 34).

Deflection of curved beams of small radius. For a sharply curved beam, i.e., the radius of curvature is less than 10 times the depth, the stress distribution is not linear across the depth. The expression for the complementary energy of flexure is given by

$$U_f = \int \frac{M^2}{2AEeR} R\,dx = \int \frac{M^2}{2AEe}\,dx \tag{9.2-1}$$

where A is the cross-sectional area, E is the modulus of elasticity, and e is the distance from the centroidal axis to the neutral axis as given in Table 9.1. The differential change in angle dx is the same as is used in the previous example. See Fig. 9.1. Also keep clearly in mind that the bending in the plane of the curvature must be about a *principal* axis or the restraints described in the first paragraph of Sec. 9.1 must be present.

For all cross sections the value of the product AeR approaches the value of the moment of inertia I when the radius of curvature becomes greater than 10 times the depth. This is seen clearly in the following table where values of the ratio AeR/I are given for several sections and curvatures.

Case no.	Section	R/d			
		1	3	5	10
1	Solid rectangle	1.077	1.008	1.003	1.001
2	Solid circle	1.072	1.007	1.003	1.001
5	Triangle (base inward)	0.927	0.950	0.976	0.988
6	Triangle (base outward)	1.268	1.054	1.030	1.014

For curved beams of large radius the effect on deflections of the shear stresses due to V and the circumferential normal stresses due to N were small unless the length was small. For sharply curved beams the effects of these stresses must be considered. Only the effects of the radial stresses σ_r will be neglected. The expression for the complementary energy including all but the radial stresses is given by

$$U_f = \int \frac{M^2}{2AEe}\,dx + \int \frac{FV^2R}{2AG}\,dx + \int \frac{N^2R}{2AE}\,dx - \int \frac{MN}{AE}\,dx \qquad (9.2\text{-}2)$$

where all the quantities are defined in the notation at the top of Table 9.2.

The last term, hereafter referred to as the *coupling* term, involves the complementary energy developed from coupling the strains from the bending moment M and the normal force N. A positive bending moment M produces a negative strain at the position of the *centroidal* axis in a curved beam, and the resultant normal force N passes through the centroid. Reasons have been given for and against including the coupling term in attempts to improve the accuracy of calculated deformations (see Refs. 3 and 29). Ken Tepper, Ref. 30, called attention to the importance of the coupling term for sharply curved beams. The equations in Tables 9.2 and 9.3 have been modified and now include the effect of the coupling term. With this change, the formulas given in Tables 9.2 and 9.3 for the indeterminate reactions and for the deformations are no longer limited to thin rings and arches but can be used as well for thick rings and arches. As before, for thin rings and arches α and β can be set to zero with little error.

To summarize this discussion and its application to the formulas in Tables 9.2 and 9.3, one can place a given curved beam into one of three categories: a thin ring, a moderately thick ring, and a very thick or sharply curved ring. The boundaries between these categories depend upon the R/d ratio and the shape of the cross section. Reference to the preceding tabulation of the ratio AeR/I will be helpful.

For thin rings the effect of normal stress due to N and shear stress due to V can be neglected; i.e., set α and β equal to zero. For moderately thick rings and arches use the equations as they are given in Tables 9.2 and 9.3. For thick rings and arches replace the moment of inertia I with the product AeR in all equations including those for α and β. To illustrate the accuracy of this approach, the previous example problem will be repeated but for a thick ring of rectangular cross section. The rectangular cross section was chosen because a solution can be obtained by using the theory of elasticity with which to compare and evaluate the results.

EXAMPLE

Figure 9.6 represents a thick uniform bar of rectangular cross section having a curved centroidal surface of radius R. It is fixed at the lower end, and the upper end is loaded by a vertical force V, a horizontal force H, and a couple M_o. It is desired to find the vertical deflection δ_y, the horizontal deflection δ_x, and the rotation θ of the upper end. Note that the deflections δ_y and δ_x are the values at the free end and at the radial position R at which the load H is applied.

First Solution. Again Castigliano's theorem will be used. First find the moment, shear, and axial force at the angular position x.

$$\begin{aligned} M_x &= VR\sin x + HR(1-\cos x) + M_o \\ V_x &= V\cos x + H\sin x \\ N_x &= -H\cos x + V\sin x \end{aligned}$$

Since the beam is to be treated as a thick beam the expression for complementary energy is given by

$$U + \int \frac{M_x^2}{2AEe}\,dx + \int \frac{FV_x^2 R}{2AG}\,dx + \int \frac{N_x^2 R}{2AE}\,dx - \int \frac{M_x N_x}{AE}\,dx$$

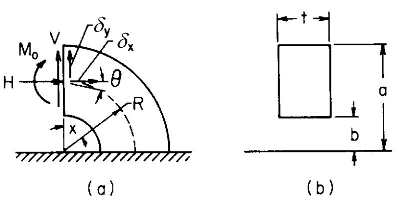


Figure 9.6

The deflections can now be calculated

$$\delta_y = \frac{\partial U}{\partial V} = \int_0^{\pi/2} \frac{M_x}{AEe}(R\sin x)dx + \int_0^{\pi/2} \frac{FV_xR}{AG}(\cos x)dx + \int_0^{\pi/2} \frac{N_xR}{AE}(\sin x)dx$$
$$- \int_0^{\pi/2} \frac{M_x}{AE}(\sin x)dx - \int_0^{\pi/2} \frac{N_x}{AE}(R\sin x)dx$$
$$= \frac{(\pi/4)VR^3 + 0.5HR^3 + M_oR^2}{EAeR} + \frac{0.5R(\pi V/2 + H)[2F(1+\nu) - 1] - M_o}{AE}$$

$$\delta_x = \frac{\partial U}{\partial H} = \int_0^{\pi/2} \frac{M_xR}{AEe}(1-\cos x)dx + \int_0^{\pi/2} \frac{FV_xR}{AG}(\sin x)dx + \int_0^{\pi/2} \frac{N_xR}{AE}(-\cos x)dx$$
$$- \int_0^{\pi/2} \frac{M_x}{AE}(-\cos x)dx - \int_0^{\pi/2} \frac{N_xR}{AE}(1-\cos x)dx$$
$$= \frac{0.5VR^3 + (3\pi/4 - 2)HR^3 + (\pi/2 - 1)M_oR^2}{EAeR}$$
$$+ \frac{0.5VR[2F(1+\nu) - 1] + (\pi/4)HR[2F(1+\nu) + 8/\pi - 1] + M_o}{EA}$$

$$\theta = \frac{\partial U}{\partial M_o} = \int_0^{\pi/2} \frac{M_x}{AEe}(1)dx + \int_0^{\pi/2} \frac{FV_xR}{AG}(0)dx + \int_0^{\pi/2} \frac{N_xR}{AE}(0)dx - \int_0^{\pi/2} \frac{M_x}{AE}(0)dx$$
$$- \int_0^{\pi/2} \frac{N_x}{AE}(1)dx$$
$$= \frac{VR^2 + (\pi/2 - 1)HR^2 + (\pi/2)M_oR}{EAeR} + \frac{H - V}{AE}$$

There is no need to reduce these expressions further in order to make a numerical calculation, but it is of interest here to compare to the solutions in the previous example. Therefore, let $\alpha = e/R$ and $\beta = FEe/GR = 2F(1+\nu)/R$ as defined previously

$$\delta_y = \frac{(\pi/4)VR^3(1 - \alpha + \beta) + 0.5HR^3(1 - \alpha + \beta) + M_oR^2(1 - \alpha)}{EAeR}$$
$$\delta_x = \frac{0.5VR^3(1 - \alpha + \beta) + HR^3[(3\pi/4 - 2) + (2 - \pi/4)\alpha + (\pi/4)\beta]}{EAeR}$$
$$+ \frac{M_oR^2(\pi/2 - 1 + \alpha)}{EAeR}$$
$$\theta = \frac{VR^2(1 - \alpha) + HR^2(\pi/2 - 1 + \alpha) + (\pi/2)M_oR}{EAeR}$$

Up to this point in the derivation, the cross section has not been specified. For a rectangular cross section having an outer radius a and an inner radius b and of thickness t normal to the surface shown in Fig. 9.6(b), the following substitutions can be made in the deformation equations. Let $\nu = 0.3$.

$$R = \frac{a + b}{2}, \qquad A = (a - b)t, \qquad F = 1.2 \quad \text{(see Sec. 8.10)}$$
$$\alpha = \frac{e}{R} = 1 - \frac{2(a - b)}{(a + b)\ln(a/b)}, \qquad \beta = 3.12\alpha$$

In the following table the value of a/b is varied from 1.1, where $R/d = 10.5$, a thin beam, to $a/b = 5.0$, where $R/d = 0.75$, a very thick beam. Three sets of numerical values are compared. The first set consists of the three deformations δ_y, δ_x, and θ evaluated from the equations just derived and due to the vertical load V. The second set consists of the same deformations due to the same loading but evaluated by applying the equations for a thin curved beam from the first example. The third set consists of the same deformations due to the same loading but evaluated by applying the theory of elasticity. See Ref. 2. The abbreviation MM in parentheses identifies the values from the *mechanics-of-materials* solutions and the abbreviation EL similarly identifies those found from the *theory of elasticity.*

		From thick-beam theory			From thin-beam theory		
a/b	R/d	$\frac{\delta_y(\text{MM})}{\delta_y(\text{EL})}$	$\frac{\delta_x(\text{MM})}{\delta_x(\text{EL})}$	$\frac{\theta(\text{MM})}{\theta(\text{EL})}$	$\frac{\delta_y(\text{MM})}{\delta_y(\text{EL})}$	$\frac{\delta_x(\text{MM})}{\delta_x(\text{EL})}$	$\frac{\theta(\text{MM})}{\theta(\text{EL})}$
1.1	10.5	0.9996	0.9990	0.9999	0.9986	0.9980	1.0012
1.3	3.83	0.9974	0.9925	0.9991	0.9900	0.9852	1.0094
1.5	2.50	0.9944	0.9836	0.9976	0.9773	0.9967	1.0223
1.8	1.75	0.9903	0.9703	0.9944	0.9564	0.9371	1.0462
2.0	1.50	0.9884	0.9630	0.9916	0.9431	0.9189	1.0635
3.0	1.00	0.9900	0.9485	0.9729	0.8958	0.8583	1.1513
4.0	0.83	1.0083	0.9575	0.9511	0.8749	0.8345	1.2304
5.0	0.75	1.0230	0.9763	0.9298	0.8687	0.8290	1.2997

If reasonable errors can be tolerated, the strength-of-materials solutions are very acceptable when proper recognition of thick and thin beams is given.

Second Solution. Table 9.3 is designed to enable one to take any angular span 2θ and any single load or combination of loads and find the necessary indeterminate reactions and the desired deflections. To demonstrate this use of Table 9.3 in this example the deflection δ_x will be found due to a load H. Use of case 12d, with load terms from case 5d and with $\theta = \pi/4$ and $\phi = \pi/4$. Both load terms LF_H and LF_V are needed since the desired deflection δ_x is not in the direction of either of the deflections given in the table. Let $c = m = s = n = 0.7071$.

$$LF_H = H\left\{\frac{\pi}{2}0.7071 + \frac{k_1}{2}\left[\frac{\pi}{2}0.7071 - 0.7071^3(2)\right] - k_2 2(0.7071)\right\}$$

$$LF_V = H\left\{-\frac{\pi}{2}0.7071 - \frac{k_1}{2}\left[\frac{\pi}{2}0.7071 + 0.7071^3(2)\right] + k_2 4(0.7071^3)\right\}$$

$$\begin{aligned}\delta_x &= (\delta_{VA} - \delta_{HA})0.7071 = \frac{-R^3}{EAeR}(LF_V - LF_H)0.7071\\ &= \frac{-R^3 H}{EAeR}\left(-\frac{\pi}{2} - \frac{k_1}{2}\frac{\pi}{2} + 2k_2\right) = \frac{R^3 H}{EAeR}\left[\frac{3\pi}{4} - 2 + \left(2 - \frac{\pi}{4}\right)\alpha + \frac{\pi}{4}\beta\right]\end{aligned}$$

This expression for δ_x is the same as the one derived directly from Castigliano's theorem. For angular spans of 90 or 180° the direct derivation is not difficult, but for odd-angle spans the use of the equations in Table 9.3 is recommended.

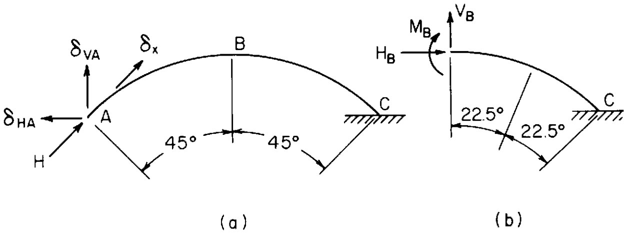


Figure 9.7

The use of the equations in Table 9.3 is also recommended when deflections are desired at positions other than the load point. For example, assume the deflections of the midspan of this arch are desired when it is loaded with the end load H as shown in Fig. 9.7(a). To do this, isolate the span from B to C and find the loads H_B, V_B, and M_B which act at point B. This gives $H_B = V_B = 0.7071H$ and $M_B = HR(1 - 0.7071)$. Now, superpose cases 12c, 12d, and 12n using these loads and $\theta = \phi = \pi/8$. In a problem with neither end fixed, a rigid-body motion may have to be superposed to satisfy the boundary conditions.

Deflection of curved beams of variable cross section and/or radius. None of the tabulated formulas applies when either the cross section or the radius of curvature varies along the span. The use of Eqs. (9.2-1) and (9.2-2), or of comparable expressions for thin curved beams, with numerical integration carried out for a finite number of elements along the span provides an effective means of solving these problems. This is best shown by example.

EXAMPLE

A rectangular beam of constant thickness and a depth varying linearly along the length is bent such that the centroidal surface follows the curve $x = 0.25y^2$ as shown in Fig. 9.8. The vertical deflection at the loaded end is desired. To keep the use of specific dimensions to a minimum let the depth of the curved beam at the fixed end = 1.0, the thickness = 0.5, and the horizontal location of the load $P = 1.0$. The beam will be subdivided into eight segments, each spanning 0.25 units in the y direction. Normally a constant length along the span is used, but using constant Δy gives shorter spans where moments are larger and curvatures are sharper. The numerical calculations are also easier. Use will be made of the following expressions in order to provide the tabulated

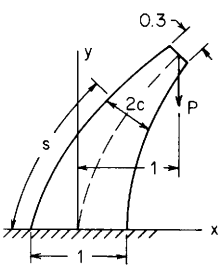


Figure 9.8

information from which the needed summation can be found. Note that y_i and x_i are used here as the y and x positions of the midlength of each segment

$$x = 0.25y^2, \qquad \frac{dx}{dy} = 0.5y, \qquad \frac{d^2x}{d^2y} = 0.5, \qquad \Delta l = \Delta y(1 + x_i)^{1/2}$$

$$R = \frac{[1 + (dx/dy)^2]^{3/2}}{d^2x/d^2y}$$

$$\frac{e}{c} = \frac{R}{c} - \frac{2}{\ln[(R/c + 1)/(R/c - 1)]} \qquad \text{for } \frac{R}{2c} < 8$$

[see Eq. (9.1-1) and case 1 of Table 9.1] or

$$\frac{e}{c} = \frac{I_c}{RAc} = \frac{t(2c)^3}{12(Rt2c^2)} = \frac{c}{3R} \qquad \text{for } \frac{R}{2c} > 8$$

[see Eq. (9.1-3)].

The desired vertical deflection of the loaded end can be determined from Castigliano's theorem, using Eq. (9.2-2) for U_f in summation form rather than integral form. This reduces to

$$\delta = \frac{\delta U}{\delta P} = \frac{P}{E}\sum\left[\frac{(M/P)^2}{eR} + F\left(\frac{V}{P}\right)^2 2(1+\nu) + \left(\frac{N}{P}\right)^2 - 2\frac{M}{P}\frac{N}{P}\right]\frac{\Delta l}{A}$$

$$= \frac{P}{E}\sum\frac{\Delta l}{A}[B]$$

where $[B]$ and $[B]\Delta l/A$ are the last two columns in the following table. The internal forces and moments can be determined from equilibrium equations as

$$\frac{M}{P} = -(1 - x_i), \quad \theta_i = \tan^{-1}\frac{dx}{dy}, \quad V = P\sin\theta_i, \quad \text{and} \quad N = -P\cos\theta_i$$

In the evaluation of the above equations for this problem, $F = 1.2$ and $\nu = 0.3$. In the table below one must fill in the first five columns in order to find the total length of the beam before the midsegment depth 2c can be found and the table completed.

Element no.	y_i	x_i	R	Δl	c	R/c
1	0.125	0.004	2.012	0.251	0.481	4.183
2	0.375	0.035	2.106	0.254	0.442	4.761
3	0.625	0.098	2.300	0.262	0.403	5.707
4	0.875	0.191	2.601	0.273	0.362	7.180
5	1.125	0.316	3.020	0.287	0.320	9.451
6	1.375	0.473	3.574	0.303	0.275	13.019
7	1.625	0.660	4.278	0.322	0.227	18.860
8	1.875	0.879	5.151	0.343	0.176	29.243
				2.295		

Element no.	e/c	M/P	V/P	N/P	$[B]$	$[B]\dfrac{\Delta l}{A}$
1	0.0809	−0.996	0.062	−0.998	11.695	6.092
2	0.0709	−0.965	0.184	−0.983	13.269	7.627
3	0.0589	−0.902	0.298	−0.954	14.370	9.431
4	0.0467	−0.809	0.401	−0.916	14.737	11.101
5	0.0354	−0.684	0.490	−0.872	14.007	12.569
6	0.0256	−0.527	0.567	−0.824	11.856	13.105
7	0.0177	−0.340	0.631	−0.776	8.049	11.431
8	0.0114	−0.121	0.684	−0.730	3.232	6.290
						77.555

Therefore, the deflection at the load and in the direction of the load is $77.56P/E$ in whatever units are chosen as long as the depth at the fixed end is unity. If one maintains the same length-to-depth ratio and the same shape, the deflection can be expressed as $\delta = 77.56P/(E2t_o)$, where t_o is the constant thickness of the beam.

Michael Plesha (Ref. 33) provided a finite-element solution for this configuration and obtained for the load point a vertically downward deflection of 72.4 units and a horizontal deflection of 88.3 units. The 22 elements he used were nine-node, quadratic displacement, Lagrange elements. The reader is invited to apply a horizontal dummy load and verify the horizontal deflection.

9.3 Circular Rings and Arches

In large pipelines, tanks, aircraft, and submarines the circular ring is an important structural element, and for correct design it is often necessary to calculate the stresses and deflections produced in such a ring under various conditions of loading and support. The circular arch of uniform section is often employed in buildings, bridges, and machinery.

Rings. A closed circular ring may be regarded as a *statically indeterminate beam* and analyzed as such by the use of Castigliano's second theorem. In Table 9.2 are given formulas thus derived for the bending moments, tensions, shears, horizontal and vertical deflections, and rotations of the load point in the plane of the ring for various loads and supports. By superposition, these formulas can be combined so as to cover almost any condition of loading and support likely to occur.

The ring formulas are based on the following assumptions: (1) The ring is of uniform cross section and has symmetry about the plane of curvature. An exception to this requirement of symmetry can be made if moment restraints are provided to prevent rotation of each cross section out of its plane of curvature. Lacking the plane of symmetry and any external constraints, out-of-plane deformations will accompany in-plane loading. Meck, in Ref. 21, derives expressions concern-ing the coupling of in-plane and out-of-plane deformations of circular rings of arbitrary compact cross section and resulting instabilities. (2) All loadings are applied at the radial position of the centroid of the cross section. For thin rings this is of little concern, but for radially thick rings a concentrated load acting in other than a radial direction and not at the centroidal radius must be replaced by a statically equivalent load at the centroidal radius and a couple. For case 15, where the loading is due to gravity or a constant linear acceleration, and for case 21, where the loading is due to rotation around an axis normal to the plane of the ring, the proper distribution of loading through the cross section is accounted for in the formulas. (3) It is nowhere stressed beyond the elastic limit. (4) It is not so severely deformed as to lose its essentially circular shape. (5) Its deflection is due primarily to bending, but for thicker rings the deflections due to deformations caused by axial tension or compression in the ring and/or by transverse shear stresses in the ring may be included. To include these effects, we can evaluate first the coefficients α and β, the *axial stress deformation factor*, and the *transverse shear deformation factor*, and then the constants k_1 and k_2. Such corrections are more often necessary when composite or sandwich construction is employed. If no axial or shear stress corrections are desired, α and β are set equal to zero and the values of k are set equal to unity. (6) In the case of pipes acting as beams between widely spaced supports, the distribution of shear stress across the section of the pipe is in accordance with Eq. (8.1-2), and the direction of the resultant shear stress at any point of the cross section is tangential.

Note carefully the deformations given regarding the point or points of loading as compared with the deformations of the horizontal and vertical diameters. For many of the cases listed, the numerical values of load and deflection coefficients have been given for several positions of the loading. These coefficients do not include the effect of axial and shear deformation.

No account has been taken in Table 9.2 of the effect of radial stresses in the vicinity of the concentrated loads. These stresses and the local deformations they create can have a significant effect on overall ring deformations and peak stresses. In case 1 a reference is made to Sec. 14.3 in which thick-walled rollers or rings are loaded on the outer ends of a diameter. The stresses and deflections given here are different from those predicted by the equations in case 1. If a concentrated load is used only for purposes of superposition, as is often the case, there is no cause for concern, but if an actual applied load is concentrated over a small region and the ring is sharply curved with thick walls, then one must be aware of the possible errors.

EXAMPLES

1. A pipe with a diameter of 13 ft and thickness of $\frac{1}{2}$ in is supported at intervals of 44 ft by rings, each ring being supported at the extremities of its horizontal diameter by vertical reactions acting at the centroids of the ring sections. It is required to determine the bending moments in a ring at the bottom, sides, and top, and the maximum bending moment when the pipe is filled with water.

Solution. We use the formulas for cases 4 and 20 of Table 9.2. Taking the weight of the water as 62.4 lb/ft^3 and the weight of the shell as 20.4 lb/ft^2, the total weight W of 44 ft of pipe carried by one ring is found to be 401,100 lb. Therefore, for case 20, $W = 401{,}100$ lb; and for case 4, $W = 250{,}550$ lb and $\theta = \pi/2$. Assume a thin ring, $\alpha = \beta = 0$.

At bottom:

$$\begin{aligned} M = M_C &= 0.2387(401{,}100)(6.5)(12) - 0.50(200{,}550)(78) \\ &= 7.468(10^6) - 7.822(10^6) = -354{,}000 \text{ lb-in} \end{aligned}$$

At top:

$$\begin{aligned} M &= M_A = 0.0796(401{,}100)(78) - 0.1366(200{,}550)(78) = 354{,}000 \text{ lb-in} \\ N &= N_A = 0.2387(401{,}100) - 0.3183(200{,}500) = 31{,}900 \text{ lb} \\ V &= V_A = 0 \end{aligned}$$

At sides:

$$M = M_A - N_A R(1 - u) + V_A R z + LT_M$$

where for $x = \pi/2$, $u = 0$, $z = 1$, and $LT_M = (WR/\pi)(1 - u - xz/2) = [401{,}100(78)/\pi]\ (1 - \pi/4) = 2.137(10^6)$ for case 20, and $LT_M = 0$ for case 4 since $z - s = 0$. Therefore

$$M = 354{,}000 - 31{,}900(78)(1 - 0) + 0 + 2.137(10^6) = 2800 \text{ lb-in}$$

The value of 2800 lb-in is due to the small differences in large numbers used in the superposition. An exact solution would give zero for this value. It is apparent that at least four digits must be carried.

To determine the location of maximum bending moment let $0 < x < \pi/2$ and examine the expression for M:

$$\begin{aligned} M &= M_A - N_A R(1 - \cos x) + \frac{WR}{\pi}\left(1 - \cos x - \frac{x \sin x}{2}\right) \\ \frac{dM}{dx} &= -N_A R \sin x + \frac{WR}{\pi} \sin x - \frac{WR}{2\pi} \sin x - \frac{WRx}{2\pi} \cos x \\ &= 31{,}950 R \sin x - 63{,}800 R x \cos x \end{aligned}$$

At $x = x_1$, let $dM/dx = 0$ or $\sin x_1 = 2x_1 \cos x_1$, which yields $x_1 = 66.8^\circ(1.166\,\text{rad})$. At $x = x_1 = 66.8^\circ$,

$$M = 354{,}00 - 31{,}900(78)(1 - 0.394) + \frac{401{,}100(78)}{\pi}\left[1 - 0.394 - \frac{1.166(0.919)}{2}\right]$$
$$= -455{,}000\,\text{lb-in} \quad \text{(max negative moment)}$$

Similarly, at $x = 113.2^\circ$, $M = 455{,}000\,\text{lb-in}$ (max positive moment).

By applying the supporting reactions outside the center line of the ring at a distance a from the centroid of the section, side couples that are each equal to $Wa/2$ would be introduced. The effect of these, found by the formulas for case 3, would be to reduce the maximum moments, and it can be shown that the optimum condition obtains when $a = 0.04R$.

2. The pipe of Example 1 rests on soft ground, with which it is in contact over 150° of its circumference at the bottom. The supporting pressure of the soil may be assumed to be radial and uniform. It is required to determine the bending moment at the top and bottom and at the surface of the soil. Also the bending stresses at these locations and the change in the horizontal diameter must be determined.

Solution. A section of pipe 1 in long is considered. The loading may be considered as a combination of cases 12, 15, and 16. Owing to the weight of the pipe (case 15, $w = 0.1416\,\text{lb/in}$), and letting $K_T = k_1 = k_2 = 1$, and $\alpha = \beta = 0$,

$$M_A = \frac{0.1416(78)^2}{2} = 430\,\text{lb-in}$$
$$N_A = \frac{0.1416(78)}{2} = 5.52\,\text{lb}$$
$$V_A = 0$$

and at $x = 180 - \frac{150}{2} = 105^\circ = 1.833$ rad,

$$LT_M = -0.1416(78^2)[1.833(0.966) - 0.259 - 1] = -440\,\text{lb-in}$$

Therefore

$$M_{105^\circ} = 430 - 5.52(78)(1 + 0.259) - 440 = -552\,\text{lb-in}$$
$$M_C = 1.5(0.1416)(78) = 1292\,\text{lb-in}$$

Owing to the weight of contained water (case 16, $\rho = 0.0361\,\text{lb/in}^3$),

$$M_A = \frac{0.0361(78^3)}{4} = 4283\,\text{lb-in/in}$$
$$N_A = \frac{0.0361(78^2)(3)}{4} = 164.7\,\text{lb/in}$$
$$V_A = 0$$

and at $x = 105^\circ$,

$$LT_M = 0.0361(78^3)\left[1 + 0.259 - \frac{1.833(0.966)}{2}\right] = 6400\,\text{lb-in/in}$$

Therefore

$$M_{105^\circ} = 4283 - 164.7(78)(1 + 0.259) + 6400 = -5490\,\text{lb-in/in}$$

$$M_C = \frac{0.0361(78^3)(3)}{4} = 12{,}850\,\text{lb-in/in}$$

Owing to earth pressure and the reversed reaction (case 12, $\theta = 105^\circ$),

$$2wR\sin\theta = 2\pi R(0.1416) + 0.0361\pi R^2 = 759\,\text{lb} \quad (w = 5.04\,\text{lb/in})$$

$$M_A = \frac{-5.04(78^2)}{\pi}[0.966 + (\pi - 1.833)(-0.259) - 1(\pi - 1.833 - 0.966)]$$

$$= -2777\,\text{in-lb}$$

$$N_A = \frac{-5.04(78)}{\pi}[0.966 + (\pi - 1.833)(-0.259)] = -78.5\,\text{lb}$$

$$V_A = 0$$

$$LT_M = 0$$

$$M_{105^\circ} = -2777 + 78.5(78)(1.259) = 4930\,\text{lb-in}$$

$$M_C = -5.04(78^2)\frac{1.833(1 - 0.259)}{\pi} = -13{,}260\,\text{lb-in}$$

Therefore, for the 1 in section of pipe

$$M_A = 430 + 4283 - 2777 = 1936\,\text{lb-in}$$

$$\sigma_A = \frac{6M_A}{t^2} = 46{,}500\,\text{lb/in}^2$$

$$M_{105^\circ} = -552 - 5490 + 4930 = -1112\,\text{lb-in}$$

$$\sigma_{105^\circ} = 26{,}700\,\text{lb/in}^2$$

$$M_C = 1292 + 12{,}850 - 13{,}260 = 882\,\text{lb-in}$$

$$\sigma_C = 21{,}200\,\text{lb/in}^2$$

The change in the horizontal diameter is found similarly by superimposing the three cases. For E use $30(10^6)/(1 - 0.285^2) = 32.65(10^6)\,\text{lb/in}^2$, since a plate is being bent instead of a narrow beam (see page 169). For I use the moment of inertia of a 1-in-wide piece, 0.5 in thick:

$$I = \tfrac{1}{12}(1)(0.5^3) = 0.0104\,\text{in}^4, \qquad EI = 340{,}000\,\text{lb-in}^2$$

From case 12:

$$\Delta D_H = \frac{-5.04(78^4)}{340{,}000}\left[\frac{(\pi - 1.833)(-0.259) + 0.966}{2} - \frac{2}{\pi}(\pi - 1.833 - 0.966)\right]$$

$$= \frac{-5.04(78)^4}{340{,}000}(0.0954) = -52.37\,\text{in}$$

From case 15:

$$\Delta D_H = \frac{0.4292(0.1416)78^4}{340{,}000} = 6.616\,\text{in}$$

From case 16:

$$\Delta D_H = \frac{0.2146(0.0361)78^5}{340{,}000} = 65.79\,\text{in}$$

The total change in the horizontal diameter is 20 in. It must be understood at this point that the anwers are somewhat in error since this large a deflection does violate the assumption that the loaded ring is very nearly circular. This was expected when the stresses were found to be so large in such a thin pipe.

Arches. Table 9.3 gives formulas for end reactions and end deformations for circular arches of constant radius of curvature and constant cross section under 18 different loadings and with 14 combinations of end conditions. The corrections for axial stress and transverse shear are accomplished as they were in Table 9.2 by the use of the constants α and β. Once the indeterminate reactions are known, the bending moments, axial loads, and transverse shear forces can be found from equilibrium equations. If deformations are desired for points away from the ends, the unit-load method [Eq. (8.1-6)] can be used or the arch can be divided at the position where the deformations are desired and either portion analyzed again by the formulas in Table 9.3. Several examples illustrate this last approach. Note that in many instances the answer depends upon the difference of similar large terms, and so appropriate attention to accuracy must be given.

EXAMPLES

1. A $WT4 \times 6.5$ structural steel T-beam is formed in the plane of its web into a circular arch of 50-in radius spanning a total angle of 120°. The right end is fixed, and the left end has a pin which is constrained to follow a horizontal slot in the support. The load is applied through a vertical bar welded to the beam, as shown in Fig. 9.9. Calculate the movement of the pin at the left end, the maximum bending stress, and the rotation of the bar at the point of attachment to the arch.

Solution. The following material and cross-sectional properties may be used for this beam. $E = 30(10^6)\,\text{lb/in}^2$, $G = 12(10^6)\,\text{lb/in}^2$, $I_x = 2.90\,\text{in}^4$, $A = 1.92\,\text{in}^2$, flange thickness $= 0.254$ in, and web thickness $= 0.230$ in. The loading on the

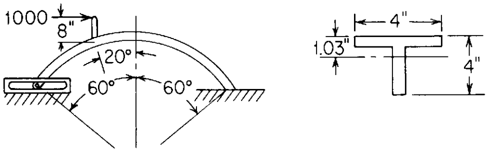


Figure 9.9

arch can be replaced by a concentrated moment of 8000 lb-in and a horizontal force of 1000 lb at a position indicated by $\phi = 20^\circ$ (0.349 rad). $R = 50$ in and $\theta = 60^\circ$ (1.047 rad). For these loads and boundary conditions, cases 9b and 9n of Table 9.3 can be used.

Since the radius of 50 in is only a little more than 10 times the depth of 4 in, corrections for axial load and shear will be considered. The axial-stress deformation factor $\alpha = I/AR^2 = 2.9/1.92(50^2) = 0.0006$. The transverse-shear deformation factor $\beta = FEI/GAR^2$, where F will be approximated here by using $F = 1$ and $A =$ web area $= 4(0.23) = 0.92$. This gives $\beta = 1(30)(10^6)(2.90)/12(10^6)(0.92)(50^2) = 0.003$. The small values of α and β indicate that bending governs the deformations, and so the effect of axial load and transverse shear will be neglected. Note that $s = \sin 60^\circ$, $c = \cos 60^\circ$, $n = \sin 20^\circ$, and $m = \cos 20^\circ$.

For case 9b,

$$LF_H = 1000\left[\frac{1.0472 + 0.3491}{2}(1 + 2\cos 20^\circ \cos 60^\circ) - \frac{\sin 60^\circ \cos 60^\circ}{2} - \frac{\sin 20^\circ \cos 20^\circ}{2} - \cos 20^\circ \sin 60^\circ - \sin 20^\circ \cos 60^\circ\right]$$
$$= 1000(-0.00785) = -7.85 \text{ lb}$$

Similarly,

$$LF_V = 1000(-0.1867) = -186.7 \text{ lb} \quad \text{and} \quad LF_M = 1000(-0.1040) = -104.0 \text{ lb}$$

For the case 9n,

$$LF_H = \frac{8000}{50}(-0.5099) = -81.59 \text{ lb}$$
$$LF_V = \frac{8000}{50}(-1.6489) = -263.8 \text{ lb}$$
$$LF_M = \frac{8000}{50}(-1.396) = -223.4 \text{ lb}$$

Also,

$$B_{VV} = 1.0472 + 2(1.0472)\sin^2 60^\circ - \sin 60^\circ \cos 60^\circ = 2.1850 \text{ lb}$$
$$B_{HV} = 0.5931 \text{ lb}$$
$$B_{MV} = 1.8138 \text{ lb}$$

Therefore,

$$V_A = -\frac{186.7}{2.1850} - \frac{263.8}{2.1850} = -85.47 - 120.74 = -206.2\,\text{lb}$$

$$\delta_{HA} = \frac{50^3}{30(10^6)(2.9)}[0.5931(-206.2) + 7.85 + 81.59] = -0.0472\,\text{in}$$

The expression for the bending moment can now be obtained by an equilibrium equation for a position located by an angle x measured from the left end:

$$M_x = V_A R[\sin\theta - \sin(\theta - x)] + 8000\langle x - (\theta - \phi)\rangle^0$$
$$- 1000R[\cos(\theta - x) - \cos\phi]\langle x - (\theta - \phi)\rangle^0$$

At $x = 40^\circ-$ $\quad M_x = -206.2(50)[\sin 60^\circ - \sin(60^\circ - 40^\circ)] = -5403\,\text{lb-in}$

At $x = 40^\circ+$ $\quad M_x = -5403 + 8000 = 2597\,\text{lb-in}$

At $x = 60^\circ$ $\quad M_x = -206.2(50)(0.866) + 8000 - 1000(50)(1 - 0.940)$
$= -3944\,\text{lb-in}$

At $x = 120^\circ$ $\quad M_x = 12{,}130\,\text{lb-in}$

The maximum bending stress is therefore

$$\sigma = \frac{12{,}130(4 - 1.03)}{2.9} = 12{,}420\,\text{lb/in}^2$$

To obtain the rotation of the arch at the point of attachment of the bar, we first calculate the loads on the portion to the right of the loading and then establish an equivalent symmetric arch (see Fig. 9.10). Now from cases 12a, 12b, and 12n, where $\theta = \phi = 40^\circ(0.698\text{ rad})$, we can determine the load terms:

For case 12a $\quad LF_M = -148[2(0.698)(0.643)] = -133\,\text{lb}$

For case 12b $\quad LF_M = 1010[0.643 + 0.643 - 2(0.698)(0.766)] = 218\,\text{lb}$

For case 12n $\quad LF_M = \dfrac{2597}{50}(-0.698 - 0.698) = -72.5\,\text{lb}$

Therefore, the rotation at the load is

$$\psi_A = \frac{-50^2}{30(10^6)(2.9)}(-133 + 218 - 72.5) = -0.00036\,\text{rad}$$

We would not expect the rotation to be in the opposite direction to the applied moment, but a careful examination of the problem shows that the point on the arch where the bar is fastened moves to the right 0.0128 in. Therefore, the net motion in the direction of the 1000-lb load on the end of the 8-in bar is 0.0099 in, and so the applied load does indeed do positive work on the system.

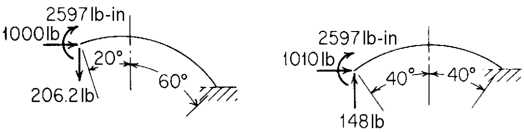


Figure 9.10

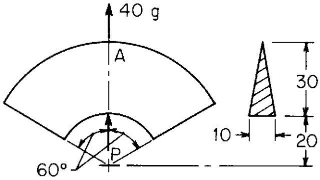


Figure 9.11

2. The deep circular arch of titanium alloy has a triangular cross section and spans 120° as shown in Fig. 9.11. It is driven by the central load P to produce an acceleration of $40g$. The tensile stress at A and the deformations at the extreme ends are required. All dimensions given and used in the formulas are in centimeters.

Solution. This is a statically determinate problem, so the use of information from Table 9.3 is needed only to obtain the deformations. Superposing the central load and the forces needed to produce the acceleration on the total span can be accomplished readily by using cases 3a and 3h. This solution, however, will provide only the horizontal and rotational deformations of the ends. Using the symmetry one can also superpose the loadings from cases 12h and 12i on the left half of the arch and obtain all three deformations. Performing both calculations provides a useful check. All dimensions are given in centimeters and used with expressions from Table 9.1, case 5, to obtain the needed factors for this section. Thus, $b = 10$, $d = 30$, $A = 150$, $c = 10$, $R = 30$, $R/c = 3$, $e/c = 0.155$, $e = 1.55$ and for the peak stresses, $k_i = 1.368$ and $k_o = 0.697$. The titanium alloy has a modulus of elasticity of 117 GPa $[11.7(10^6)\text{N/cm}^2]$, a Poisson's ratio of 0.33, and a mass density of $4470\,\text{kg/m}^3$, or $0.00447\,\text{kg/cm}^3$. One g of acceleration is $9.81\,\text{m/s}^2$, and 1 cm of arc length at the centroidal radius of 30 cm will have a volume of $150\,\text{cm}^3$ and a mass of 0.6705 kg. This gives a loading parallel to the driving force P of $0.6705(40)(9.81) = 263$ N/cm of centroidal arc length. Since this is a very sharply curved beam, $R/d = 1$, one must recognize that the resultant load of 263 N/cm does not act through the centroid of the cross-sectional area but instead acts through the mass center of the differential length. The radius to this point is given as R_{cg} and is found from the expression $R_{cg}/R = 1 + I/AR^2$, where I is the area moment of inertia about the centroidal axis of the cross section. Therefore, $R_{cg}/R = 1 + (bd^3/36)/(bd/2)R^2 = 1.056$. Again due to the sharp curvature the axial- and shear-stress contributions to deformation must be considered. From the introduction to Table 9.3 we find that $\alpha = h/R = 0.0517$ and $\beta = 2F(1+\nu)h/R = 0.1650$, where $F = 1.2$ for a triangular cross section as given in Sec. 8.10. Therefore, $k_1 = 1 - \alpha + \beta = 1.1133$, and $k_2 = 1 - \alpha = 0.9483$.

For a first solution use the full span and superpose cases 3a and 3h. To obtain the load terms LP_H and LP_M use cases 1a and 1h.

For case 1a, $W = -263(30)(2\pi/3) = -16{,}525$ N, $\theta = 60°$, $\phi = 0°$, $s = 0.866$, $c = 0.500$, $n = 0$, and $m = 1.000$.

$$LP_H = -16{,}525\left[\frac{\pi}{3}(0.866)(0.5) - 0 + \frac{1.1133}{2}(0.5^2 - 1.0^2) + 0.9483(0.5)(0.5 - 1.0)\right]$$

$$= -16{,}525(-0.2011) = 3323\,\text{N}$$

Similarly, $LP_M = 3575$ N.

For case $1h$, $w = 263\,\mathrm{N/cm}$, $R = 30$, $R_{cg}/R = 1.056$, $\theta = 60^\circ$, $s = 0.866$, and $c = 0.5000$.

$$LP_H = 263(30)(-0.2365) = -1866\,\mathrm{N}$$

and

$$LP_M = 263(30)(-0.2634) = -2078\,\mathrm{N}$$

Returning now to case 3 where M_A and H_A are zero, one finds that $V_A = 0$ by superposing the loadings. To obtain δ_{HA} and ψ_A we superpose cases a and h and substitute AhR for I because of the sharp curvature

$$\delta_{HA1} = -30^3\frac{3323 - 1866}{11.7(10^6)(150)(1.55)(30)} = -482(10^{-6})\,\mathrm{cm}$$

$$\psi_{A1} = -30^2\frac{3575 - 2078}{8.161(10^{10})} = -16.5(10^{-6})\,\mathrm{rad}$$

Now for the second and more complete solution, use will be made of cases 12h and 12i. The left half spans 60°, so $\theta = 30^\circ$, $s = 0.5000$, and $c = 0.8660$. In this solution the central symmetry axis of the left half being used is inclined at 30° to the gravitational loading of 263 N/cm. Therefore, for case 5h, $w = 263\cos 30^\circ = 227.8\,\mathrm{N/cm}$

$$\begin{aligned} LF_H = 227.8(30)\Bigg\{&\frac{1.1133}{2}\left[2\left(\frac{\pi}{6}\right)(0.866^2) - \frac{\pi}{6} - 0.5(0.866)\right] \\ &+ 0.9483(1.056 + 1)\left[\frac{\pi}{6} - 0.5(0.866)\right] + 1.056(2)(0.866)\left(\frac{\pi}{6}0.866 - 0.5\right)\Bigg\} \\ = 227.8(30)(-0.00383) = -26.2\,\mathrm{N}& \end{aligned}$$

Similarly

$$LF_V = 227.8(30)(0.2209) = 1510\,\mathrm{N} \quad \text{and} \quad LF_M = 227.8(30)(0.01867) = 1276\,\mathrm{N}$$

For case 5i, $w = -263\sin 30^\circ = -131.5\,\mathrm{N/cm}$ and again $\theta = 30^\circ$

$$LF_H = -131.5(30)(0.0310) = -122.3\,\mathrm{N}$$

$$LF_V = -131.5(30)(-0.05185) = 204.5\,\mathrm{N}$$

$$LF_M = -131.5(30)(-0.05639) = 222.5\,\mathrm{N}$$

Using case 12 and superposition of the loadings gives

$$\delta_{HA2} = -30^3\frac{-26.2 - 122.3}{8.161(10^{10})} = 49.1(10^{-6})\,\mathrm{cm}$$

$$\delta_{VA2} = -30^3\frac{1510 + 204.5}{8.161(10^{10})} = -567(10^{-6})\,\mathrm{cm}$$

$$\psi_{A2} = -30^2\frac{1276 + 222.5}{8.161(10^{10})} = -16.5(10^{-6})\ \mathrm{rad}$$

Although the values of ψ_A from the two solutions check, one further step is needed to check the horizontal and vertical deflections of the free ends. In the last solution the reference axes are tilted at 30°. Therefore, the horizontal and vertical deflections of the left end are given by

$$\delta_{HA} = \delta_{HA2}(0.866) + \delta_{VA2}(0.5) = -241(10^{-6})\,\text{cm}$$

$$\delta_{VA} = \delta_{HA2}(-0.5) + \delta_{VA2}(0.866) = -516(10^{-6})\,\text{cm}$$

Again the horizontal deflection of -0.000241 cm for the left half of the arch checks well with the value of -0.000482 cm for the entire arch. With the two displacements of the centroid and the rotation of the end cross section now known, one can easily find the displacements of any other point on the end cross section.

To find the tensile stress at point A we need the bending moment at the center of the arch. This can be found by integration as

$$M = \int_{\pi/6}^{\pi/2} -263R\,d\theta(R_{cg}\cos\theta) = -263RR_{cg}\sin\theta\Big|_{\pi/6}^{\pi/2} = -125{,}000\,\text{N-cm}$$

Using the data from Table 9.1, the stress in the outer fiber at the point A is given by

$$\sigma_A = \frac{k_o Mc}{I} = \frac{0.697(125{,}000)(20)}{10(30^3)/36} = 232\,\text{N/cm}^2$$

9.4 Elliptical Rings

For an elliptical ring of semiaxes a and b, under equal and opposite forces W (Fig. 9.12), the bending moment M_1 at the extremities of the major axis is given by $M_1 = K_1 Wa$, and for equal and opposite outward forces applied at the ends of the minor axis, the moment M_1 at the ends of the major axis is given by $M_1 = -K_2 Wa$, where K_1 and K_2 are coefficients which depend on the ratio a/b and have the following values:

a/b	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
K_1	0.318	0.295	0.274	0.255	0.240	0.227	0.216	0.205
K_2	0.182	0.186	0.191	0.195	0.199	0.203	0.206	0.208

a/b	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5
K_1	0.195	0.185	0.175	0.167	0.161	0.155	0.150	0.145
K_2	0.211	0.213	0.215	0.217	0.219	0.220	0.222	0.223

Burke (Ref. 6) gives charts by which the moments and tensions in elliptical rings under various conditions of concentrated loading can be found; the preceding values of K were taken from these charts.

Timoshenko (Ref. 13) gives an analysis of an elliptical ring (or other ring with two axes of symmetry) under the action of a uniform

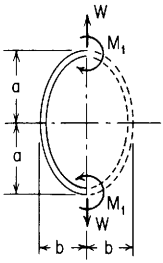


Figure 9.12

outward pressure, which would apply to a tube of elliptical section under internal pressure. For this case $M = Kpa^2$, where M is the bending moment at a section a distance x along the ring from the end of the minor axis, p is the outward normal pressure per linear inch, and K is a coefficient that depends on the ratios b/a and x/S, where S is one-quarter of the perimeter of the ring. Values of K are given in the following table; M is positive when it produces tension at the inner surface of the ring:

x/S \ b/a	0.3	0.5	0.6	0.7	0.8	0.9
0	−0.172	−0.156	−0.140	−0.115	−0.085	−0.045
0.1	−0.167	−0.152	−0.135	−0.112	−0.082	−0.044
0.2	−0.150	−0.136	−0.120	−0.098	−0.070	−0.038
0.4	−0.085	−0.073	−0.060	−0.046	−0.030	−0.015
0.6	0.020	0.030	0.030	0.028	0.022	0.015
0.7	0.086	0.090	0.082	0.068	0.050	0.022
0.8	0.160	0.150	0.130	0.105	0.075	0.038
0.9	0.240	0.198	0.167	0.130	0.090	0.046
1.0	0.282	0.218	0.180	0.140	0.095	0.050

Values of M calculated by the preceding coefficients are correct only for a ring of uniform moment of inertia I; if I is not uniform, then a correction ΔM must be added. This correction is given by

$$\Delta M = \frac{-\int_0^x \frac{M}{I}\,dx}{\int_0^x \frac{dx}{I}}$$

The integrals can be evaluated graphically. Reference 12 gives charts for the calculation of moments in elliptical rings under uniform radial loading; the preceding values of K were taken from these charts.

9.5 Curved Beams Loaded Normal to Plane of Curvature

This type of beam usually presents a statically indeterminate problem, the degree of indeterminacy depending upon the manner of loading and support. Both bending and twisting occur, and it is necessary to distinguish between an analysis that is applicable to compact or flangeless sections (circular, rectangular, etc.) in which torsion does not produce secondary bending and one that is applicable to flanged sections (I-beams, channels, etc.) in which torsion may be accompanied by such secondary bending (see Sec. 10.3). It is also necessary to distinguish among three types of constraints that may or may not occur at the supports, namely: (1) the beam is prevented from *sloping*, its horizontal axis held horizontal by a bending couple; (2) the beam is prevented from *rolling*, its vertical axis held vertical by a twisting couple; and (3) in the case of a flanged section, the flanges are prevented from turning about their vertical axes by horizontal secondary bending couples. These types of constraints will be designated here as (1) fixed as to slope, (2) fixed as to roll, and (3) flanges fixed.

Compact sections. Table 9.4 treats the curved beam of uniform cross section under concentrated and distributed loads normal to the plane of curvature, out-of-plane concentrated bending moments, and concentrated and distributed torques. Expressions are given for transverse shear, bending moment, twisting moment, deflection, bending slope, and roll slope for 10 combinations of end conditions. To keep the presentation to a reasonable size, use is made of the singularity functions discussed in detail previously and an extensive list of constants and functions is given. In previous tables the representative functional values have been given, but in Table 9.4 the value of β depends upon both bending and torsional properties, and so a useful set of tabular values would be too large to present. The curved beam or ring of circular cross section is so common, however, that numerical coefficients are given in the table for $\beta = 1.3$ which will apply to a solid or hollow circular cross section of material for which Poisson's ratio is 0.3.

Levy (Ref. 14) has treated the closed circular ring of arbitrary compact cross section for six loading cases. These cases have been chosen to permit apropriate superposition in order to solve a large number of problems, and both isolated and distributed out-of-plane loads are discussed. Hogan (Ref. 18) presents similar loadings and supports. In a similar way the information in Table 9.4 can be used by appropriate superposition to solve most out-of-plane loading problems on closed rings of compact cross section if strict attention is given to the symmetry and boundary conditions involved. Several simple examples of this reasoning are described in the following three cases.

1. If a closed circular ring is supported on any number of equally spaced simple supports (two or more) and if identical loading on each span is symmetrically placed relative to the center of the span, then each span can be treated by boundary condition f of Table 9.4. This boundary condition has both ends with no deflection or slope, although they are free to roll as needed.
2. If a closed circular ring is supported on any even number of equally spaced simple supports and if the loading on any span is antisymmetrically placed relative to the center line of each span and symmetrically placed relative to each support, then boundary condition f can be applied to each full span. This problem can also be solved by applying boundary condition g to each half span. Boundary condition g has one end simply supported and slope-guided and the other end simply supported and roll-guided.
3. If a closed circular ring is supported on any even number of equally spaced simple supports (four or more) and if each span is symmetrically loaded relative to the center of the span with adjacent spans similarly loaded in opposite directions, then boundary condition i can be applied to each span. This boundary condition has both ends simply supported and roll-guided.

Once any indeterminate reaction forces and moments have been found and the indeterminate internal reactions found at at least one location in the ring, all desired internal bending moment, torques, and transverse shears can be found by equilibrium equations. If a large number of such calculations need be made, one should consider using a theorem published in 1922 by Biezeno. For details of this theorem see Ref. 32. A brief illustration of this work for loads normal to the plane of the ring is given in Ref. 29.

A treatment of curved beams on elastic foundations is beyond the scope of this book. See Ref. 20.

The following examples illustrate the applications of the formulas in Table 9.4 to both curved beams and closed rings with out-of-plane loads.

EXAMPLES

1. A piece of 8-in standard pipe is used to carry water across a passageway 40 ft wide. The pipe must come out of a wall normal to the surface and enter normal to a parallel wall at a position 16.56 ft down the passageway at the same elevation. To accomplish this a decision was made to bend the pipe into two opposite arcs of 28.28-ft radius with a total angle of 45° in each arc. If it is assumed that both ends are rigidly held by the walls, determine the maximum combined stress in the pipe due to its own weight and the weight of a full pipe of water.

Solution. An 8-in standard pipe has the following properties: $A = 8.4\,\text{in}^2$, $I = 72.5\,\text{in}^4$, $w = 2.38\,\text{lb/in}$, $E = 30(10^6)\,\text{lb/in}^2$, $\nu = 0.3$, $J = 145\,\text{in}^4$, $\text{OD} = 8.625\,\text{in}$, $\text{ID} = 7.981\,\text{in}$, and $t = 0.322\,\text{in}$. The weight of water in a 1-in length of pipe is 1.81 lb. Owing to the symmetry of loading it is apparent that at the center of the span where the two arcs meet there is neither slope nor roll. An examination of Table 9.4 reveals that a curved beam that is fixed at the right end and roll- and slope-guided at the left end is not included among the 10 cases. Therefore, a solution will be carried out by considering a beam that is fixed at the right end and free at the left end with a uniformly distributed load over the entire span and both a concentrated moment and a concentrated torque on the left end. (These conditions are covered in cases 2a, 3a, and 4a.)

Since the pipe is round, $J = 2I$; and since $G = E/2(1+\nu)$, $\beta = 1.3$. Also note that for all three cases $\phi = 45^\circ$ and $\theta = 0^\circ$. For these conditions, numerical values of the coefficients are tabulated and the following expressions for the deformations and moments can be written directly from superposition of the three cases:

$$y_A = 0.3058\frac{M_o R^2}{EI} - 0.0590\frac{T_o R^2}{EI} - 0.0469\frac{(2.38+1.81)R^4}{EI}$$

$$\Theta_A = -0.8282\frac{M_o R}{EI} - 0.0750\frac{T_o R}{EI} + 0.0762\frac{4.19R^3}{EI}$$

$$\psi_A = 0.0750\frac{M_o R}{EI} + 0.9782\frac{T_o R}{EI} + 0.0267\frac{4.19R^3}{EI}$$

$$V_B = 0 + 0 - 4.19R(0.7854)$$

$$M_B = 0.7071M_o - 0.7071T_o - 0.2929(4.19)R^2$$

$$T_B = 0.7071M_o + 0.7071T_o - 0.0783(4.19)R^2$$

Since both Θ_A and ψ_A are zero and $R = 28.28(12) = 339.4\,\text{in}$,

$$0 = -0.8282M_o - 0.0750T_o + 36{,}780$$

$$0 = 0.0750M_o + 0.9782T_o + 12{,}888$$

Solving these two equations gives $M_o = 45{,}920\,\text{lb-in}$ and $T_o = -16{,}700\,\text{lb-in}$. Therefore,

$$y_A = -0.40\,\text{in}, \qquad M_B = -97{,}100\,\text{lb-in}$$

$$T_B = -17{,}000\,\text{lb-in}, \qquad V_B = -1120\,\text{lb}$$

The maximum combined stress would be at the top of the pipe at the wall where $\sigma = Mc/I = 97{,}100(4.3125)/72.5 = 5575\,\text{lb/in}^2$ and $\tau = Tr/J = 17{,}100(4.3125)/145 = 509\,\text{lb/in}^2$

$$\sigma_{\max} = \frac{5775}{2} + \sqrt{\left(\frac{5775}{2}\right)^2 + 509^2} = 5819\,\text{lb/in}^2$$

2. A hollow steel rectangular beam 4 in wide, 8 in deep, and with 0.1-in wall thickness extends over a loading dock to be used as a crane rail. It is fixed to a warehouse wall at one end and is simply supported on a post at the other. The beam is curved in a horizontal plane with a radius of 15 ft and covers a total angular span of 60°. Calculate the torsional and bending stresses at the wall when a load of 3000 lb is 20° out from the wall. Neglect the weight of the beam.

Solution. The beam has the following properties: $R = 180$ in; $\phi = 60°(\pi/3 \text{ rad})$; $\theta = 40°$; $\phi - \theta = 20°(\pi/9 \text{ rad})$; $I = \frac{1}{12}[4(8^3) - 3.8(7.8^3)] = 20.39 \text{ in}^4$; $K = 2(0.1^2)(7.9^2)(3.9^2)/[8(0.1) + 4(0.1) - 2(0.1^2)] = 16.09 \text{ in}^4$ (see Table 10.1, case 16); $E = 30(10^6)$; $G = 12(10^6)$; and $\beta = 30(10^6)(20.39)/12(10^6)(16.09) = 3.168$. Equations for a curved beam that is fixed at one end and simply supported at the other with a concentrated load are found in Table 9.4, case 1b. To obtain the bending and twisting moments at the wall requires first the evaluation of the end reaction V_A, which, in turn, requires the following constants:

$$C_3 = -3.168\left(\frac{\pi}{3} - \sin 60°\right) - \frac{1 + 3.168}{2}\left(\frac{\pi}{3}\cos 60° - \sin 60°\right) = 0.1397$$

$$C_{a3} = -3.168\left(\frac{\pi}{9} - \sin 20°\right) - C_{a2} = 0.006867$$

Similarly,

$$C_6 = C_1 = 0.3060, \qquad C_{a6} = C_{a1} = 0.05775$$

$$C_9 = C_2 = -0.7136, \qquad C_{a9} = C_{a2} = -0.02919$$

Therefore,

$$V_A = 3000\frac{-0.02919(1 - \cos 60°) - 0.05775 \sin 60° + 0.006867}{-0.7136(1 - \cos 60°) - 0.3060 \sin 60° + 0.1397} = 359.3 \text{ lb}$$

$$M_B = 359.3(180)(\sin 60°) - 3000(180)(\sin 20°) = -128{,}700 \text{ lb-in}$$

$$T_B = 359.3(180)(1 - \cos 60°) - 3000(180)(1 - \cos 20°) = -230 \text{ lb-in}$$

At the wall,

$$\sigma = \frac{Mc}{I} = \frac{128{,}700(4)}{20.39} = 25{,}240 \text{ lb/in}^2$$

$$\tau = \begin{cases} \dfrac{VA'\bar{y}}{Ib} = \dfrac{(3000 - 359.3)[4(4)(2) - 3.9(3.8)(1.95)]}{20.39(0.2)} = 2008 \text{ lb/in}^2 & \text{(due to transverse shear)} \\[2ex] \dfrac{T}{2t(a - t)(b - t)} = \dfrac{230}{2(0.1)(7.9)(3.9)} = 37.3 \text{ lb/in}^2 & \text{(due to torsion)} \end{cases}$$

3. A solid round aluminum bar is in the form of a horizontal closed circular ring of 100-in radius resting on three equally spaced simple supports. A load of 1000 lb is placed midway between two supports, as shown in Fig. 9.13(*a*). Calculate the deflection under this load if the bar is of such diameter as to make the maximum normal stress due to combined bending and torsion equal to 20,000 lb/in^2. Let $E = 10(10^6)$ lb/in^2 and $\nu = 0.3$.

Solution. The reactions R_B, R_C, and R_D are statically determinate, and a solution yields $R_B = -333.3$ lb and $R_C = R_D = 666.7$ lb. The internal bending

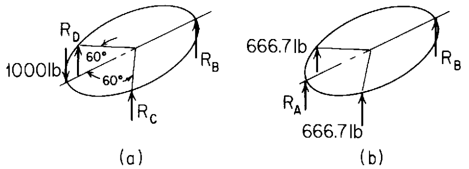


Figure 9.13

and twisting moments are statically indeterminate, and so an energy solution would be appropriate. However, there are several ways that Table 9.4 can be used by superimposing various loadings. The method to be described here is probably the most straightforward.

Consider the equivalent loading shown in Fig. 9.13(*b*), where $R_B = -333.3\,\text{lb}$ and $R_A = -1000\,\text{lb}$. The only difference is in the point of zero deflection. Owing to the symmetry of loading, one-half of the ring can be considered slope-guided at both ends, points A and B. Case 1f gives tabulated values of the necessary coefficients for $\phi = 180^\circ$ and $\theta = 60^\circ$. We can now solve for the following values:

$$V_A = -666.7(0.75) = -500\,\text{lb}$$

$$M_A = -666.7(100)(-0.5774) = 38{,}490\,\text{lb-in}$$

$$\psi_A = \frac{-666.7(100^2)}{EI}(-0.2722) = \frac{1.815(10^6)}{EI}$$

$$T_A = 0 \qquad y_A = 0 \quad \Theta_A = 0$$

$$M_B = -666.7(100)(-0.2887) = 19{,}250\,\text{lb-in}$$

$$M_{60^\circ} = -666.7(100)(0.3608) = -24{,}050\,\text{lb-in}$$

The equations for M and T can now be examined to determine the location of the maximum combined stress:

$$M_x = -50{,}000 \sin x + 38{,}490 \cos x + 66{,}667 \sin(x - 60^\circ)\langle x - 60^\circ\rangle^0$$

$$T_x = -50{,}000(1 - \cos x) + 38{,}490 \sin x + 66{,}667[1 - \cos(x - 60^\circ)]\langle x - 60^\circ\rangle^0$$

A careful examination of the expression for M shows no maximum values except at the ends and at the position of the load The torque, however, has a maximum value of 13,100 in-lb at $x = 37.59^\circ$ and a minimum value of -8790 in-lb at $x = 130.9^\circ$. At these same locations the bending moments are zero. At the position of the load, the torque $T = 8330$ lb-in. Nowhere is the combined stress larger than the bending stress at point A. Therefore,

$$\sigma_A = 20{,}000 = \frac{M_A c}{I} = \frac{38{,}490 d/2}{(\pi/64)d^4} = \frac{392{,}000}{d^3}$$

which gives

$$d = 2.70\,\text{in} \quad \text{and} \quad I = 2.609\,\text{in}^4$$

To obtain the deflection under the 1000-lb load in the original problem, first we must find the deflection at the position of the load of 666.7 lb in Fig. 9.13(*b*). At $x = 60^\circ$,

$$y_x = 0 + 0 + \frac{1.815(10^6)(100)}{10(10^6)2.609}(1 - \cos 60^\circ) + \frac{38{,}490(100^2)}{10(10^6)(2.609)}F_1 + 0 + \frac{-500(100^3)}{10(10^6)(2.609)}F_3$$

where

$$F_1 = \frac{1 + 1.3}{2}\frac{\pi}{3}\sin 60^\circ - 1.3(1 - \cos 60^\circ) = 0.3029 \quad \text{and} \quad F_3 = 0.1583$$

Therefore,

$$y_{60} = 3.478 + 5.796 - 3.033 = 6.24\,\text{in}$$

If the entire ring were now rotated as a rigid body about point B in order to lower points C and D by 6.24 in, point A would be lowered a distance of $6.24(2)/(1 + \cos 60^\circ) = 8.32$ in, which is the downward deflection of the 1000-lb load.

The use of a fictitious support, as was done in this problem at point A, is generalized for asymmeric loadings, both in-plane and out-of-plane, by Barber in Ref. 35.

Flanged sections. The formulas in Table 9.4 for flangeless or compact sections apply also to flanged sections when the ends are fixed as to slope only or when fixed as to slope and roll but not as to flange bending and if the loads are distributed or applied only at the ends. If the flanges are fixed or if concentrated loads are applied within the span, the additional torsional stiffness contributed by the bending resistance of the flanges [*warping restraint* (see Sec. 10.3)] may appreciably affect the value and distribution of twisting and bending moments. The flange stresses caused by the secondary bending or warping may exceed the primary bending stresses. References 15 to 17 and 22 show methods of solution and give some numerical solutions for simple concentrated loads on curved I-beams with both ends fixed completely. Brookhart (Ref. 22) also includes results for additional boundary conditions and uniformly distributed loads. Results are compared with cases where the warping restraint was not considered.

Dabrowski (Ref. 23) gives a thorough presentation of the theory of curved thin-walled beams and works out many examples including multispan beams and beams with open cross sections, closed cross sections, and cross sections which contain both open and closed elements; an extensive bibliography is included. Vlasov (Ref. 27) also gives a very thorough derivation and discusses, among many other topics, vibrations, stability, laterally braced beams of open cross section, and thermal stresses. He also examines the corrections necessary to account for shear deformation in flanges being warped. Verden (Ref. 24) is primarily concerned with multispan curved beams and works out many examples. Sawko and Cope (Ref. 25) and Meyer (Ref. 26) apply finite-element analysis to curved box girder bridges.

9.6 Tables

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve

NOTATION: $R =$ radius of curvature measured to centroid of section; $c =$ distance from centroidal axis to extreme fiber on concave side of beam; $A =$ area of section; $e =$ distance from centroidal axis to neutral axis measured toward center of curvature; $I =$ moment of inertia of cross section about centroidal axis perpendicular to plane of curvature; and $k_i = \sigma_i/\sigma$ and $k_o = \sigma_o/\sigma$ where $\sigma_i =$ actual stress in exteme fiber on concave side, $\sigma_o =$ actual stress in extreme fiber on convex side, and $\sigma =$ fictitious unit stress in corresponding fiber as computed by ordinary flexure formula for a straight beam

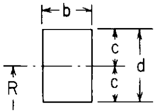
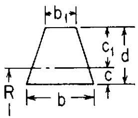
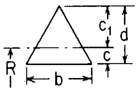
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
1. Solid rectangular section 	$\frac{e}{c} = \frac{R}{c} - \frac{2}{\ln\frac{R/c+1}{R/c-1}}$ (*Note*: e/c, k_i, and k_o are independent of the width b) $k_i = \frac{1}{3e/c}\frac{1-e/c}{R/c-1}$ $k_o = \frac{1}{3e/c}\frac{1+e/c}{R/c+1}$ $\int_{\text{area}} \frac{dA}{r} = b\frac{R/c+1}{R/c-1}$	$\frac{R}{c} =$ 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c} =$ 0.366 0.284 0.236 0.204 0.180 0.115 0.085 0.056 0.042 0.033 $k_i =$ 2.888 2.103 1.798 1.631 1.523 1.288 1.200 1.124 1.090 1.071 $k_o =$ 0.566 0.628 0.671 0.704 0.730 0.810 0.853 0.898 0.922 0.937
2. Trapezoidal section 	$\frac{d}{c} = \frac{3(1+b_1/b)}{1+2b_1/b}$, $\frac{c_1}{c} = \frac{d}{c} - 1$ $\frac{e}{c} = \frac{R}{c} - \frac{\frac{1}{2}(1+b_1/b)(d/c)^2}{\left[\frac{R}{c}+\frac{c_1}{c}-\frac{b_1}{b}\left(\frac{R}{c}-1\right)\right]\ln\left(\frac{R/c+c_1/c}{R/c-1}\right)-\left(1-\frac{b_1}{b}\right)\frac{d}{c}}$ $k_i = \frac{1}{2e/c}\frac{1-e/c}{R/c-1}\frac{1+4b_1/b+(b_1/b)^2}{(1+2b_1/b)^2}$ $k_o = \frac{c_1/c}{2e/c}\frac{c_1/c+h/c}{R/c+c_1/c}\frac{1+4b_1/b+(b_1/b)^2}{(2+b_1/b)^2}$ (*Note*: while $e/c, k_i, k_o$ depend upon the width ratio b_1/b, they are independent of the width b)	(When $b_1/b = \frac{1}{2}$) $\frac{R}{c} =$ 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c} =$ 0.403 0.318 0.267 0.232 0.206 0.134 0.100 0.067 0.050 0.040 $k_i =$ 3.011 2.183 1.859 1.681 1.567 1.314 1.219 1.137 1.100 1.078 $k_o =$ 0.544 0.605 0.648 0.681 0.707 0.790 0.836 0.885 0.911 0.927

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

3. Triangular section, base inward



$$\frac{e}{c}=\frac{R}{c}-\frac{4.5}{\left(\frac{R}{c}+2\right)\ln\left(\frac{R/c+2}{R/c-1}\right)-3}, \quad c=\frac{d}{3}$$

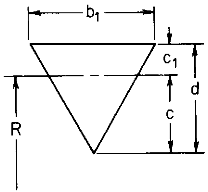
$$k_i=\frac{1}{2e/c}\frac{1-e/c}{R/c-1} \qquad \int_{\text{area}}\frac{dA}{r}=b\left[\left(\frac{R}{3c}+\frac{2}{3}\right)\ln\left(\frac{R/c+2}{R/c-1}\right)-1\right]$$

$$k_o=\frac{1}{4e/c}\frac{2+e/c}{R/c+2}$$

(*Note:* e/c, k_i, and k_o are independent of the width b)

R/c	1.20	1.40	1.60	1.80	2.00	3.00	4.00	6.00	8.00	10.00
e/c	0.434	0.348	0.296	0.259	0.232	0.155	0.117	0.079	0.060	0.048
k_i	3.265	2.345	1.984	1.784	1.656	1.368	1.258	1.163	1.120	1.095
k_o	0.438	0.497	0.539	0.573	0.601	0.697	0.754	0.821	0.859	0.883

4. Triangular section, base outward



$$\frac{e}{c}=\frac{R}{c}-\frac{1.125}{1.5-\left(\frac{R}{c}-1\right)\ln\frac{R/c+0.5}{R/c-1}}, \quad c=\frac{2d}{3}$$

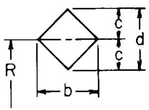
$$k_i=\frac{1}{8e/c}\frac{1-e/c}{R/c-1} \qquad \int_{\text{area}}\frac{dA}{r}=b_1\left[1-\frac{2}{3}\left(\frac{R}{c}-1\right)\ln\frac{R/c+0.5}{R/c-1}\right]$$

$$k_o=\frac{1}{4e/c}\frac{2e/c+1}{2R/c+1}$$

(*Note:* e/c, k_i, and k_o are independent of the width b_1)

R/c	1.20	1.40	1.60	1.80	2.00	3.00	4.00	6.00	8.00	10.00
e/c	0.151	0.117	0.097	0.083	0.073	0.045	0.033	0.022	0.016	0.013
k_i	3.527	2.362	1.947	1.730	1.595	1.313	1.213	1.130	1.094	1.074
k_o	0.636	0.695	0.735	0.765	0.788	0.857	0.892	0.927	0.945	0.956

5. Diamond



$$\frac{e}{c}=\frac{R}{c}-\frac{1}{\frac{R}{c}\ln\left[1-\left(\frac{c}{R}\right)^2\right]+\ln\frac{R/c+1}{R/c-1}}$$

$$k_i=\frac{1}{6e/c}\frac{1-e/c}{R/c-1}$$

$$k_o=\frac{1}{6e/c}\frac{1+e/c}{R/c+1}$$

$$\int_{\text{area}}\frac{dA}{r}=b\left[\frac{R}{c}\ln\left[1-\left(\frac{c}{R}\right)^2\right]+\ln\frac{R/c+1}{R/c-1}\right]$$

(*Note:* e/c, k_i, and k_o are independent of the width b)

R/c	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
e/c	0.175	0.138	0.116	0.100	0.089	0.057	0.042	0.028	0.021	0.017
k_i	3.942	2.599	2.118	1.866	1.709	1.377	1.258	1.159	1.115	1.090
k_o	0.510	0.572	0.617	0.652	0.681	0.772	0.822	0.875	0.904	0.922

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

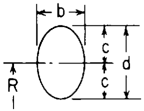
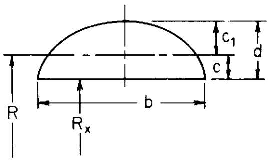
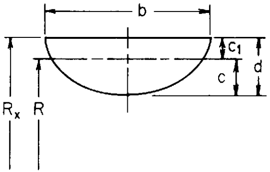
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
6. Solid circular or elliptical section 	$\frac{e}{c}=\frac{1}{2}\left[\frac{R}{c}-\sqrt{\left(\frac{R}{c}\right)^2-1}\right]$ $k_i=\frac{1}{4e/c}\frac{1-e/c}{R/c-1}$ $k_o=\frac{1}{4e/c}\frac{1+e/c}{R/c+1}$, $\int_{\text{area}}\frac{dA}{r}=\pi b\left[\frac{R}{c}-\sqrt{\left(\frac{R}{c}\right)^2-1}\right]$ (*Note:* e/c, k_i, and k_o are independent of the width b)	$\frac{R}{c}=$ 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.000 $\frac{e}{c}=$ 0.268 0.210 0.176 0.152 0.134 0.086 0.064 0.042 0.031 0.025 $k_i=$ 3.408 2.350 1.957 1.748 1.616 1.332 1.229 1.142 1.103 1.080 $k_o=$ 0.537 0.600 0.644 0.678 0.705 0.791 0.837 0.887 0.913 0.929
7. Solid semicircle or semiellipse, base inward  (*Note:* For a semicircle, $b/2=d$)	$R=R_x+c$, $\frac{d}{c}=\frac{3\pi}{4}$ (*Note:* e/c, k_i and k_o are independent of the width b) $k_i=\frac{0.3879}{e/c}\frac{1-e/c}{R/c-1}$ $k_o=\frac{0.2860}{e/c}\frac{e/c+1.3562}{R/c+1.3562}$ For $R_x \geqslant d$: $R/c \geqslant 3.356$ and $\int_{\text{area}}\frac{dA}{r}=\frac{\pi R_x b}{2d}-b-\frac{b}{d}\sqrt{R_x^2-d^2}\left(\frac{\pi}{2}-\sin^{-1}\frac{d}{R_x}\right)$ $\frac{e}{c}=\frac{R}{c}-\frac{(d/c)^2/2}{\frac{R}{c}-2.5-\sqrt{\left(\frac{R}{c}-1\right)^2-\left(\frac{d}{c}\right)^2}\left(1-\frac{2}{\pi}\sin^{-1}\frac{d/c}{R/c-1}\right)}$ For $R_x < d$: $R/c < 3.356$ and $\int_{\text{area}}\frac{dA}{r}=\frac{\pi R_x b}{2d}-b+\frac{b}{d}\sqrt{d^2-R_x^2}\ln\frac{d+\sqrt{d^2-R_x^2}}{R_x}$ $\frac{e}{c}=\frac{R}{c}-\frac{(d/c)^2/2}{\frac{R}{c}-2.5+\frac{2}{\pi}\sqrt{\left(\frac{d}{c}\right)^2-\left(\frac{R}{c}-1\right)^2}\ln\frac{d/c+\sqrt{(d/c)^2-(R/c-1)^2}}{R/c-1}}$	$\frac{R}{c}=$ 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}=$ 0.388 0.305 0.256 0.222 0.197 0.128 0.096 0.064 0.048 0.038 $k_i=$ 3.056 2.209 1.878 1.696 1.579 1.321 1.224 1.140 1.102 1.080 $k_o=$ 0.503 0.565 0.609 0.643 0.671 0.761 0.811 0.867 0.897 0.916

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

8. Solid semicircle or semiellipse, base outward



(*Note:* for a semicircle, $b/2 = d$)

$$R = R_x - c_1, \quad \frac{d}{c} = \frac{3\pi}{3\pi - 4}, \quad \frac{c_1}{c} = \frac{4}{3\pi - 4}$$

(*Note:* e/c, k_i, and k_o are independent of the width b)

$$\int_{\text{area}} \frac{dA}{r} = \frac{\pi R_x b}{2d} + b - \frac{b}{d}\sqrt{R_x^2 - d^2}\left(\frac{\pi}{2} + \sin^{-1}\frac{d}{R_x}\right)$$

$$\frac{e}{c} = \frac{R}{c} - \frac{(d/c)^2/2}{\dfrac{R}{c} + \dfrac{10}{3\pi - 4} - \sqrt{\left(\dfrac{R}{c} + \dfrac{c_1}{c}\right)^2 - \left(\dfrac{d}{c}\right)^2}\left(1 + \dfrac{2}{\pi}\sin^{-1}\dfrac{d/c}{R/c + c_1/c}\right)}$$

$$k_i = \frac{0.2109}{e/c}\,\frac{1 - e/c}{R/c - 1}$$

$$k_o = \frac{0.2860}{e/c}\,\frac{e/c + 0.7374}{R/c + 0.7374}$$

$R/c =$	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
$e/c =$	0.244	0.189	0.157	0.135	0.118	0.075	0.055	0.036	0.027	0.021
$k_i =$	3.264	2.262	1.892	1.695	1.571	1.306	1.210	1.130	1.094	1.073
$k_o =$	0.593	0.656	0.698	0.730	0.755	0.832	0.871	0.912	0.933	0.946

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

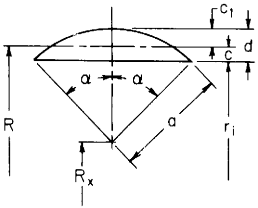
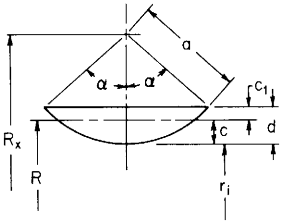
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
9. Segment of a solid circle, base inward  *Note:* $R, c, e/c, k_i$, and k_o are all independent of the width of the segment provided all horizontal elements of the segment change width proportionately. Therefore to use these expressions for a segment of an ellipse, find the circle which has the same radial dimensions R_x, r_i, and d and evaluate $e/c, k_i$, and k_o which have the same values for both the circle and ellipse. To find dA/r for the ellipse, multiply the value of dA/r for the circle by the ratio of the horizontal to vertical semiaxes for the ellipse. See the example.	$R = R_x + c + a\cos\alpha$ $\frac{a}{c} = \frac{3\alpha - 3\sin\alpha\cos\alpha}{3\sin\alpha - 3\alpha\cos\alpha - \sin^3\alpha}$, $\quad \frac{c_1}{c} = \frac{3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha}{3\sin\alpha - 3\alpha\cos\alpha - \sin^3\alpha}$ $k_i = \frac{I}{Ac^2}\frac{1}{e/c}\frac{1-e/c}{R/c-1}$ where expressions for I and A are found in Table A.1, case 19 $k_o = \frac{I}{Ac^2}\frac{1}{(e/c)(c_1/c)}\frac{e/c+c_1/c}{R/c+c_1 c}$ For $R_x \geqslant a : R/c \geqslant (a/c)(1+\cos\alpha)+1$ and $\int_{\text{area}}\frac{dA}{r} = 2R_x\alpha - 2a\sin\alpha - 2\sqrt{R_x^2-a^2}\left(\frac{\pi}{2} - \sin^{-1}\frac{a+R_x\cos\alpha}{R_x+a\cos\alpha}\right)$ $\frac{e}{c} = \frac{R}{c} - \frac{(\alpha-\sin\alpha\cos\alpha)a/c}{\frac{2\alpha R_x}{a} - 2\sin\alpha - 2\sqrt{\left(\frac{R_x}{a}\right)^2-1}\left(\frac{\pi}{2} - \sin^{-1}\frac{1+(R_x/a)\cos\alpha}{R_x/a+\cos\alpha}\right)}$ (*Note:* Values of $\sin^{-1}$ between $-\pi/2$ and $\pi/2$ are to be taken in above expressions.) For $R_x < a$: $R/c < (a/c)(1+\cos\alpha)+1$ and $\int_{\text{area}}\frac{dA}{r} = 2R_x\alpha - 2a\sin\alpha + 2\sqrt{a^2-R_x^2}\ln\frac{\sqrt{a^2-R_x^2}\sin\alpha + a + R_x\cos\alpha}{R_x+a\cos\alpha}$ $\frac{e}{c} = \frac{R}{c}$ $- \frac{(\alpha-\sin\alpha\cos\alpha)a/c}{\frac{2\alpha R_x}{a} - 2\sin\alpha + 2\sqrt{1-\left(\frac{R_x}{a}\right)^2}\ln\frac{\sqrt{1-(R_x/a)^2}\sin\alpha + 1 + (R_x/a)\cos\alpha}{R_x/a+\cos\alpha}}$	For $\alpha = 60^\circ$: $\frac{R}{c}$ = 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}$ = 0.401 0.317 0.266 0.232 0.206 0.134 0.101 0.067 0.051 0.041 k_i = 3.079 2.225 1.891 1.707 1.589 1.327 1.228 1.143 1.104 1.082 k_o = 0.498 0.560 0.603 0.638 0.665 0.755 0.806 0.862 0.893 0.913 For $\alpha = 30^\circ$: $\frac{R}{c}$ = 1.200 1.400 1.600 1.800 2.000 3.000 4.000 6.000 8.000 10.000 $\frac{e}{c}$ = 0.407 0.322 0.271 0.236 0.210 0.138 0.103 0.069 0.052 0.042 k_i = 3.096 2.237 1.900 1.715 1.596 1.331 1.231 1.145 1.106 1.083 k_o = 0.495 0.556 0.600 0.634 0.662 0.752 0.803 0.860 0.891 0.911

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

10. Segment of a solid circle, base outward



Note: R, c, e/c, k_i, and k_o are all independent of the width of the segment provided all horizontal elements of the segment change width proportionately. To use these expressions for a segment of an ellipse, refer to the explanation in case 9.

$$R = R_x + c - a$$

$$\frac{a}{c} = \frac{3\alpha - 3\sin\alpha\cos\alpha}{3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha}, \qquad \frac{c_1}{c} = \frac{3\sin\alpha - 3\alpha\cos\alpha - \sin^3\alpha}{3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha}$$

$$k_i = \frac{I}{Ac^2}\frac{1}{e/c}\frac{1 - e/c}{R/c - 1}$$

where expressions for I and A are found in Table A.1, case 19

$$k_o = \frac{I}{Ac^2}\frac{1}{(e/c)(c_1/c)}\frac{e/c + c_1/c}{R/c + c_1/c}$$

$$\int_{\text{area}} \frac{dA}{r} = 2R_x\alpha + 2a\sin\alpha - 2\sqrt{R_x^2 - a^2}\left(\frac{\pi}{2} + \sin^{-1}\frac{a - R_x\cos\alpha}{R_x - a\cos\alpha}\right)$$

$$\frac{e}{c} = \frac{R}{c} - \frac{(\alpha - \sin\alpha\cos\alpha)a/c}{\dfrac{2\alpha R_x}{a} + 2\sin\alpha - 2\sqrt{\left(\dfrac{R_x}{a}\right)^2 - 1}\left(\dfrac{\pi}{2} + \sin^{-1}\dfrac{1 - (R_x/a)\cos\alpha}{R_x/a - \cos\alpha}\right)}$$

(*Note:* Values of $\sin^{-1}$ between $-\pi/2$ and $\pi/2$ are to be taken in above expressions.)

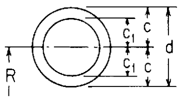
For $\alpha = 60°$:

$\frac{R}{c} =$	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
$\frac{e}{c} =$	0.235	0.181	0.150	0.129	0.113	0.071	0.052	0.034	0.025	0.020
$k_i =$	3.241	2.247	1.881	1.686	1.563	1.301	1.207	1.127	1.092	1.072
$k_o =$	0.598	0.661	0.703	0.735	0.760	0.836	0.874	0.914	0.935	0.948

For $\alpha = 30°$:

$\frac{R}{c} =$	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
$\frac{e}{c} =$	0.230	0.177	0.146	0.125	0.110	0.069	0.051	0.033	0.025	0.020
$k_i =$	3.232	2.241	1.876	1.682	1.560	1.299	1.205	1.126	1.091	1.072
$k_o =$	0.601	0.663	0.706	0.737	0.763	0.838	0.876	0.916	0.936	0.948

11. Hollow circular section



$$\frac{e}{c} = \frac{1}{2}\left[\frac{2R}{c} - \sqrt{\left(\frac{R}{c}\right)^2 - 1} - \sqrt{\left(\frac{R}{c}\right)^2 - \left(\frac{c_1}{c}\right)^2}\right]$$

$$k_i = \frac{1}{4e/c}\frac{1 - e/c}{R/c - 1}\left[1 + \left(\frac{c_1}{c}\right)^2\right]$$

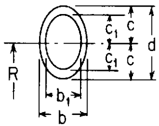
$$k_o = \frac{1}{4e/c}\frac{1 + e/c}{R/c + 1}\left[1 + \left(\frac{c_1}{c}\right)^2\right]$$

(*Note:* For thin-walled tubes the discussion on page 277 should be considered)

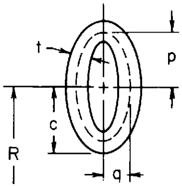
(When $c_1/c = \frac{1}{2}$)

$\frac{R}{c} =$	1.20	1.40	1.60	1.80	2.00	3.00	4.00	6.00	8.00	10.00
$k_i =$	3.276	2.267	1.895	1.697	1.573	1.307	1.211	1.130	1.094	1.074
$\frac{e}{c} =$	0.323	0.256	0.216	0.187	0.166	0.107	0.079	0.052	0.039	0.031
$k_o =$	0.582	0.638	0.678	0.708	0.733	0.810	0.852	0.897	0.921	0.936

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
12. Hollow elliptical section 12a. Inner and outer perimeters are ellipses, wall thickness is not constant 	$\frac{e}{c} = \frac{R}{c} - \frac{\frac{1}{2}[1-(b_1/b)(c_1/c)]}{\frac{R}{c} - \sqrt{\left(\frac{R}{c}\right)^2 - 1} - \frac{b_1/b}{c_1/c}\left[\frac{R}{c} - \sqrt{\left(\frac{R}{c}\right)^2 - \left(\frac{c_1}{c}\right)^2}\right]}$ $k_i = \frac{1}{4e/c}\frac{1-e/c}{R/c-1}\frac{1-(b_1/b)(c_1/c)^3}{1-(b_1/b)(c_1/c)}$ $k_o = \frac{1}{4e/c}\frac{1+e/c}{R/c+1}\frac{1-(b_1/b)(c_1/c)^3}{1-(b_1/b)(c_1/c)}$ (*Note:* While $e/c, k_i$, and k_o depend upon the width ratio b_1/b, they are independent of the width b)	(When $b_1/b = \frac{3}{5}$, $c_1/c = \frac{4}{5}$) $\frac{R}{c}$ = 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c}$ = 0.345 0.279 0.233 0.202 0.178 0.114 0.085 0.056 0.042 0.034 k_i = 3.033 2.154 1.825 1.648 1.535 1.291 1.202 1.125 1.083 1.063 k_o = 0.579 0.637 0.677 0.709 0.734 0.812 0.854 0.899 0.916 0.930

12b. Constant wall thickness, midthickness perimeter is an ellipse (shown dashed)



Note: There is a limit on the maximum wall thickness allowed in this case. Cusps will form on the inner perimeter at the ends of the major axis if this maximum is exceeded. If $p/q \leqslant 1$, then $t_{max} = 2p^2/q$. If $p/q \geqslant 1$, then $t_{max} = 2q^2/p$

There is no closed-form solution for this case, so numerical solutions were run for the ranges $1.2 < R/c < 5$; $0 < t < t_{max}$; $0.2 < p/q < 5$. Results are expressed below in terms of the solutions for case 12a for which $c = p + t/2$, $c_1 = p - t/2$, $b = 2q + t$, and $b_1 = 2q - t$.

$\frac{e}{c} = K_1\left(\frac{e}{c}\text{ from case 12a}\right)$, $k_i = K_2(k_i$ from case 12a)

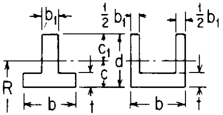
$k_o = K_3(k_o$ from case 12a)

where K_1, K_2, and K_3 are given in the following table and are essentially independent of t and R/c.

p/q	0.200	0.333	0.500	0.625	1.000	1.600	2.000	3.000	4.000	5.000
K_1	0.965	0.985	0.995	0.998	1.000	1.002	1.007	1.027	1.051	1.073
K_2	1.017	1.005	1.002	1.001	1.000	1.000	1.000	0.998	0.992	0.985
K_3	0.982	0.992	0.998	0.999	1.000	1.002	1.004	1.014	1.024	1.031

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

13. T-beam or channel section



$$\frac{d}{c}=\frac{2[b_1/b+(1-b_1/b)(t/d)]}{b_1/b+(1-b_1/b)(t/d)^2}, \qquad \frac{c_1}{c}=\frac{d}{c}-1$$

$$\frac{e}{c}=\frac{R}{c}-\frac{(d/c)[b_1/b+(1-b_1/b)(t/d)]}{\dfrac{b_1}{b}\ln\dfrac{d/c+R/c-1}{(d/c)(t/d)+R/c-1}+\ln\dfrac{(d/c)(t/d)+R/c-1}{R/c-1}}$$

$$k_i=\frac{I_c}{Ac^2(R/c-1)}\frac{1-e/c}{e/c}$$

$$\text{where } \frac{I_c}{Ac^2}=\frac{1}{3}\left(\frac{d}{c}\right)^2\left[\frac{b_1/b+(1-b_1/b)(t/d)^3}{b_1/b+(1-b_1/b)(t/d)}\right]-1$$

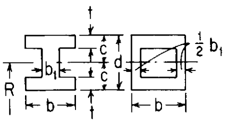
$$k_o=\frac{I_c}{Ac^2(e/c)}\frac{d/c+e/c-1}{R/c+d/c-1}\frac{1}{d/c-1}$$

(*Note:* While e/c, k_i, and k_o depend upon the width ratio b_1/b, they are independent of the width b)

(When $b_1/b=\frac{1}{4}$, $t/d=\frac{1}{4}$)

R/c	1.200	1.400	1.600	1.800	2.000	3.000	4.000	6.000	8.000	10.000
e/c	0.502	0.419	0.366	0.328	0.297	0.207	0.160	0.111	0.085	0.069
k_i	3.633	2.538	2.112	1.879	1.731	1.403	1.281	1.176	1.128	1.101
k_o	0.583	0.634	0.670	0.697	0.719	0.791	0.832	0.879	0.905	0.922

14. Symmetrical I-beam or hollow rectangular section



$$\frac{e}{c}=\frac{R}{c}-\frac{2[t/c+(1-t/c)(b_1/b)]}{\ln\dfrac{R/c^2+(R/c+1)(t/c)-1}{(R/c)^2-(R/c-1)(t/c)-1}+\dfrac{b_1}{b}\ln\dfrac{R/c-t/c+1}{R/c+t/c-1}}$$

$$k_i=\frac{I_c}{Ac^2(R/c-1)}\frac{1-e/c}{e/c}$$

$$\text{where } \frac{I_c}{Ac^2}=\frac{1}{3}\frac{1-(1-b_1/b)(1-t/c)^3}{1-(1-b_1/b)(1-t/c)}$$

$$k_o=\frac{I_c}{Ac^2(R/c+1)}\frac{1+e/c}{e/c}$$

(*Note:* While e/c, k_i, and k_o depend upon the width ratio b_1/b, they are independent of the width b)

(When $b_1/b=\frac{1}{3}$, $t/d=\frac{1}{6}$)

R/c	1.20	1.40	1.60	1.80	2.00	3.00	4.00	6.00	8.00	10.00
e/c	0.489	0.391	0.330	0.287	0.254	0.164	0.122	0.081	0.060	0.048
k_i	2.156	1.876	1.630	1.496	1.411	1.225	1.156	1.097	1.071	1.055
k_o	0.666	0.714	0.747	0.771	0.791	0.853	0.886	0.921	0.940	0.951

TABLE 9.1 Formulas for curved beams subjected to bending in the plane of the curve (*Continued*)

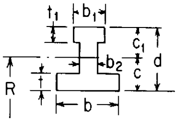
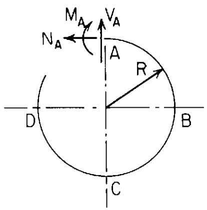
Form and dimensions of cross section, reference no.	Formulas	Values of $\frac{e}{c}$, k_i, and k_o for various values of $\frac{R}{c}$
15. Unsymmetrical I-beam section 	$A = bd[b_1/b + (1 - b_2/b)(t/d) - (b_1/b - b_2/b)(1 - t_1/d)]$ $\frac{d}{c} = \frac{2A/bd}{(b_1/b - b_2/b)(2 - t_1/d)(t_1/d) + (1 - b_2/b)(t/d)^2 + b_2/b}$ $\frac{e}{c} = \frac{R}{c} - \frac{(A/bd)(d/c)}{\ln \frac{R/c + t/c - 1}{R/c - 1} + \frac{b_2}{b} \ln \frac{R/c + c_1/c - t_1/c}{R/c + t/c - 1} + \frac{b_1}{b} \ln \frac{R/c + c_1/c}{R/c + c_1/c - t_1/c}}$ $k_i = \frac{I_c}{Ac^2(R/c - 1)} \frac{1 - e/c}{e/c}$ where $\frac{I_c}{Ac^2} = \frac{1}{3}\left(\frac{d}{c}\right)^2 \left[\frac{b_1/b + (1 - b_2/b)(t/d)^3 - (b_1/b - b_2/b)(1 - t_1/d)^3}{b_1/b + (1 - b_2/b)(t/d) - (b_1/b - b_2/b)(1 - t_1/d)}\right] - 1$ $k_o = \frac{I_c}{Ac^2(e/c)} \frac{d/c + e/c - 1}{R/c + d/c - 1} \frac{1}{d/c - 1}$ (*Note:* While e/c, k_i, and k_o depend upon the width ratios b_1/b and b_2/b, they are independent of the width b)	(When $b_1/b = \frac{2}{3}$, $b_2/b = \frac{1}{6}$, $t_1/d = \frac{1}{6}$, $t/d = \frac{1}{3}$) $\frac{R}{c} =$ 1.20 1.40 1.60 1.80 2.00 3.00 4.00 6.00 8.00 10.00 $\frac{e}{c} =$ 0.491 0.409 0.356 0.318 0.288 0.200 0.154 0.106 0.081 0.066 $k_i =$ 3.589 2.504 2.083 1.853 1.706 1.385 1.266 1.165 1.120 1.094 $k_o =$ 0.671 0.721 0.754 0.779 0.798 0.856 0.887 0.921 0.938 0.950

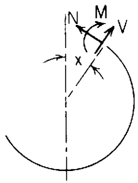
TABLE 9.2 Formulas for circular rings

NOTATION: W = load (force); w and v = unit loads (force per unit of circumferential length); ρ = unit weight of contained liquid (force per unit volume); M_o = applied couple (force-length). M_A, M_B, M_C, and M are internal moments at A, B, C, and x, respectively, positive as shown. N_A, N, V_A, and V are internal forces, positive as shown. E = modulus of elasticity (force per unit area); ν = Poisson's ratio; A = cross-sectional area (length squared); R = radius to the centroid of the cross section (length); I = area moment of inertia of ring cross section about the principal axis perpendicular to the plane of the ring (length^4). [Note that for a pipe or cylinder, a representative segment of unit axial length may be used by replacing EI by $Et^3/12(1-\nu^2)$.] e = positive distance measured radially inward from the centroidal axis of the cross section to the neutral axis of pure bending (see Sec. 9.1). θ, x, and ϕ are angles (radians) and are limited to the range zero to π for all cases except 18 and 19; $s = \sin\theta, c = \cos\theta$, $z = \sin x$, $u = \cos x$, $n = \sin\phi$, and $m = \cos\phi$.

ΔD_V and ΔD_H are changes in the vertical and horizontal diameters, respectively, and an increase is positive. ΔL is the change in the lower half of the vertical diameter or the vertical motion relative to point C of a line connecting points B and D on the ring. Similarly ΔL_W is the vertical motion relative to point C of a horizontal line connecting the load points on the ring. ΔL_{WH} is the change in length of a horizontal line connecting the load points on the ring. ψ is the angular rotation (radians) of the load point in the plane of the ring and is positive in the direction of positive θ. For the distributed loadings the load points just referred to are the points where the distributed loading starts, i.e., the position located by the angle θ. The reference to points A, B, and C and to the diameters refer to positions on a circle of radius R passing through the centroids of the several sections; i.e., diameter $= 2R$. It is important to consider this when dealing with thick rings. Similarly, all concentrated and distributed loadings are assumed to be applied at the radial position of the centroid with the exception of the cases where the ring is loaded by its own weight or by dynamic loading, cases 15 and 21. In these two cases the actual radial distribution of load is considered. If the loading is on the outer or inner surfaces of thick rings, an equivalent loading at the centroidal radius R must be used. See the examples to determine how this might be accomplished.

The hoop-stress deformation factor is $\alpha = I/AR^2$ for thin rings or $\alpha = e/R$ for thick rings. The transverse (radial) shear deformation factor is $\beta = FEI/GAR^2$ for thin rings or $\beta = 2F(1+\nu)e/R$ for thick rings, where G is the shear modulus of elasticity and F is a shape factor for the cross section (see Sec. 8.10). The following constants are defined to simplify the expressions whcih follow. Note that these constants are unity if no correction for hoop stress or shear stress is necessary or desired for use with thin rings. $k_1 = 1 - \alpha + \beta$, $k_2 = 1 - \alpha$.





General formulas for moment, hoop load, and radial shear

$$M = M_A - N_A R(1-u) + V_A Rz + LT_M$$
$$N = N_A u + V_A z + LT_N$$
$$V = -N_A z + V_A u + LT_V$$

where LT_M, LT_N, and LT_V are load terms given below for several types of load.

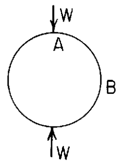
Note: Due to symmetry in most of the cases presented, the loads beyond 180° are not included in the load terms. Only for cases 16, 17, and 19 should the equations for M, N, and V be used beyond 180°.

Note: The use of the bracket $\langle x-\theta\rangle^0$ is explained on page 131 and has a value of zero unless $x > \theta$

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

1.



$LT_M = \dfrac{-WRz}{2} \quad LT_N = \dfrac{-Wz}{2}$

$LT_v = \dfrac{-Wu}{2}$

$$M_A = \frac{WRk_2}{\pi}$$

$$N_A = 0$$

$$V_A = 0$$

$$\Delta D_H = \frac{WR^3}{EI}\left(\frac{k_1}{2} - k_2 + \frac{2k_2^2}{\pi}\right)$$

$$\Delta D_V = \frac{-WR^3}{EI}\left(\frac{\pi k_1}{4} - \frac{2k_2^2}{\pi}\right)$$

$\text{Max} + M = M_A = 0.3183WRk_2$

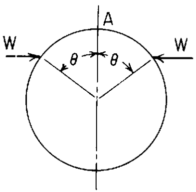
$\text{Max} - M = M_B = -(0.5 - 0.3183k_2)WR$

If $\alpha = \beta = 0$,

$$\Delta D_H = 0.1366\frac{WR^3}{EI} \quad \text{and} \quad \Delta D_V = -0.1488\frac{WR^3}{EI}$$

Note: For concentrated loads on thick-walled rings, study the material in Sec. 14.3 on hollow pins and rollers. Radial stresses under the concentrated loads have a significant effect not considered here.

2.



$LT_M = -WR(c-u)\langle x-\theta\rangle^0$

$LT_N = Wu\langle x-\theta\rangle^0$

$LT_V = -Wz\langle x-\theta\rangle^0$

$$M_A = \frac{-WR}{\pi}[(\pi-\theta)(1-c) - s(k_2 - c)]$$

$$M_C = \frac{-WR}{\pi}[\theta(1+c) - s(k_2 + c)]$$

$$N_A = \frac{-W}{\pi}[\pi - \theta + sc]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-WR^3}{EI\pi}[0.5\pi k_1(\theta - sc) + 2k_2\theta c - 2k_2^2 s] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-WR^3}{EI\pi}[0.5\pi k_1(\pi - \theta + sc) - 2k_2(\pi-\theta)c - 2k_2^2 s] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI}\left[\frac{k_1 s^2}{2} - k_2\left(1 - c + \frac{2\theta c}{\pi}\right) + \frac{2k_2^2 s}{\pi}\right]$$

$$\Delta L = \begin{cases} \dfrac{WR^3}{EI}\left[\dfrac{\theta c}{2} + \dfrac{k_1(\theta - sc)}{2\pi} - k_2\left(\dfrac{\theta c}{\pi} + \dfrac{s}{2}\right) + \dfrac{k_2^2 s}{\pi}\right] & \text{if } \theta \leqslant \pi/2 \\ \dfrac{WR^3}{EI}\left[\dfrac{(\pi-\theta)c}{2} + \dfrac{k_1(\theta - sc - \pi c^2)}{2\pi} - k_2\left(1 + \dfrac{\theta c}{\pi} - \dfrac{s}{2}\right) + k_2^2 s/\pi\right] & \text{if } \theta \geqslant \pi/2 \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI\pi}[(\pi-\theta)\theta sc + 0.5k_1 s^2(\theta - sc) + k_2(2\theta s^2 - \pi s^2 - \theta c - \theta) + k_2^2 s(1+c)]$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[(\pi-\theta)2\theta c^2 - k_1(\pi sc + s^2c^2 - 2\theta sc - \pi\theta + \theta^2) - 2k_2 sc(\pi - 2\theta) - 2k_2^2 s^2]$$

$$\Delta\psi = \frac{-WR^2}{EI\pi}[(\pi-\theta)\theta c - k_2 s(sc + \pi - 2\theta)]$$

$$\text{Max} + M = \frac{WRs(k_2 - c^2)}{\pi} \quad \text{at } x = \theta$$

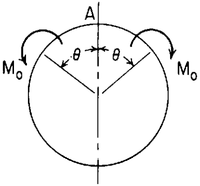
$$\text{Max} - M = \begin{cases} M_A & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ M_C & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D} WR^3/EI$, $\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	45°	60°
K_{M_A}	−0.0903	−0.1538	−0.1955
K_{M_θ}	0.0398	0.1125	0.2068
K_{N_A}	−0.9712	−0.9092	−0.8045
$K_{\Delta D_H}$	−0.0157	−0.0461	−0.0891
$K_{\Delta D_V}$	0.0207	0.0537	0.0930
$K_{\Delta L}$	0.0060	0.0179	0.0355
$K_{\Delta L_W}$	0.0119	0.0247	0.0391
$K_{\Delta L_{WH}}$	−0.0060	−0.0302	−0.0770
$K_{\Delta\psi}$	0.0244	0.0496	0.0590

TABLE 9.2 Formulas for circular rings (*Continued*)

3.



$LT_M = M_o\langle x-\theta\rangle^0$
$LT_N = 0$
$LT_V = 0$

$$M_A = \frac{-M_o}{\pi}\left(\pi-\theta-\frac{2sk_2}{k_1}\right)$$

$$M_C = \frac{M_o}{\pi}\left(\theta-\frac{2sk_2}{k_1}\right)$$

$$N_A = \frac{M_o}{R\pi}\left(\frac{2sk_2}{k_1}\right)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{M_oR^2}{EI}k_2\left(\dfrac{2\theta}{\pi}-s\right) & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{M_oR^2}{EI}k_2\left(\dfrac{2\theta}{\pi}-2+s\right) & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{M_oR^2}{EI}k_2\left(\frac{2\theta}{\pi}-1+c\right)$$

$$\Delta L = \begin{cases} \dfrac{-M_oR^2}{EI}\left[\dfrac{\theta}{2}-\dfrac{k_2(\theta+s)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-M_oR^2}{EI}\left[\dfrac{\pi-\theta}{2}-\dfrac{k_2(\theta+s+\pi c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{-M_oR^2}{EI\pi}[(\pi-\theta)\theta s-k_2(s^3+\theta+\theta c)]$$

$$\Delta L_{WH} = \frac{M_oR^2}{EI\pi}[2\theta c(\pi-\theta)+2k_2s(2\theta-\pi-sc)]$$

$$\Delta\psi = \frac{M_oR}{EI\pi}\left[\theta(\pi-\theta)-\frac{2s^2k_2^2}{k_1}\right]$$

$\text{Max} + M = \dfrac{M_o}{\pi}\left(\theta+\dfrac{2sck_2}{k_1}\right)$ at x just greater than θ

$\text{Max} - M = \dfrac{-M_o}{\pi}\left(\pi-\theta-\dfrac{2sck_2}{k_1}\right)$ at x just less than θ

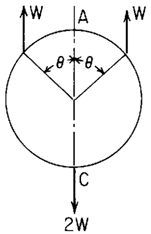
If $\alpha=\beta=0$, $M=K_MM_o$, $N=K_NM_o/R$, $\Delta D=K_{\Delta D}M_oR^2/EI$, $\Delta\psi=K_{\Delta\psi}M_oR/EI$, etc.

θ	30°	45°	60°	90°
K_{M_A}	−0.5150	−0.2998	−0.1153	0.1366
K_{N_A}	0.3183	0.4502	0.5513	0.6366
K_{M_θ}	−0.5577	−0.4317	−0.3910	−0.5000
$K_{\Delta D_H}$	−0.1667	−0.2071	−0.1994	0.0000
$K_{\Delta D_V}$	0.1994	0.2071	0.1667	0.0000
$K_{\Delta L}$	0.0640	0.0824	0.0854	0.0329
$K_{\Delta L_W}$	0.1326	0.1228	0.1022	0.0329
$K_{\Delta L_{WH}}$	−0.0488	−0.0992	−0.1180	0.0000
$K_{\Delta\psi}$	0.2772	0.2707	0.2207	0.1488

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

4.



$LT_M = WR(z - s)\langle x - \theta\rangle^0$
$LT_N = Wz\langle x - \theta\rangle^0$
$LT_V = Wu\langle x - \theta\rangle^0$

$$M_A = \frac{-WR}{\pi}[s(s - \pi + \theta) + k_2(1 + c)]$$

$$M_C = \frac{-WR}{\pi}[s\theta - s^2 + k_2(1 + c)]$$

$$N_A = \frac{-W}{\pi}s^2$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-WR^3}{EI\pi}\left[\pi k_1\left(1 - \dfrac{s^2}{2}\right) - 2k_2(\pi - \theta s) + 2k_2^2(1 + c)\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-WR^3}{EI\pi}\left[\dfrac{\pi k_1 s^2}{2} - 2sk_2(\pi - \theta) + 2k_2^2(1 + c)\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI\pi}\left[\frac{\pi k_1(\pi - \theta - sc)}{2} + k_2 s(\pi - 2\theta) - 2k_2^2(1 + c)\right]$$

$$\Delta L = \begin{cases} \dfrac{WR^3}{2EI}\left[\theta s + k_1\left(\dfrac{\pi}{2} - \dfrac{s^2}{\pi}\right) - k_2\left(1 - c + \dfrac{2\theta s}{\pi}\right) - \dfrac{2k_2^2(1 + c)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{WR^3}{2EI}\left[s(\pi - \theta) + k_1\left(\pi - \theta - sc - \dfrac{s^2}{\pi}\right) - k_2\left(1 + c + \dfrac{2\theta s}{\pi}\right) - \dfrac{2k_2^2(1 + c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI\pi}\left[\theta s^2(\pi - \theta) + \frac{\pi k_1(\pi - \theta - sc - s^4/\pi)}{2} + k_2 s(\pi c - 2\theta - 2\theta c) - k_2^2(1 + c)^2\right]$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[2\theta sc(\pi - \theta) + k_1 s^2(\theta - sc) - 2k_2(\pi s^2 - \theta s^2 + \theta c + \theta c^2) + 2k_2^2 s(1 + c)]$$

$$\Delta\psi = \frac{WR^2}{EI\pi}[-\theta s(\pi - \theta) + k_2(\theta + \theta c + s^3)]$$

Max $+M$ occurs at an angular position $x_1 = \tan^{-1}\dfrac{-\pi}{s^2}$ if $\theta < 106.3^\circ$

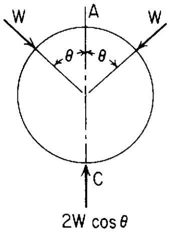
Max $+M$ occurs at the load if $\theta \geqslant 106.3^\circ$

Max $-M = M_C$

If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D} WR^3/EI$, $\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	−0.2569	−0.1389	−0.1366	−0.1092	−0.0389
K_{N_A}	−0.0796	−0.2387	−0.3183	−0.2387	−0.0796
K_{M_C}	−0.5977	−0.5274	−0.5000	−0.4978	−0.3797
K_{M_θ}	−0.2462	−0.0195	0.1817	0.2489	0.1096
$K_{\Delta D_H}$	−0.2296	−0.1573	−0.1366	−0.1160	−0.0436
$K_{\Delta D_V}$	0.2379	0.1644	0.1488	0.1331	0.0597
$K_{\Delta L}$	0.1322	0.1033	0.0933	0.0877	0.0431
$K_{\Delta L_W}$	0.2053	0.1156	0.0933	0.0842	0.0271
$K_{\Delta L_{WH}}$	−0.0237	−0.0782	−0.1366	−0.1078	−0.0176
$K_{\Delta\psi}$	0.1326	0.1022	0.0329	−0.0645	−0.0667

5.



$LT_M = -WR\sin(x-\theta)\langle x-\theta\rangle^0$
$LT_N = -W\sin(x-\theta)\langle x-\theta\rangle^0$
$LT_V = -W\cos(x-\theta)\langle x-\theta\rangle^0$

$$M_A = \frac{-WR}{\pi}[s(\pi-\theta) - k_2(1+c)]$$

$$M_C = \frac{-WR}{\pi}[s\theta - k_2(1+c)]$$

$$N_A = \frac{-W}{\pi}s(\pi-\theta)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-WR^3}{EI}\left[k_1\left(\dfrac{\theta s}{2} - c\right) + 2k_2c - \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-WR^3}{EI}\left[\dfrac{k_1 s(\pi-\theta)}{2} - \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{WR^3}{EI}\left[\frac{k_1(s-\pi c+\theta c)}{2} - k_2 s + \frac{2k_2^2(1+c)}{\pi}\right]$$

$$\Delta L = \begin{cases} \dfrac{WR^3}{2EI}\left[k_1\left(\dfrac{\theta s}{\pi} - \dfrac{\pi c}{2}\right) - k_2(1-c) + \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{WR^3}{2EI}\left[k_1\left(\dfrac{\theta s}{\pi} - \pi c + \theta c\right) + k_2(1+c-2s) + \dfrac{2k_2^2(1+c)}{\pi}\right] & \text{if } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta L_W = \frac{WR^3}{EI}\left\{k_1\frac{s - s^3(1-\theta/\pi) - c(\pi-\theta)}{2} + k_2\left[\frac{\theta s(1+c)}{\pi} - s\right] + \frac{k_2^2(1+c)^2}{\pi}\right\}$$

$$\Delta L_{WH} = \frac{-WR^3}{EI\pi}[k_1 s(\pi-\theta)(\theta - sc) + 2\theta c k_2(1+c) - 2sk_2^2(1+c)]$$

$$\Delta\psi = \frac{WR^2}{EI\pi}[\pi s^2 - \theta(1+c+s^2)]k_2$$

Max $+M = M_C$ if $\theta \leqslant 60^\circ$

Max $+M$ occurs at the load if $\theta > 60^\circ$ where

$$M_\theta = \frac{WR}{\pi}[k_2(1+c) - sc(\pi-\theta)]$$

$$\text{Max } -M = \begin{cases} M_C & \text{if } \theta \geqslant 90^\circ \\ M_A & \text{if } 60^\circ \leqslant \theta \leqslant 90^\circ \end{cases}$$

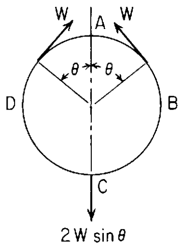
Max $-M$ occurs at an angular

position $x_1 = \tan^{-1}\dfrac{-\pi c}{\theta s}$ if $\theta \leqslant 60^\circ$

If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D} WR^3/EI$, $\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	0.1773	−0.0999	−0.1817	−0.1295	−0.0407
K_{N_A}	−0.4167	−0.5774	−0.5000	−0.2887	−0.0833
K_{M_C}	0.5106	0.1888	−0.1817	−0.4182	−0.3740
K_{M_θ}	0.2331	0.1888	0.3183	0.3035	0.1148
$K_{\Delta D_H}$	0.1910	0.0015	−0.1488	−0.1351	−0.0456
$K_{\Delta D_V}$	−0.1957	−0.0017	0.1366	0.1471	0.0620
$K_{\Delta L}$	−0.1115	−0.0209	0.0683	0.0936	0.0447
$K_{\Delta L_W}$	−0.1718	−0.0239	0.0683	0.0888	0.0278
$K_{\Delta L_{WH}}$	0.0176	−0.0276	−0.1488	−0.1206	− 0.0182
$K_{\Delta\psi}$	−0.1027	0.0000	0.0000	0.0833	−0.0700

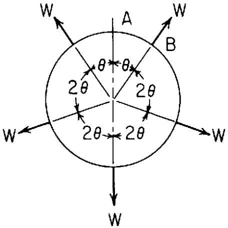
TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
6.  $LT_M = -WR[1-\cos(x-\theta)]\langle x-\theta\rangle^0$ $LT_N = W\cos(x-\theta)\langle x-\theta\rangle^0$ $LT_V = -W\sin(x-\theta)\langle x-\theta\rangle^0$	$M_A = \frac{-WR}{\pi}[s(1+k_2)-(\pi-\theta)(1-c)]$ $M_C = \frac{-WR}{\pi}[s(k_2-1)+\theta(1+c)]$ $N_A = \frac{-W}{\pi}[s+(\pi-\theta)c]$ $V_A = 0$ $\Delta D_H = \begin{cases} \frac{-WR^3}{EI}\left[\frac{k_1(s+\theta c)}{2} - 2k_2\left(s-\frac{\theta}{\pi}\right)+\frac{2k_2^2 s}{\pi}\right] & \text{if } \theta \leqslant \frac{\pi}{2} \\ \frac{-WR^3}{EI}\left[\frac{k_1(s+\pi c-\theta c)}{2} - 2k_2\left(1-\frac{\theta}{\pi}\right)+\frac{2k_2^2 s}{\pi}\right] & \text{if } \theta \geqslant \frac{\pi}{2} \end{cases}$ $\Delta D_V = \frac{WR^3}{EI}\left[\frac{k_1 s(\pi-\theta)}{2} + k_2\left(1-c-\frac{2\theta}{\pi}\right) - \frac{2k_2^2 s}{\pi}\right]$ $\Delta L = \begin{cases} \frac{WR^3}{EI}\left[\frac{\theta}{2}+\frac{k_1(\pi^2 s+2\theta c-2s)}{4\pi} - k_2\left(\frac{s}{2}+\frac{\theta}{\pi}\right)-\frac{k_2^2 s}{\pi}\right] & \text{if } \theta \leqslant \frac{\pi}{2} \\ \frac{WR^3}{EI}\left[\frac{\pi}{2}-\frac{\theta}{2}+\frac{k_1(\pi s-\theta s+\theta c/\pi - s/\pi - c)}{2} - k_2\left(\frac{\theta}{\pi}+\frac{s}{2}+c\right)-\frac{k_2^2 s}{\pi}\right] & \text{if } \theta \geqslant \frac{\pi}{2} \end{cases}$ $\Delta L_W = \frac{WR^3}{EI\pi}\left[\theta s(\pi-\theta)+\frac{k_1 s(\theta sc - s^2 - sc\pi + \pi^2 - \theta\pi)}{2} - k_2\theta(1+s^2+c) - k_2^2 s(1+c)\right]$ $\Delta L_{WH} = \frac{-WR^3}{EI\pi}[2\theta c(\pi-\theta) - k_1(sc^2\pi - 2\theta sc^2 + s^2 c - \theta c\pi + \theta^2 c - \theta s^3) - 2k_2 s(\pi-\theta+\theta c) + 2k_2^2 s^2]$ $\Delta\psi = \frac{-WR^2}{EI\pi}[\theta(\pi-\theta) - k_2 s(\theta+s+\pi c-\theta c)]$ $\text{Max} + M = \frac{WR}{\pi}[\pi s\sin x_1 - (s-\theta c)\cos x_1 - k_2 s - \theta]$ at an angular position $x_1 = \tan^{-1}\frac{-\pi s}{s-\theta c}$ (*Note*: $x_1 > \theta$ and $x_1 > \pi/2$) $\text{Max} - M = M_C$ If $\alpha=\beta=0$, $M = K_M WR$, $N = K_N W$, $\Delta D = K_{\Delta D} WR^3/EI$, $\Delta\psi = K_{\Delta\psi} WR^2/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	−0.2067	−0.2180	−0.1366	−0.0513	−0.0073
K_{N_A}	−0.8808	−0.6090	−0.3183	−0.1090	−0.0148
K_{M_C}	−0.3110	−0.5000	−0.5000	−0.3333	−0.1117
$K_{\Delta D_H}$	−0.1284	−0.1808	−0.1366	−0.0559	−0.0083
$K_{\Delta D_V}$	0.1368	0.1889	0.1488	0.0688	0.0120
$K_{\Delta L}$	0.0713	0.1073	0.0933	0.0472	0.0088
$K_{\Delta L_W}$	0.1129	0.1196	0.0933	0.0460	0.0059
$K_{\Delta L_{WH}}$	−0.0170	−0.1063	−0.1366	−0.0548	− 0.0036
$K_{\Delta\psi}$	0.0874	0.1180	0.0329	−0.0264	−0.0123

TABLE 9.2 Formulas for circular rings (*Continued*)

7. Ring under any number of equal radial forces equally spaced



For $0 < x < \theta$ $\quad M = \dfrac{WR(u/s - k_2/\theta)}{2} \qquad N = \dfrac{Wu}{2s} \qquad V = \dfrac{-Wz}{2s}$

$\text{Max} + M = M_A = \dfrac{WR(1/s - k_2/\theta)}{2} \qquad \text{Max} - M = \dfrac{-WR}{2}\left(\dfrac{k_2}{\theta} - \dfrac{c}{s}\right)$ at each load position

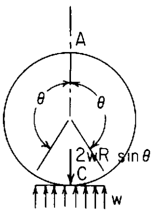
Radial displacement at each load point $= \Delta R_B = \dfrac{WR^3}{EI}\left[\dfrac{k_1(\theta - sc)}{4s^2} + \dfrac{k_2 c}{2s} - \dfrac{k_2^2}{2\theta}\right]$

Radial displacement at $x = 0, 2\theta, \ldots = \Delta R_A = \dfrac{-WR^3}{EI}\left[\dfrac{k_1(s - \theta c)}{4s^2} - \dfrac{k_2}{2s} + \dfrac{k_2^2}{2\theta}\right]$

If $\alpha = \beta = 0$, $M = K_M WR$, $\Delta R = K_{\Delta R} WR^3/EI$

θ	15°	30°	45°	60°	90°
K_{M_A}	0.02199	0.04507	0.07049	0.09989	0.18169
K_{M_B}	−0.04383	−0.08890	−0.13662	−0.18879	−0.31831
$K_{\Delta R_B}$	0.00020	0.00168	0.00608	0.01594	0.07439
$K_{\Delta R_A}$	−0.00018	−0.00148	−0.00539	−0.01426	−0.06831

8.



(*Note*: $\theta \geqslant \dfrac{\pi}{2}$)

$LT_M = \dfrac{-wR^2}{2}(z - s)^2\langle x - \theta\rangle^0$

$LT_N = -wRz(z - s)\langle x - \theta\rangle^0$

$LT_V = -wRu(z - s)\langle x - \theta\rangle^0$

$$M_A = \frac{wR^2}{2\pi}\left[\pi(s^2 - 0.5) - \frac{sc - \theta}{2} - s^2\left(\theta + \frac{2s}{3}\right) - k_2(2s + sc - \pi + \theta)\right]$$

$$M_C = \frac{-wR^2}{2\pi}\left[\frac{\pi}{2} + \frac{sc}{2} - \frac{\theta}{2} + \theta s^2 - \frac{2s^3}{3} + k_2(2s + sc - \pi + \theta)\right]$$

$$N_A = \frac{-wRs^3}{3\pi}$$

$$V_A = 0$$

$$\Delta D_H = \frac{-wR^4}{2EI\pi}\left[\frac{k_1\pi s^3}{3} + k_2(\pi - 2\pi s^2 - \theta + 2\theta s^2 + sc) + 2k_2^2(2s + sc - \pi + \theta)\right]$$

$$\Delta D_V = \frac{wR^4}{2EI\pi}\left[k_1\pi\left(\pi s - \theta s - \tfrac{2}{3} - c + \frac{c^3}{3}\right) - k_2(\pi c^2 + sc - \theta + 2\theta s^2) - 2k_2^2(2s + sc - \pi + \theta)\right]$$

$$\Delta L = \frac{wR^4}{4EI\pi}\left[\pi(\pi - \theta)\frac{2s^2 - 1}{2} - \frac{\pi sc}{2} - 2\pi k_1\left(\tfrac{2}{3} - \pi s + c + \theta s - \frac{c^3}{3} + \frac{s^3}{3\pi}\right) - k_2(sc + \pi - \theta + 2\theta s^2 + 2s\pi - \pi^2 + \pi\theta + \pi sc) - 2k_2^2(2s + sc - \pi + \theta)\right]$$

Max $+ M$ occurs at an angular position x_1 where $x_1 > \theta$, $x_1 > 123.1°$, and

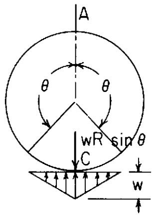
$$\tan x_1 + \frac{3\pi(s - \sin x_1)}{s^3} = 0$$

Max $- M = M_C$

If $\alpha = \beta = 0$, $M = K_M wR^2$, $N = K_N wR$, $\Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	90°	120°	135°	150°
K_{M_A}	−0.0494	−0.0329	−0.0182	−0.0065
K_{N_A}	−0.1061	−0.0689	−0.0375	−0.0133
K_{M_C}	−0.3372	−0.2700	−0.1932	−0.1050
$K_{\Delta D_H}$	−0.0533	−0.0362	−0.0204	−0.0074
$K_{\Delta D_V}$	0.0655	0.0464	0.0276	0.0108
$K_{\Delta L}$	0.0448	0.0325	0.0198	0.0080

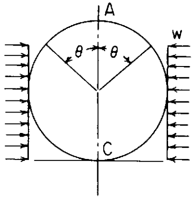
TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
9.  $\left(Note\colon\ \theta \geqslant \frac{\pi}{2}\right)$ $LT_M = \frac{wR^2}{6s}(z-s)^3\langle x-\theta\rangle^0$ $LT_N = \frac{wRz}{s^2}(z-s)^2\langle x-\theta\rangle^0$ $LT_V = \frac{wRu}{2s}(z-s)^2\langle x-\theta\rangle^0$	$M_A = \frac{wR^2}{36\pi s}\{(\pi-\theta)(6s^3-9s)-3s^4+8+8c-5s^2c-6k_2[3s(s-\pi+\theta)+s^2c+2+2c]\}$ $M_C = \frac{-wR^2}{36\pi s}\{9s(\pi-\theta)+6\theta s^3-3s^4-8-8c+5s^2c+6k_2[3s(s-\pi+\theta)+s^2c+2+2c]\}$ $N_A = \frac{-wRs^3}{12\pi}$ $V_A = 0$ $\Delta D_H = \frac{-wR^4}{18EI\pi}\left\{\frac{3k_1\pi s^3}{4} - k_2\left[(\pi-\theta)(6s^2-9)+\frac{8(1+c)}{s}-5sc\right]+6k_2^2\left[sc+\frac{2(1+c)}{s}-3(\pi-\theta-s)\right]\right\}$ $\Delta D_V = \frac{wR^4}{18EI\pi}\left\{18\pi k_1\left[\left(\frac{s}{4}+\frac{1}{16s}\right)(\pi-\theta)-\frac{13c}{48}-\frac{s^2c}{24}-\frac{1}{3}\right]+k_2\left[(\pi-\theta)(3s^2-9)-3s^2\theta+\frac{8(1+c)}{s}-5sc\right]-6k_2^2\left[sc+\frac{2(1+c)}{s}-3(\pi-\theta-s)\right]\right\}$ $\Delta L = \frac{wR^4}{EI}\left[(\pi-\theta)\left(\frac{s^2}{12}-\frac{1}{8}\right)+\frac{1+c}{9s}-\frac{5sc}{72}+k_1\frac{(\pi-\theta)(12s+3/s)-13c-2s^2c-16-2s^3/\pi}{48} - k_2\frac{(1+c)(2\pi-\frac{8}{3})/s-3(\pi-\theta)(\pi-1)+2\theta s^2+3\pi s+sc(\pi+\frac{5}{3})}{12\pi}-k_2^2\frac{3(s-\pi+\theta)+2(1+c)s+sc}{6\pi}\right]$ Max $+M$ occurs at an angular position x_1 where $x_1>\theta, x_1>131.1^\circ$, and $\tan x_1+\frac{6\pi(s-\sin x_1)^2}{s^4}=0$ Max $-M=M_C$ If $\alpha=\beta=0, M=K_MwR^2, N=K_NwR, \Delta D=K_{\Delta D}wR^4/EI$, etc.

θ	90°	120°	135°	150°
K_{M_A}	−0.0127	−0.0084	−0.0046	−0.0016
K_{N_A}	−0.0265	−0.0172	−0.0094	−0.0033
K_{M_C}	−0.1263	−0.0989	−0.0692	−0.0367
$K_{\Delta D_H}$	−0.0141	−0.0093	−0.0052	−0.0019
$K_{\Delta D_V}$	0.0185	0.0127	0.0074	0.0028
$K_{\Delta L}$	0.0131	0.0092	0.0054	0.0021

TABLE 9.2 Formulas for circular rings (*Continued*)

10.



$$LT_M = \frac{-wR^2}{2}(c-u)^2\langle x-\theta\rangle^0$$
$$LT_N = wRu(c-u)\langle x-\theta\rangle^0$$
$$LT_V = -wRz(c-u)\langle x-\theta\rangle^0$$

$$M_A = \frac{-wR^2}{4\pi}\left[(\pi-\theta)(4c+2s^2-1)+s\left(4-\frac{4s^2}{3}-c\right)-2k_2(\pi-\theta+sc)\right]$$

$$M_C = \frac{-wR^2}{4\pi}\left[3\pi+\theta+4\theta c-2\theta s^2-4s-sc+\frac{4s^3}{3}-2k_2(\pi-\theta+sc)\right]$$

$$N_A = \frac{-wR}{\pi}\left(\pi c+s-\theta c-\frac{s^3}{3}\right)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{6EI\pi}[\pi k_1(s^3+3\theta c+4-3s)+3k_2(\pi-\theta+2\theta c^2-sc)-6k_2^2(\pi-\theta+sc)] & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{2EI\pi}\left\{\pi k_1\left[c(\pi-\theta)+s-\dfrac{s^3}{3}\right]+k_2[(\pi-\theta)(2s^2-1)-sc]-2k_2^2(\pi-\theta+sc)\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{3EI\pi}\{\pi k_1(2-c^3+3c)+3k_2[2\theta s^2-\theta+sc-\pi(1+2c+s^2)]+6k_2^2(\pi-\theta+sc)\}$$

$$\Delta L = \begin{cases} \dfrac{wR^4}{12EI\pi}[1.5\pi(\theta-2\theta s^2-sc)+2k_1(2\pi+s^3+3\theta c-3s)+3k_2(sc+\theta\pi+2\theta s^2-3\pi-\theta-\pi sc)+6k_2^2(\pi-\theta+sc)] & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{12EI\pi}\{1.5\pi[(\pi-\theta)(1-2s^2)+sc]+2k_1(2\pi+s^3+3\theta c-3s-\pi c^3)+3k_2[(\pi+1)(\pi-\theta+sc)+2\theta s^2-4\pi(1+c)]+6k_2^2(\pi-\theta+sc)\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 90^\circ$, and $x_1 = \cos^{-1}\dfrac{s^3/3+\theta c-s}{\pi}$

Max $-M = M_C$

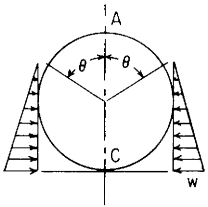
If $\alpha = \beta = 0, M = K_M wR^2, N = K_N wR, \Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	0°	30°	45°	60°	90°	120°	135°	150°
K_{M_A}	−0.2500	−0.2434	−0.2235	−0.1867	−0.0872	−0.0185	−0.0052	−0.00076
K_{N_A}	−1.0000	−0.8676	−0.7179	−0.5401	−0.2122	−0.0401	−0.0108	−0.00155
K_{M_C}	−0.2500	−0.2492	−0.2448	−0.2315	−0.1628	−0.0633	−0.0265	−0.00663
$K_{\Delta D_H}$	−0.1667	−0.1658	−0.1610	−0.1470	−0.0833	−0.0197	−0.0057	−0.00086
$K_{\Delta D_V}$	0.1667	0.1655	0.1596	0.1443	0.0833	0.0224	0.0071	0.00118
$K_{\Delta L}$	0.0833	0.0830	0.0812	0.0756	0.0486	0.0147	0.0049	0.00086

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

11.



$$LT_M = \frac{-wR^2}{6(1+c)}(c-u)^3\langle x-\theta\rangle^0$$

$$LT_N = \frac{wRu}{2(1+c)}(c-u)^2\langle x-\theta\rangle^0$$

$$LT_V = \frac{-wRz}{2(1+c)}(c-u)^2\langle x-\theta\rangle^0$$

$$M_A = \frac{-wR^2}{(1+c)\pi}\left[(\pi-\theta)\frac{3+12c^2+2c+4cs^2}{24} - \frac{3s^3c-3s-5s^3}{36} + \frac{5sc}{8} - k_2\left(\frac{\pi c}{2}-\frac{\theta c}{2}+\frac{s^3}{3}+\frac{sc^2}{2}\right)\right]$$

$$M_C = \frac{-wR^2}{(1+c)\pi}\left[(\pi-\theta)\frac{-3-12c^2+2c+4cs^2}{24} + \frac{\pi(1+c)^3}{6} + \frac{3s^3c+3s+5s^3}{36} - \frac{5sc}{8} - k_2\left(\frac{\pi c}{2}-\frac{\theta c}{2}+\frac{s^3}{3}+\frac{sc^2}{2}\right)\right]$$

$$N_A = \frac{-wR}{(1+c)\pi}\left[(\pi-\theta)\frac{1+4c^2}{8} + \frac{5sc}{8} - \frac{s^3c}{12}\right]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{EI(1+c)\pi}\left\{\pi k_1\left(\dfrac{\theta+4\theta c^2-5sc}{16}+\dfrac{s^3c+16c}{24}\right) - k_2\left(\dfrac{5sc^2+3\theta c+6\theta s^2c-8s}{18}-\dfrac{\pi c}{2}\right) - k_2^2\left[c(\pi-\theta)+\dfrac{2s^3}{3}+sc^2\right]\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{EI(1+c)\pi}\left\{\pi k_1\left[(\pi-\theta)\dfrac{1+4c^2}{16}+\dfrac{5sc}{16}-\dfrac{s^3c}{24}\right] - k_2\left[\dfrac{5sc^2-8s}{18}-(\pi-\theta)\dfrac{c+2s^2c}{6}\right] - k_2^2\left[c(\pi-\theta)+s-\dfrac{s^3}{3}\right]\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{EI(1+c)}\left\{k_1\left[\frac{(1+c)^2}{6}-\frac{s^4}{24}\right] + k_2\left(\frac{5sc^2+3\theta c+6\theta s^2c-8s}{18\pi}+\frac{s^2}{2}+\frac{c^3}{6}-c-\frac{2}{3}\right) + k_2^2\frac{c(\pi-\theta)+s-s^3/3}{\pi}\right\}$$

$$\Delta L = \begin{cases} \dfrac{wR^4}{EI(1+c)}\left\{\dfrac{3s+5s^3+6\theta c^3-9\theta c-16}{72} + k_1\left(\dfrac{c}{3}+\dfrac{1}{16}+\dfrac{12\theta c^2+3\theta+2s^3c-15sc}{48\pi}\right) + k_2\left[\dfrac{1-s^3}{6}-\dfrac{c(3+sc-\theta)}{4}+\dfrac{3\theta c+6\theta s^2c-3s-5s^3}{36\pi}\right] + k_2^2\dfrac{c(\pi-\theta)/2+sc^2/2+s^3/3}{\pi}\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{EI(1+c)}\left\{\dfrac{-(\pi-\theta)c(1+2s^2)}{24}-\dfrac{sc^2}{24}-\dfrac{s^3}{9} + k_1\left(\dfrac{c}{3}+\dfrac{1}{16}-\dfrac{c^4}{24}+\dfrac{12\theta c^2+3\theta+2s^3c-15sc}{48\pi}\right) + k_2\left[\dfrac{2s-2+sc^2}{12}+\dfrac{c(\pi-\theta-3-2c)}{4}+\dfrac{3\theta c+6\theta s^2c-3s-5s^3}{36\pi}\right] + k_2^2\dfrac{3c(\pi-\theta)+2s+sc^2}{6\pi}\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

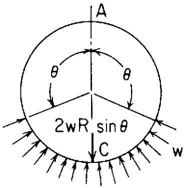
Max $+M$ occurs at an angular position x_1 where $x_1 > \theta$, $x_1 > 96.8^\circ$, and $x_1 = \arccos\left\{c - \left[(c^2+0.25)\left(1-\dfrac{\theta}{\pi}\right)+\dfrac{sc(5-2s^2/3)}{4\pi}\right]^{1/2}\right\}$

Max $-M = M_C$

If $\alpha = \beta = 0$, $M = K_M wR^2$, $N = K_N wR$, $\Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	0°	30°	45°	60°	90°	120°	135°	150°
K_{M_A}	−0.1042	−0.0939	−0.0808	−0.0635	−0.0271	− 0.0055	−0.0015	−0.00022
K_{N_A}	−0.3125	−0.2679	−0.2191	−0.1628	−0.0625	− 0.0116	−0.0031	−0.00045
K_{M_C}	−0.1458	−0.1384	−0.1282	−0.1129	−0.0688	−0.0239	−0.0096	−0.00232
$K_{\Delta D_H}$	−0.8333	−0.0774	−0.0693	−0.0575	−0.0274	− 0.0059	−0.0017	−0.00025
$K_{\Delta D_V}$	0.0833	0.0774	0.0694	0.0579	0.0291	0.0071	0.0022	0.00035
$K_{\Delta L}$	0.0451	0.0424	0.0387	0.0332	0.0180	0.0048	0.0015	0.00026

12.



$LT_M = -wR^2[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

$LT_N = -wR[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

$LT_V = -wR\sin(x-\theta)\langle x-\theta\rangle^0$

$$M_A = \frac{-wR^2}{\pi}[s+\pi c-\theta c-k_2(\pi-\theta-s)]$$

$$M_C = \frac{-wR^2}{\pi}[\pi-s+\theta c-k_2(\pi-\theta-s)]$$

$$N_A = \frac{-wR}{\pi}(s+\pi c-\theta c)$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{EI}\left[\dfrac{k_1(s+\theta c)}{2}+2k_2(1-s)-\dfrac{2k_2^2(\pi-\theta-s)}{\pi}\right] & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{EI}\left[\dfrac{k_1(s+\pi c-\theta c)}{2}-\dfrac{2k_2^2(\pi-\theta-s)}{\pi}\right] & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{EI}\left[\frac{k_1 s(\pi-\theta)}{2}-k_2(1+c)+\frac{2k_2^2(\pi-\theta-s)}{\pi}\right]$$

$$\Delta L = \begin{cases} \dfrac{wR^4}{2EI\pi}\left[k_1\left(\dfrac{\pi^2 s}{2}-s+\theta c\right)+k_2\pi(\theta-s-2)+2k_2^2(\pi-\theta-s)\right] & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{2EI\pi}[k_1(\pi^2 s-\pi\theta s-\pi c-s+\theta c)+k_2\pi(\pi-\theta-s-2-2c)+2k_2^2(\pi-\theta-s)] & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 90^\circ$, and $x_1 = \tan^{-1}\dfrac{-\pi s}{s-\theta c}$

Max $-M = M_C$

If $\alpha=\beta=0$, $M = K_M wR^2$, $N = K_N wR$, $\Delta D = K_{\Delta D} wR^4/EI$, etc.

θ	30°	60°	90°	120°	150°
K_{M_A}	−0.2067	−0.2180	−0.1366	−0.0513	−0.0073
K_{N_A}	−0.8808	−0.6090	−0.3183	−0.1090	−0.0148
K_{M_C}	−0.3110	−0.5000	−0.5000	−0.3333	−0.1117
$K_{\Delta D_H}$	−0.1284	−0.1808	−0.1366	−0.0559	−0.0083
$K_{\Delta D_V}$	0.1368	0.1889	0.1488	0.0688	0.0120
$K_{\Delta L}$	0.0713	0.1073	0.0933	0.0472	0.0088

TABLE 9.2 Formulas for circular rings (*Continued*)

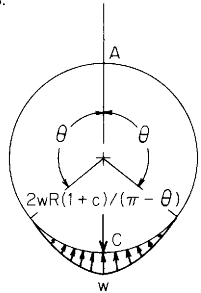
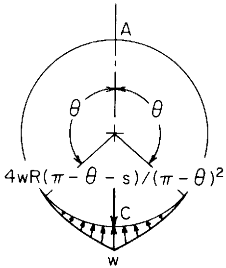
Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
13.  The radial pressure w_x varies linearly with x from 0 at $x=\theta$ to w at $x=\pi$. $LT_M = \frac{-wR^2}{\pi-\theta}(x-\theta-zc+us)\langle x-\theta\rangle^0$ $LT_N = \frac{-wR}{\pi-\theta}(x-\theta-zc+us)\langle x-\theta\rangle^0$ $LT_V = \frac{-wR}{\pi-\theta}(1-uc-zs)\langle x-\theta\rangle^0$	$M_A = \frac{-wR^2}{\pi(\pi-\theta)}\left\{2+2c-s(\pi-\theta)+k_2\left[1+c-\frac{(\pi-\theta)^2}{2}\right]\right\}$ $M_C = \frac{-wR^2}{\pi(\pi-\theta)}\left\{\pi(\pi-\theta)-2-2c-s\theta+k_2\left[1+c-\frac{(\pi-\theta)^2}{2}\right]\right\}$ $N_A = \frac{-wR}{\pi(\pi-\theta)}[2+2c-s(\pi-\theta)]$ $V_A = 0$ Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 103.7^\circ$, and x_1 is found from $\left(1+c+\frac{s\theta}{2}\right)\sin x_1 + c\cos x_1 - 1 = 0$ Max $-M = M_C$ $\Delta D_H = \begin{cases} \frac{-wR^4}{EI(\pi-\theta)}\left\{k_1\left(1-\frac{s\theta}{2}\right)+k_2(\pi-2\theta-2c)+k_2^2\frac{2+2c-(\pi-\theta)^2}{\pi}\right\} & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{-wR^4}{EI(\pi-\theta)}\left\{k_1\left[1+c-\frac{s(\pi-\theta)}{2}\right]+k_2^2\frac{2+2c-(\pi-\theta)^2}{\pi}\right\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$ $\Delta D_V = \frac{wR^4}{EI(\pi-\theta)}\left\{k_1\frac{s+c(\pi-\theta)}{2}-k_2(\pi-\theta-s)-k_2^2\frac{2+2c-(\pi-\theta)^2}{\pi}\right\}$ $\Delta L = \begin{cases} \frac{wR^4}{2EI\pi(\pi-\theta)}\left\{k_1\left(\frac{\pi^2 c}{2}-2c+2\pi-2-\theta s\right)-k_2\pi\left[2(\pi-\theta)-1+c-\frac{\pi^2}{4}+\frac{\theta^2}{2}\right]-k_2^2[2+2c-\pi-\theta)^2\right\} & \text{for } \theta \leqslant \frac{\pi}{2} \\ \frac{wR^4}{2EI\pi(\pi-\theta)}\left\{k_1[\pi c(\pi-\theta)+2\pi s-2c-2-\theta s]-k_2\pi\left[2(\pi-\theta)+1+c-2s-\frac{(\pi-\theta)^2}{2}\right]-k_2^2[2+2c-(\pi-\theta)^2]\right\} & \text{for } \theta \geqslant \frac{\pi}{2} \end{cases}$

TABLE 9.2 Formulas for circular rings (*Continued*)

14.



The radial pressure w_x varies with $(x-\theta)^2$ from 0 at $x=\theta$ to w at $x=\pi$

$$LT_M = \frac{-wR^2}{(\pi-\theta)^2}[(x-\theta)^2 - 2 + 2uc + 2zs]\langle x-\theta\rangle^0$$

$$LT_N = \frac{-wR}{(\pi-\theta)^2}[(x-\theta)^2 - 2 + 2uc + 2zs]\langle x-\theta\rangle^0$$

$$LT_V = \frac{-2wR}{(\pi-\theta)^2}(x-\theta-zc+us) \times \langle x-\theta\rangle^0$$

$$M_A = \frac{-wR^2}{\pi(\pi-\theta)^2}\left\{2(\pi-\theta)(2-c) - 6s + k_2\left[2(\pi-\theta-s) - \frac{(\pi-\theta)^3}{3}\right]\right\}$$

$$M_C = \frac{-wR^2}{\pi(\pi-\theta)^2}\left\{2\theta(2-c) + 6s - 6\pi + \pi(\pi-\theta)^2 + k_2\left[2(\pi-\theta-s) - \frac{(\pi-\theta)^3}{3}\right]\right\}$$

$$N_A = \frac{-wR}{\pi(\pi-\theta)^2}[2(\pi-\theta)(2-c) - 6s]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{-wR^4}{EI(\pi-\theta)^2}\left\{k_1\left(\dfrac{\pi^2}{4} - 6 - \theta^2 s + 3s - 3\theta c + \dfrac{3\pi}{2} + c + \theta s - 2\theta\right) + k_2\left(\dfrac{\pi^2}{2} - 4 + 4s - 2\theta\pi + 2\theta^2\right) - 2k_2^2\dfrac{(\pi-\theta)^3 - 6(\pi-\theta-s)}{3\pi}\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-wR^4}{EI(\pi-\theta)^2}\left\{k_1[(2-c)(\pi-\theta) - 3s] - 2k_2^2\dfrac{(\pi-\theta)^3 - 6(\pi-\theta-s)}{3\pi}\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{wR^4}{EI(\pi-\theta)^2}\left\{k_1[2+2c-s(\pi-\theta)] + k_2(2+2c-(\pi-\theta)^2] + 2k_2^2\frac{(\pi-\theta)^3 - 6(\pi-\theta-s)}{3\pi}\right\}$$

Max $+M$ occurs at an angular position x_1 where $x_1 > \theta, x_1 > 108.6^\circ$, and x_1 is found from

$$(x_1 - \theta + s\cos x_1) + (3s - 2\pi + 2\theta - \theta c)\sin x_1 = 0$$

Max $-M = M_C$

TABLE 9.2 Formulas for circular rings (*Continued*)

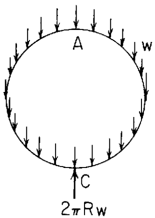
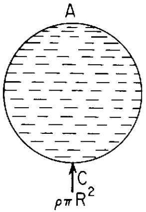
Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
14. Continued	$\Delta L = \begin{cases} \dfrac{wR^4}{EI(\pi-\theta)^2}\left\{k_1 \dfrac{3s+2\theta-\theta c}{\pi} + \pi - 2\theta - \dfrac{\pi s}{2}\right) + k_2\left[2+s-\theta+\dfrac{\pi^3}{8}+\dfrac{\theta^3}{6}-\dfrac{\theta\pi^2}{4}-(\pi-\theta)^2\right] + k_2^2\dfrac{(\pi-\theta)^3-6(\pi-\theta-s)}{3\pi}\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{wR^4}{EI(\pi-\theta)^2}\left\{k_1\left[\dfrac{3s+2\theta-\theta c}{\pi} - s(\pi-\theta)+3c\right] + k_2\left[\dfrac{(\pi-\theta)^3}{6}-(\pi-\theta)^2-\pi+\theta+s+2+2c\right] + k_2^2\dfrac{(\pi-\theta)^3-6(\pi-\theta-s)}{3\pi}\right\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$
15. Ring supported at base and loaded by own weight per unit length of circumference w  $LT_M = -wR^2[xz + K_T(u-1)]$ $LT_N = -wRxz$ $LT_V = -wRxu$	$M_A = wR^2\left[k_2 - 0.5 - \dfrac{(K_T-1)\beta}{k_1}\right]$ where $K_T = 1 + \dfrac{I}{AR^2}$ $M_C = wR^2\left[k_2 + 0.5 + \dfrac{(K_T-1)\beta}{k_1}\right]$ $N_A = wR\left[0.5 + \dfrac{(K_T-1)k_2}{k_1}\right]$ $V_A = 0$ $\Delta D_H = \dfrac{wR^3}{EAe}\left(\dfrac{k_1\pi}{2} - k_2\pi + 2k_2^2\right)$ $\Delta D_V = \dfrac{-wR^3}{EAe}\left(\dfrac{k_1\pi^2}{4} - 2k_2^2\right)$ $\Delta L = \dfrac{-wR^3}{EAe}\left[1 + \dfrac{3k_1\pi^2}{16} - \dfrac{k_2\pi}{2} - k_2^2 + (K_T-1)\alpha\right]$ *Note:* The constant K_T accounts for the radial distribution of mass in the ring. Max $+M = M_C$ Max $-M$ occurs at an angular position x_1 where $\dfrac{x_1}{\tan x_1} = -0.5 + \dfrac{(K_T-1)\beta}{k_1}$ For a thin ring where $K_T \approx 1$, Max $-M = -wR^2(1.6408 - k_2)$ at $x = 105.23^\circ$ If $\alpha = \beta = 0$, $M_A = \dfrac{wR^2}{2}$ $N_A = \dfrac{wR}{2}$ $\Delta D_H = 0.4292\dfrac{wR^4}{EI}$ $\Delta D_V = -0.4674\dfrac{wR^4}{EI}$ $\Delta L = -0.2798\dfrac{wR^4}{EI}$ Max $+M = \frac{3}{2}wR^2$ at C

TABLE 9.2 Formulas for circular rings (*Continued*)

16. Unit axial segment of pipe filled with liquid of weight per unit volume ρ and supported at the base



$$LT_M = \rho R^3\left(1 - u - \frac{xz}{2}\right)$$

$$LT_N = \rho R^2\left(1 - u - \frac{xz}{2}\right)$$

$$LT_V = \rho R^2\left(\frac{z}{2} - \frac{xu}{2}\right)$$

$$M_A = \rho R^3\left(0.75 - \frac{k_2}{2}\right)$$

$$M_C = \rho R^3\left(1.25 - \frac{k_2}{2}\right)$$

$$N_A = 0.75\rho R^2$$

$$V_A = 0$$

$$\Delta D_H = \frac{\rho R^5 12(1-\nu^2)}{Et^3}\left[\frac{k_1\pi}{4} + k_2\left(2 - \frac{\pi}{2}\right) - k_2^2\right]$$

$$\Delta D_V = \frac{-\rho R^5 12(1-\nu^2)}{Et^3}\left(\frac{k_1\pi^2}{8} - 2k_2 + k_2^2\right)$$

$$\Delta L = \frac{-\rho R^5 12(1-\nu^2)}{Et^3}\left[\frac{k_1 3\pi^2}{32} - k_2\left(0.5 + \frac{\pi}{4}\right) + \frac{k_2^2}{2}\right]$$

Note: For this case and case 17,

$$\alpha = \frac{t^2}{12R^2(1-\nu^2)}$$

$$\beta = \frac{t^2}{6R^2(1-\nu)} \quad \text{where } t = \text{pipe wall thickness}$$

Max $+M = M_C$

Max $-M = -\rho R^3\left(\frac{k_2}{2} - 0.1796\right)$ at $x = 105.23^\circ$

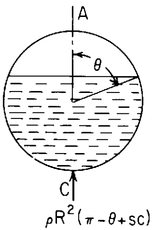
If $\alpha = \beta = 0$,

$$\Delta D_H = 0.2146\frac{\rho R^5 12(1-\nu^2)}{Et^3}$$

$$\Delta D_V = -0.2337\frac{\rho R^5 12(1-\nu^2)}{Et^3}$$

$$\Delta L = -0.1399\frac{\rho R^5 12(1-\nu^2)}{Et^3}$$

17. Unit axial segment of pipe partly filled with liquid of weight per unit volume ρ and supported at the base



Note: see case 16 for expressions for α and β

$$M_A = \frac{\rho R^3}{4\pi}\{2\theta s^2 + 3sc - 3\theta + \pi + 2\pi c^2 + 2k_2[sc - 2s + (\pi-\theta)(1-2c)]\}$$

$$N_A = \frac{\rho R^2}{4\pi}[3sc + (\pi-\theta)(1+2c^2)]$$

$$V_A = 0$$

$$\Delta D_H = \begin{cases} \dfrac{\rho R^5 3(1-\nu^2)}{2Et^3\pi}\left\{k_1\pi(sc + 2\pi - 3\theta + 2\theta c^2) + 8k_2\pi\left(2c - sc - \dfrac{\pi}{2} + \theta\right) + 8k_2^2[(\pi-\theta)(1-2c) + sc - 2s]\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{\rho R^5 3(1-\nu^2)}{2Et^3\pi}\{k_1\pi[(\pi-\theta)(1+2c^2) + 3sc] + 8k_2^2[(\pi-\theta)(1-2c) + sc - 2s]\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$$

$$\Delta D_V = \frac{-\rho R^5 3(1-\nu^2)}{2Et^3\pi}\{k_1\pi[s^2 + (\pi-\theta)(\pi - \theta + 2sc)] - 4k_2\pi(1+c)^2 - 8k_2^2[(\pi-\theta)(1-2c) + sc - 2s]\}$$

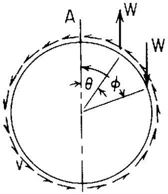
TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values
17. Continued $LT_M = \dfrac{\rho R^3}{2}[2c - z(x - \theta + sc) - u(1 + c^2)]\langle x - \theta\rangle^0$ $LT_N = \dfrac{\rho R^2}{2}[2c - z(x - \theta + sc) - u(1 + c^2)]\langle x - \theta\rangle^0$ $LT_V = \dfrac{\rho R^2}{2}[zc^2 - u(x - \theta + sc)] \times \langle x - \theta\rangle^0$	$\Delta L = \begin{cases} \dfrac{-\rho R^5 3(1-\nu^2)}{2Et^3\pi}\left\{k_1\left[2\theta c^2 + \theta - 3sc + \pi^2\left(sc - \theta + \dfrac{3\pi}{4}\right)\right] + 2k_2\pi[2 + 2\theta c - 2s - 4c - \pi + \theta - sc] - 4k_2^2[(\pi-\theta)(1-2c) + sc - 2s]\right\} & \text{for } \theta \leqslant \dfrac{\pi}{2} \\ \dfrac{-\rho R^5 3(1-\nu^2)}{2Et^3\pi}\{k_1[2\theta c^2 + \theta - 3sc + \pi(\pi-\theta)(\pi - \theta + 2sc) - 3\pi c^2] + 2k_2\pi[2s - 2(1+c)^2 - sc - (\pi-\theta)(1-2c)] - 4k_2^2[(\pi-\theta)(1-2c) + sc - 2s]\} & \text{for } \theta \geqslant \dfrac{\pi}{2} \end{cases}$ $\text{Max} + M = M_C = \dfrac{\rho R^3}{4\pi}\{4\pi c + \pi + 2\theta c^2 + \theta - 3sc + 2k_2[(\pi-\theta)(1-2c) + sc - 2s]\}$ Max $-M$ occurs at an angular position where $x_1 > \theta, x_1 > 105.23^\circ$, and x_1 is found from $(\theta + 2\theta c^2 - 3sc - \pi)\tan x_1 + 2\pi(\theta - sc - x_1) = 0$ If $\alpha = \beta = 0, M = K_M\rho R^3, N = K_N\rho R^2, \Delta D = K_{\Delta D}\rho R^5 12(1-\nu^2)/Et^3$, etc.

θ	0°	30°	45°	60°	90°	120°	135°	150°
K_{M_A}	0.2500	0.2290	0.1935	0.1466	0.0567	0.0104	0.0027	0.00039
K_{N_A}	0.7500	0.6242	0.4944	0.3534	0.1250	0.0216	0.0056	0.00079
K_{M_C}	0.7500	0.7216	0.6619	0.5649	0.3067	0.0921	0.0344	0.00778
$K_{\Delta D_H}$	0.2146	0.2027	0.1787	0.1422	0.0597	0.0115	0.0031	0.00044
$K_{\Delta D_V}$	−0.2337	−0.2209	−0.1955	−0.1573	−0.0700	−0.0150	−0.0043	−0.00066
$K_{\Delta L}$	−0.1399	−0.1333	−0.1198	−0.0986	−0.0465	−0.0106	−0.0031	−0.00050

TABLE 9.2 Formulas for circular rings (*Continued*)

18.



$$v = \frac{W}{2\pi R}(\sin\phi - \sin\theta)$$

$$LT_M = -\frac{WR}{2\pi}(n-s)(x-z) + WR(z-s)\langle x-\theta\rangle^0 - WR(z-n)\langle x-\phi\rangle^0$$

$$LT_N = \frac{W}{2\pi}(n-s)z + Wz\langle x-\theta\rangle^0 - Wz\langle x-\phi\rangle^0$$

$$LT_V = \frac{-W}{2\pi}(n-s)(1-u) + Wu\langle x-\theta\rangle^0 - Wu\langle x-\phi\rangle^0$$

$$M_A = \frac{WR}{2\pi}[n^2 - s^2 - (\pi-\phi)n + (\pi-\theta)s - k_2(c-m)]$$

$$N_A = \frac{W}{2\pi}(n^2 - s^2)$$

$$V_A = \frac{W}{2\pi}(\theta - \phi + s - n + sc - nm)$$

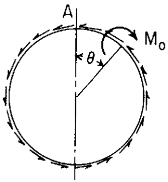
If $\alpha = \beta = 0$, $M = K_M WR$, $N = K_N W$, $V = K_V W$

θ	$\phi-\theta$	30°	45°	60°	90°	120°	135°	150°	180°
0°	K_{M_A}	−0.1899	−0.2322	−0.2489	−0.2500	−0.2637	−0.2805	−0.2989	−0.3183
	K_{N_A}	0.0398	0.0796	0.1194	0.1592	0.1194	0.0796	0.0398	0.0000
	K_{V_A}	−0.2318	−0.3171	−0.3734	−0.4092	−0.4022	−0.4080	−0.4273	−0.5000
30°	K_{M_A}	−0.0590	−0.0613	−0.0601	−0.0738	−0.1090	−0.1231	−0.1284	−0.1090
	K_{N_A}	0.0796	0.1087	0.1194	0.0796	−0.0000	−0.0291	−0.0398	−0.0000
	K_{V_A}	−0.1416	−0.1700	−0.1773	−0.1704	−0.1955	−0.2279	−0.2682	−0.3408
45°	K_{M_A}	−0.0190	−0.0178	−0.0209	−0.0483	−0.0808	−0.0861	−0.0808	−0.0483
	K_{N_A}	0.0689	0.0796	0.0689	0.0000	−0.0689	−0.0796	−0.0689	−0.0000
	K_{V_A}	−0.0847	−0.0920	−0.0885	−0.0908	−0.1426	−0.1829	−0.2231	−0.2749
60°	K_{M_A}	−0.0011	−0.0042	−0.0148	−0.0500	−0.0694	−0.0641	−0.0500	−0.0148
	K_{N_A}	0.0398	0.0291	−0.0000	−0.0796	−0.1194	−0.1087	−0.0796	0.0000
	K_{V_A}	−0.0357	−0.0322	−0.0288	−0.0539	−0.1266	−0.1668	−0.1993	−0.2243
90°	K_{M_A}	−0.0137	−0.0305	−0.0489	−0.0683	−0.0489	−0.0305	−0.0137	0.0000
	K_{N_A}	−0.0398	−0.0796	−0.1194	−0.1592	−0.1194	−0.0796	−0.0398	0.0000
	K_{V_A}	0.0069	0.0012	−0.0182	−0.0908	−0.1635	−0.1829	−0.1886	−0.1817

TABLE 9.2 Formulas for circular rings (*Continued*)

Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values

19.



$v = \dfrac{M_o}{2\pi R^2}$

$$LT_M = \frac{-M_o}{2\pi}(x - z) + M_o\langle x - \theta\rangle^0$$

$$LT_N = \frac{M_o z}{2\pi R}$$

$$LT_V = \frac{-M_o}{2\pi R}(1 - u)$$

$$M_A = \frac{-M_o}{2\pi}\left(\pi - \theta - \frac{2k_2 s}{k_1}\right)$$

$$N_A = \frac{M_o}{\pi R}\left(\frac{k_2 s}{k_1}\right)$$

$$V_A = \frac{-M_o}{2\pi R}\left(1 + \frac{2k_2 c}{k_1}\right)$$

$\text{Max} + M = \dfrac{M_o}{2}$ for x just greater than θ

$\text{Max} - M = \dfrac{-M_o}{2}$ for x just less than θ

At $x = \theta + 180^\circ$, $M = 0$

Other maxima are, for $\alpha = \beta = 0$

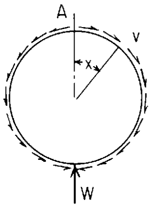
$$M\begin{cases} -0.1090M_o & \text{at } x = \theta + 120^\circ \\ 0.1090M_0 & \text{at } x = \theta + 240^\circ \end{cases}$$

If $\alpha = \beta = 0$, $M = k_M M_o$, $N = K_N M_o/R$, $V = K_V M_o/R$

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
K_{M_A}	−0.5000	−0.2575	−0.1499	−0.0577	0.0683	0.1090	0.1001	0.0758	0.0000
K_{N_A}	0.0000	0.1592	0.2251	0.2757	0.3183	0.2757	0.2250	0.1592	0.0000
K_{V_C}	−0.4775	−0.4348	−0.3842	−0.3183	−0.1592	0.0000	0.0659	0.1165	0.1592

TABLE 9.2 Formulas for circular rings (*Continued*)

20. Bulkhead or supporting ring in pipe, supported at bottom and carrying total load W transferred by tangential shear v distributed as shown



$$v = \frac{W \sin x}{\pi R}$$

$$LT_M = \frac{WR}{\pi}\left(1 - u - \frac{xz}{2}\right)$$

$$LT_N = \frac{-W}{2\pi} xz$$

$$LT_V = \frac{W}{2\pi}(z - xu)$$

for $0 < x < 180^\circ$

$$M_A = \frac{WR}{2\pi}(k_2 - 0.5)$$

$$M_C = \frac{WR}{2\pi}(k_2 + 0.5)$$

$$N_A = \frac{0.75W}{\pi}$$

$$V_A = 0$$

$$\Delta D_H = \frac{WR^3}{EI}\left(\frac{k_1}{4} - \frac{k_2}{2} + \frac{k_2^2}{\pi}\right)$$

$$\Delta D_V = \frac{-WR^3}{EI}\left(\frac{k_1 \pi}{8} - \frac{k_2^2}{\pi}\right)$$

$$\Delta L = \frac{-WR^3}{4EI\pi}\left[4 + k_1 \frac{3\pi^2}{8} - k_2(\pi + 2) - 2k_2^2\right]$$

Max $+M = M_C$

Max $-M = \frac{-WR}{4\pi}(3.2815 - 2k_2)$ at $x = 105.2^\circ$

If $\alpha = \beta = 0$,

$$M_A = 0.0796WR$$

$$N_A = 0.2387W$$

$$V_A = 0$$

$$\Delta D_H = 0.0683\frac{WR^3}{EI}$$

$$\Delta D_V = -0.0744\frac{WR^3}{EI}$$

$$\Delta R = -0.0445\frac{WR^3}{EI}$$

TABLE 9.2 Formulas for circular rings (*Continued*)

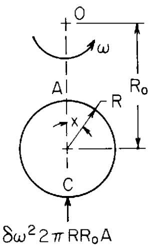
Reference no., loading, and load terms	Formulas for moments, loads, and deformations and some selected numerical values	
21. Ring rotating at angular rate ω rad/s about an axis perpendicular to the plane of the ring. Note the requirement of symmetry of the cross section in Sec. 9.3.  $\delta =$ mass density of ring material $LT_M = \delta\omega^2 AR^3 \left\{ K_T(1-u) - \frac{R_o}{R}[xz - K_T(1-u)] \right\}$ $LT_N = \delta\omega^2 AR^2 \left[K_T(1-u) - \frac{R_o}{R} xz \right]$ $LT_V = \delta\omega^2 AR^2 \left[zK_T(2u-1) - \frac{R_o}{R} xu \right]$	$M_A = \delta\omega^2 AR^3 \left\{ K_T\alpha + \frac{R_o}{R}\left[k_2 - 0.5 - \frac{(K_T-1)\beta}{k_1} \right] \right\}$ where $K_T = 1 + \frac{I}{AR^2}$ $M_C = \delta\omega^2 AR^3 \left\{ K_T\alpha + \frac{R_o}{R}\left[k_2 + 0.5 + \frac{(K_T-1)\beta}{k_1} \right] \right\}$ $N_A = \delta\omega^2 AR^2 \left\{ K_T + \frac{R_o}{R}\left[0.5 + (K_T-1)\frac{k_2}{k_1} \right] \right\}$ $V_A = 0$ $\Delta D_H = \frac{\delta\omega^2 R^4}{Ee}\left[2K_T k_2 \alpha + \frac{R_o}{R}\left(\frac{k_1\pi}{2} - k_2\pi + 2k_2^2 \right) \right]$ $\Delta D_V = \frac{\delta\omega^2 R^4}{Ee}\left[2K_T k_2 \alpha - \frac{R_o}{R}\left(\frac{k_1\pi^2}{4} - 2k_2^2 \right) \right]$ $\Delta L = \frac{\delta\omega^2 R^4}{Ee}\left[K_T k_2 \alpha - \frac{R_o}{R}\left(\frac{k_1 3\pi^2}{16} + k_2 - \frac{k_2\pi}{2} - k_2^2 + K_T\alpha \right) \right]$ *Note:* The constant K_T accounts for the radial distribution of mass in the ring.	Max $+M = M_C$ Max $-M$ occurs at an angular position x_1 where $\frac{x_1}{\tan x_1} = -0.5 + \frac{(K_T-1)\beta}{k_1}$ For a thin ring where $K_T \approx 1$, Max $-M = -\delta\omega^2 AR^3 \left[\frac{R_o}{R}(1.6408 - k_2) - \alpha \right]$ at $x = 105.23°$

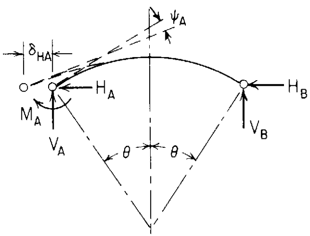
TABLE 9.3 Reaction and deformation formulas for circular arches

NOTATION: $W=$ load (force); $w=$ unit load (force per unit of circumferential length); $M_o=$ applied couple (force-length). $\theta_o=$ externally created concentrated angular displacement (radians); $\Delta_o=$ externally created concentrated radial displacement; $T-T_o=$ uniform temperature rise (degrees); T_1 and $T_2=$ temperatures on outside and inside, respectively (degrees). H_A and H_B are the horizontal end reactions at the left and right, respectively, and are positive to the left; V_A and V_B are the vertical end reactions at the left and right ends, respectively, and are positive upward; M_A and M_B are the reaction moments at the left and right, respectively, and are positive clockwise. $E=$ modulus of elasticity (force per unit area); $\nu=$ Poisson's ratio; A is the cross-sectional area; R is the radius ot the centroid of the cross section; $I=$ area moment of inertia of arch cross section about the principal axis perpendicular to the plane of the arch. [Note that for a wide curved plate or a sector of a cylinder, a representative segment of unit axial length may be used by replacing EI by $Et^3/12(1-\nu^2)$.] e is the positive distance measured radially inward from the centroidal axis of the cross section to the neutral axis of pure bending (see Sec. 9.1). θ (radians) is one-half of the total subtended angle of the arch and is limited to the range zero to π. For an angle θ close to zero, round-off errors may cause troubles; for an angle θ close to π, the possibility of static or elastic instability must be considered. Deformations have been assumed small enough so as to not affect the expressions for the internal bending moments, radial shear, and circumferential normal forces. Answers should be examined to be sure that such is the case before accepting them. ϕ (radians) is the angle measured counterclockwise from the midspan of the arch to the position of a concentrated load or the start of a distributed load. $s=\sin\theta$, $c=\cos\theta$, $n=\sin\phi$, and $m=\cos\phi$. $\gamma=$ temperature coefficient of expansion.

The references to end points A and B refer to positions on a circle of radius R passing through the centroids of the several sections. It is important to note this carefully when dealing with thick rings. Similarly, all concentrated and distributed loadings are assumed to be applied at the radial position of the centroid with the exception of cases h and i where the ring is loaded by its own weight or by a constant linear acceleration. In these two cases the actual radial distribution of load is considered. If the loading is on the outer or inner surfaces of thick rings, a statically equivalent loading at the centroidal radius R must be used. See examples to determine how this might be accomplished.

The hoop-stress deformation factor is $\alpha=I/AR^2$ for thin rings or $\alpha=e/R$ for thick rings. The transverse- (radial-) shear deformation factor is $\beta=FEI/GAR^2$ for thin rings or $\beta=2F(1+\nu)e/R$ for thick rings, where G is the shear modulus of elasticity and F is a shape factor for the cross section (see Sec. 8.10). The following constants are defined to simplify the expressions which follow. Note that these constants are unity if no correction for hoop stress or shear stress is necessary or desired for use with thin rings. $k_1=1-\alpha+\beta$, $k_2=1-\alpha$.

General reaction and expressions for cases 1–4; right end pinned in all four cases, no vertical motion at the left end



Deformation equations:

Horizontal deflection at $A=\delta_{HA}=\dfrac{R^3}{EI}\left(A_{HH}H_A+A_{HM}\dfrac{M_A}{R}-LP_H\right)$

Angular rotation at $A=\psi_A=\dfrac{R^2}{EI}\left(A_{MH}H_A+A_{MM}\dfrac{M_A}{R}-LP_M\right)$

where $A_{HH}=2\theta c^2+k_1(\theta-sc)-k_2 2sc$

$A_{MH}=A_{HM}=k_2 s-\theta c$

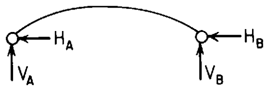
$A_{MM}=\dfrac{1}{4s^2}[2\theta s^2+k_1(\theta+sc)-k_2 2sc]$

and where LP_H and LP_M are loading terms given below for several types of load.

(*Note:* If desired, V_A, V_B, and H_B can be evaluated from equilibrium equations after calculating H_A and M_A)

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

1. Left end pinned, right end pinned



Since $\delta_{HA} = 0$ and $M_A = 0$,

$$H_A = \frac{LP_H}{A_{HH}} \quad \text{and} \quad \psi_A = \frac{R^2}{EI}(A_{MH}H_A - LP_M)$$

The loading terms are given below.

Reference no., loading	Loading terms and some selected numerical values

1a. Concentrated vertical load

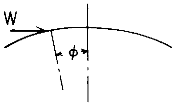
$$LP_H = W\left[\theta sc - \phi nc + \frac{k_1}{2}(c^2 - m^2) + k_2 c(c - m)\right]$$

$$LP_M = \frac{W}{2}\left[\phi n - \theta s + \frac{k_1}{2s^2}(\theta n - \phi s + snc - snm) - k_2(c - m)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	−0.0143	−0.0100	−0.1715	−0.1105	−0.5000	−0.2500
$\frac{LP_M}{W}$	−0.0639	−0.0554	−0.2034	−0.1690	−0.2854	−0.1978

1b. Concentrated horizontal load



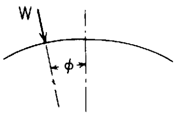
$$LP_H = W\left[\theta c^2 + \phi mc + \frac{k_1}{2}(\theta + \phi - sc - nm) - k_2 c(s + n)\right]$$

$$LP_M = \frac{W}{2}\left[-\theta c - \phi m + \frac{k_1}{2s^2}(\theta c - \theta m + sm^2 - scm) + k_2(s + n)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	0.0050	0.0057	0.1359	0.1579	0.7854	0.9281
$\frac{LP_M}{W}$	0.0201	0.0222	0.1410	0.1582	0.3573	0.4232

1c. Concentrated radial load



$$LP_H = W\left[\theta c(cn + sm) + \frac{k_1}{2}(\theta n + \phi n - scn - s^2 m) - k_2 c(1 + sn - cm)\right]$$

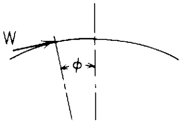
$$LP_M = \frac{W}{2}\left[-\theta(cn + sm) + \frac{k_1}{2s^2}(\theta cn - \phi sm) + k_2(1 + sn - cm)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	−0.0143	−0.0082	−0.1715	−0.0167	−0.5000	0.4795
$\frac{LP_M}{W}$	−0.0639	−0.0478	−0.2034	−0.0672	−0.2854	0.1594

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

1d. Concentrated tangential load



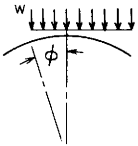
$$LP_H = W\left[\theta c(cm - sn) + \phi c + \frac{k_1}{2}(\theta m + \phi m - scm - c^2 n) - k_2 c(sm + cn)\right]$$

$$LP_M = \frac{W}{2}\left[\theta(sn - cm) - \phi + \frac{k_1}{2s^2}(\theta cm - \theta + \phi sn - sc + sm) + k_2(sm + cn)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{W}$	0.0050	0.0081	0.1359	0.1920	0.7854	0.8330
$\frac{LP_M}{W}$	0.0201	0.0358	0.1410	0.2215	0.3573	0.4391

1e. Uniform vertical load on partial span



$$LP_H = \frac{wR}{4}\left[\theta c(1 + 4sn + 2s^2) + \phi c(m^2 - n^2) - c(sc + mn) + \frac{2k_1}{3}(n^3 - 3ns^2 - 2s^3) + 2k_2 c(2cn + cs - \theta - \phi - mn)\right]$$

$$LP_M = \frac{wR}{8}\left\{mn + sc - \theta(4sn + 2s^2 + 1) - \phi(m^2 - n^2) + \frac{k_1}{s}\left[\frac{\theta}{s}(n^2 + s^2) + 2(c - m) - \tfrac{2}{3}(c^3 - m^3) + c(n^2 - s^2) - 2\phi n\right] + 2k_2(\theta + \phi + mn - sc - 2cn)\right\}$$

If $\phi = 0$ (the full span is loaded)

$$LP_H = \frac{wR}{6}[3c(2\theta s^2 + \theta - sc) - 4k_1 s^3 + 6k_2 c(sc - \theta)]$$

$$LP_M = \frac{wR}{4}[sc - \theta - 2\theta s^2 + 2k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0046	−0.0079	−0.0969	−0.1724	−0.3333	−0.6280
$\frac{LP_M}{wR}$	−0.0187	−0.0350	−0.1029	−0.2031	−0.1667	−0.3595

1f. Uniform horizontal load on left side only

$$LP_H = \frac{wR}{12}[3\theta c(1 - 6c^2 + 4c) + 3sc^2 + k_1(6\theta - 6sc - 12\theta c + 12c - 8s^3) + 6k_2 c(3sc - 2s - \theta)]$$

$$LP_M = \frac{wR}{8}\left\{6\theta c^2 - \theta - 4\theta c - sc + \frac{k_1}{3s^2}[s(2 - 3c + c^3) - 3\theta(1 - c)^2] + 2k_2(\theta + 2s - 3sc)\right\}$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LP_H}{wR}$	0.0010	0.0969	1.1187
$\frac{LP_M}{wR}$	0.0040	0.1060	0.5833

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

1g. Uniform horizontal load on right side only

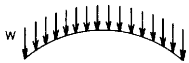
$$LP_H = \frac{wR}{12}[3\theta c(1 + 2c^2 - 4c) + 3sc^2 + 2k_1(2s^3 - 3\theta + 3sc) + 6k_2c(2s - sc - \theta)]$$

$$LP_M = \frac{wR}{8}\left\{4\theta c - 2\theta c^2 - \theta - sc - \frac{k_1}{3s^2}[s(2 - 3c + c^3) - 3\theta(1 - c)^2] + 2k_2(\theta - 2s + sc)\right\}$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LP_H}{wR}$	−0.0004	−0.0389	−0.4521
$\frac{LP_M}{wR}$	−0.0015	−0.0381	−0.1906

1h. Vertical loading uniformly distributed along the circumference (by gravity or linear acceleration)



(*Note:* The full span is loaded)

$$LP_H = wR\left\{2\theta^2 sc + \left(\frac{k_1}{2} + k_2\right)(2\theta c^2 - \theta - sc) + \frac{R_{cg}}{R}[k_2(\theta - sc) - 2c(s - \theta c)]\right\}$$

$$LP_M = wR\left[\left(\frac{R_{cg}}{R} + k_2\right)(s - \theta c) - \theta^2 s\right]$$

where R_{cg} is the radial distance to the center of mass for a differential length of the circumference for radially thicker arches. $R_{cg}/R = 1 + I_c/(AR^2)$. I_c is the area moment of inertia about the centroidal axis of the cross section. For radially thin arches let $R_{cg} = R$. See the discussion on page 333.

For $\alpha = \beta = 0$ and for $R_{cg} = R$

θ	30°	60°	90°
$\frac{LP_H}{wR}$	−0.0094	−0.2135	−0.7854
$\frac{LP_M}{wR}$	−0.0440	−0.2648	−0.4674

1i. Horizontal loading uniformly distributed along the circumference (by gravity or linear acceleration)

(*Note:* The full span is loaded)

$$LP_H = wR\theta[2\theta c^2 + k_1(\theta - sc) - 2k_2 sc]$$

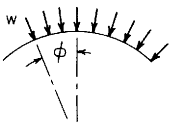
$$LP_M = \frac{wR}{2s}\left[-2\theta^2 sc + \frac{k_1}{2s}(2\theta^2 c + \theta s + s^2 c) + k_2(2\theta s^2 - \theta - sc) - \frac{R_{cg}}{R}(k_1 - k_2)(\theta + sc)\right]$$

See case 1h for a definition of the radius R_{cg}

For $\alpha = \beta = 0$ and for $R_{cg} = R$

θ	30°	60°	90°
$\frac{LP_H}{wR}$	0.0052	0.2846	2.4674
$\frac{LP_M}{wR}$	0.0209	0.2968	1.1781

1j. Partial uniformly distributed radial loading



$$LP_H = wRc\left[\theta(1 - cm + sn) + \frac{k_1}{2c}(scm + c^2 n - \theta m - \phi m) + k_2(sm + cn - \theta - \phi)\right]$$

$$LP_M = \frac{wR}{2}\left[\theta(cm - 1 - sn) + \frac{k_1}{2s^2}[\theta - \theta cm - \phi sn + sc - sm) + k_2(\theta + \phi - sm - cn)\right]$$

If $\phi = \theta$ (the full span is loaded)

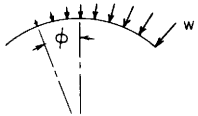
$$LP_H = wRc[2\theta s^2 - k_1(\theta - sc) - 2k_2(\theta - sc)]$$

$$LP_M = wR[-\theta s^2 + k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0050	−0.0081	−0.1359	−0.1920	−0.7854	−0.8330
$\frac{LP_M}{wR}$	−0.0201	−0.0358	−0.1410	−0.2215	−0.3573	−0.4391

1k. Partial uniformly increasing distributed radial loading



$$LP_H = \frac{wR}{\theta+\phi}\left\{\theta c(\theta+\phi-cn-sm) + \frac{k_1}{2}[n(sc-\theta-\phi)+s^2m+2c-2m] + k_2\left[scn+s^2m-\frac{c}{2}(\theta+\phi)^2+c-m\right]\right\}$$

$$LP_M = \frac{wR}{\theta+\phi}\left\{\frac{\theta}{2}(cn+sm-\theta-\phi) + \frac{k_1}{4s^2}[(\theta+\phi)(\theta+sc)+\phi sm-\theta cn-2s^2-2sn] + \frac{k_2}{4}[(\theta+\phi)^2+2cm-2sn-2]\right\}$$

If $\phi = \theta$ (the full span is loaded)

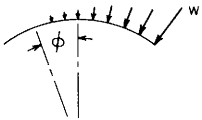
$$LP_H = \frac{wRc}{\theta}\left[\theta(\theta-cs) - \frac{k_1 s}{2c}(\theta-sc) - k_2(\theta^2-s^2)\right]$$

$$LP_M = \frac{wR}{2\theta}\left[\theta sc - \theta^2 + \frac{k_1}{2s^2}(\theta^2+\theta sc-2s^2) + k_2(\theta^2-s^2)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0018	−0.0035	−0.0518	−0.0915	−0.3183	−0.5036
$\frac{LP_M}{wR}$	−0.0072	−0.0142	−0.0516	−0.0968	−0.1366	−0.2335

1l. Partial second-order increase in distributed radial loading



$$LP_H = \frac{wRc}{(\theta+\phi)^2}\left\{\theta(\theta+\phi)^2 - 2\theta(1-cm+sn) + \frac{k_1}{c}[(\theta+\phi)(2c+m) - c(sm+cn) - 2n - 2s] + \frac{k_2}{3}[6(\theta+\phi-sm-cn)-(\theta+\phi)^3]\right\}$$

$$LP_M = \frac{wR}{(\theta+\phi)^2}\left\{\theta(1-cm+sn) - \frac{\theta}{2}(\theta+\phi)^2 + \frac{k_1}{4s^2}[(\theta+\phi)^2(\theta+sc)+2s(\phi n+3m-3c)-4s^2(\theta+\phi)-2\theta(1-cm)] + k_2\left[sm+cn-\theta-\phi+\frac{(\theta+\phi)^3}{6}\right]\right\}$$

If $\phi = \theta$ (the full span is loaded)

$$LP_H = \frac{wR}{6\theta^2}[6\theta^3c - 6\theta s^2c + 3k_1(3\theta c - 3s + s^3) + 2k_2(3\theta c - 2\theta^3c - 3sc^2)]$$

$$LP_M = \frac{wR}{2\theta^2}\left[\theta s^2 - \theta^3 + k_1\frac{\theta}{2s^2}(\theta^2+\theta sc-2s^2) + k_2\left(sc-\theta+\frac{2\theta^3}{3}\right)\right]$$

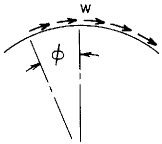
For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	−0.0010	−0.0019	−0.0276	−0.0532	−0.1736	−0.3149
$\frac{LP_M}{wR}$	−0.0037	−0.0077	−0.0269	−0.0542	−0.0726	−0.1388

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

1m. Partial uniformly distributed tangential loading



$$LP_H = \frac{wR}{2}\{2\theta c(cn+sm) - c(\theta^2 - \phi^2) + k_1[n(\theta+\phi) + c(cm - sn - 2) + e] + k_2 2c(cm - sn - 1)\}$$

$$LP_M = \frac{wR}{4}\left\{\theta^2 - \phi^2 - 2\theta(cn+sm) + \frac{k_1}{s^2}[\theta(cn - cs - \theta - \phi) - \phi s(c+m) + 2s(s+n)] + k_2 2(1 + sn - cm)\right\}$$

If $\phi = \theta$ (the full span is loaded)

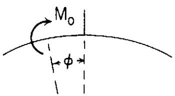
$$LP_H = wR[2\theta c^2 s + k_1 s(\theta - sc) - 2k_2 cs^2]$$

$$LP_M = wR\left[-\theta sc + \frac{k_1}{2s^2}(2s^2 - \theta sc - \theta^2) + k_2 s^2\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H}{wR}$	0.0010	0.0027	0.0543	0.1437	0.5000	1.1866
$\frac{LP_M}{wR}$	0.0037	0.0112	0.0540	0.1520	0.2146	0.5503

1n. Concentrated couple



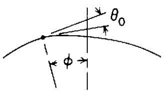
$$LP_H = \frac{M_o}{R}(\phi c - k_2 n)$$

$$LP_M = \frac{M_o}{4s^2 R}[-2s^2\phi - k_1(\theta + sc) + k_2 2sm]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LP_H R}{M_o}$	0.0000	−0.0321	0.0000	−0.2382	0.0000	−0.7071
$\frac{LP_M R}{M_o}$	0.0434	−0.1216	0.0839	−0.2552	0.1073	−0.4318

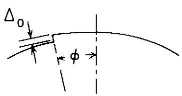
1p. Concentrated angular displacement



$$LP_H = \frac{\theta_o EI}{R^2}(m - c)$$

$$LP_M = \frac{\theta_o EI}{R^2}\left(\frac{1}{2} + \frac{n}{2s}\right)$$

1q. Concentrated radial displacement



$$LP_H = \frac{\Delta_o EI}{R^3} n$$

$$LP_M = \frac{\Delta_o EI}{R^3}\left(-\frac{m}{2s}\right)$$

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

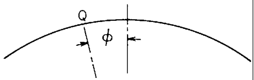
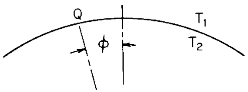
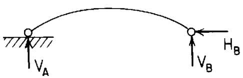
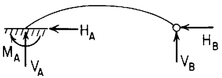
1r. Uniform temperature rise over that span to the right of point Q 	$LP_H = -(T - T_o)\dfrac{\gamma EI}{R^2}(s + n)$ $LP_M = (T - T_o)\dfrac{\gamma EI}{2R^2 s}(m - c)$ $T =$ uniform temperature $T_o =$ unloaded temperature
1s. Linear temperature differential through the thickness t for that span to the right of point Q 	$LP_H = (T_1 - T_2)\dfrac{\gamma EI}{Rt}(n + s - \theta c - \phi c)$ $LP_M = (T_1 - T_2)\dfrac{\gamma EI}{2Rts}(\theta s + \phi s - m + c)$ where t is the radial thickness and T_o, the unloaded temperature, is the temperature at the radius of the centroid
2. Left end guided horizontally, right end pinned 	Since $\psi_A = 0$ and $H_A = 0$ $M_A = \dfrac{LP_M}{A_{MM}}R$ and $\delta_{HA} = \dfrac{R^3}{EI}\left(A_{HM}\dfrac{M_A}{R} - LP_H\right)$ Use load terms given above for cases 1a–1s
3. Left end roller supported in vertical direction only, right end pinned 	Since both M_A and H_A are zero, this is a statically determinate case: $\delta_{HA} = \dfrac{-R^3}{EI}LP_H$ and $\psi_A = \dfrac{-R^2}{EI}LP_M$ Use load terms given above for cases 1a–1s

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

4. Left end fixed, right end pinned

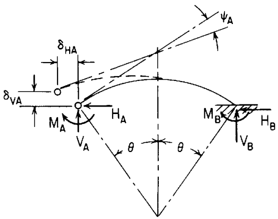


Since $\delta_{HA} = 0$ and $\psi_A = 0$,

$$H_A = \frac{A_{MM} LP_H - A_{HM} LP_M}{A_{HH} A_{MM} - A_{HM}^2} \quad \text{and} \quad \frac{M_A}{R} = \frac{A_{HH} LP_M - A_{HM} LP_H}{A_{HH} A_{MM} - A_{HM}^2}$$

Use load terms given above for cases 1a–1s

General reaction and deformation expressions for cases 5–14, right end fixed in all 10 cases.



Deformation equations:

$$\text{Horizontal deflection at } A = \delta_{HA} = \frac{R^3}{EI}\left(B_{HH} H_A + B_{HV} V_A + B_{HM} \frac{M_A}{R} - LF_H\right)$$

$$\text{Vertical deflection at } A = \delta_{VA} = \frac{R^3}{EI}\left(B_{VH} H_A + B_{VV} V_A + B_{VM} \frac{M_A}{R} - LF_V\right)$$

$$\text{Angular rotation at } A = \psi_A = \frac{R^2}{EI}\left(B_{MH} H_A + B_{MV} V_A + B_{MM} \frac{M_A}{R} - LF_M\right)$$

where $B_{HH} = 2\theta c^2 + k_1(\theta - sc) - k_2 2sc$
$B_{HV} = B_{VH} = -2\theta sc + k_2 2s^2$
$B_{HM} = B_{MH} = -2\theta c + k_2 2s$
$B_{VV} = 2\theta s^2 + k_1(\theta + sc) - k_2 2sc$
$B_{VM} = B_{MV} = 2\theta s$
$B_{MM} = 2\theta$
and where LF_H, LF_V, and LF_M are loading terms given below for several types of load

(*Note:* If desired, H_B, V_B, and M_B can be evaluated from equilibrium equations after calculating H_A, V_A, and M_A)

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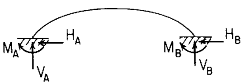
5. Left end fixed, right end fixed



Since $\delta_{HA} = 0, \delta_{VA} = 0, \psi_A = 0$, these equations must be solved simultaneously for H_A, V_A, and M_A/R
The loading terms are given in cases 5a–5s

$$B_{HH} H_A + B_{HV} V_A + B_{HM} M_A/R = LF_H$$

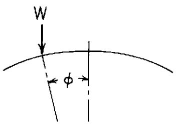
$$B_{VH} H_A + B_{VV} V_A + B_{VM} M_A/R = LF_V$$

$$B_{MH} H_A + B_{MV} V_A + B_{MM} M_A/R = LF_M$$

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

5a. Concentrated vertical load



$$LF_H = W\left[-(\theta+\phi)cn + \frac{k_1}{2}(c^2 - m^2) + k_2(1 + sn - cm)\right]$$

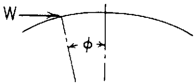
$$LF_V = W\left[(\theta+\phi)sn + \frac{k_1}{2}(\theta + \phi + sc + nm) - k_2(2sc - sm + cn)\right]$$

$$LF_M = W[(\theta+\phi)n + k_2(m - c)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{W}$	0.0090	0.0253	0.1250	0.3573	0.5000	1.4571
$\frac{LF_V}{W}$	0.1123	0.2286	0.7401	1.5326	1.7854	3.8013
$\frac{LF_M}{W}$	0.1340	0.3032	0.5000	1.1514	1.0000	2.3732

5b. Concentrated horizontal load



$$LF_H = W\left[(\theta+\phi)mc + \frac{k_1}{2}(\theta + \phi - sc - nm) - k_2(sm + cn)\right]$$

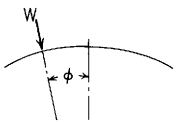
$$LF_V = W\left[-(\theta+\phi)sm + \frac{k_1}{2}(c^2 - m^2) + k_2(1 - 2c^2 + cm + sn)\right]$$

$$LF_M = W[-(\theta+\phi)m + k_2(s + n)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{W}$	−0.0013	0.0011	−0.0353	0.0326	−0.2146	0.2210
$\frac{LF_V}{W}$	−0.0208	−0.0049	−0.2819	−0.0621	−1.0708	−0.2090
$\frac{LF_M}{W}$	−0.0236	0.0002	−0.1812	0.0057	−0.5708	0.0410

5c. Concentrated radial load



$$LF_H = W\left[\frac{k_1}{2}(\theta n + \phi n - scn - s^2 m) + k_2(m - c)\right]$$

$$LF_V = W\left[\frac{k_1}{2}(\theta m + \phi m + scm + c^2 n) + k_2(s + n - 2scm - 2c^2 n)\right]$$

$$LF_M = W[k_2(1 + sn - cm)]$$

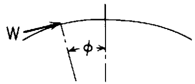
For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{W}$	0.0090	0.0248	0.1250	0.3257	0.5000	1.1866
$\frac{LF_V}{W}$	0.1123	0.2196	0.7401	1.2962	1.7854	2.5401
$\frac{LF_M}{W}$	0.1340	0.2929	0.5000	1.0000	1.0000	1.7071

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

5d. Concentrated tangential load



$$LF_H = W\left[(\theta+\phi)c + \frac{k_1}{2}(\theta m + \phi m - scm - c^2 n) - k_2(s+n)\right]$$

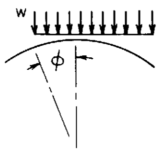
$$LF_V = W\left[-(\theta+\phi)s - \frac{k_1}{2}(\theta n + \phi n + scn + s^2 m) + k_2(2s^2 m + 2scn + c - m)\right]$$

$$LF_M = W[-\theta - \phi + k_2(sm + cn)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{W}$	−0.0013	−0.0055	−0.0353	−0.1505	−0.2146	−0.8741
$\frac{LF_V}{W}$	−0.0208	−0.0639	−0.2819	−0.8200	−1.0708	−2.8357
$\frac{LF_M}{W}$	−0.0236	−0.0783	−0.1812	−0.5708	−0.5708	−1.6491

5e. Uniform vertical load on partial span



$$LF_H = \frac{wR}{4}\left\{c[(1-2n^2)(\theta+\phi) - sc - mn] - \frac{2k_1}{3}(2s^3 + 3s^2 n - n^3) + 2k_2[s + 2n + sn^2 - c(\theta + \phi + mn)]\right\}$$

$$LF_V = \frac{wR}{4}\left\{s[(1-2m^2)(\theta+\phi) + sc + mn] + \frac{2k_1}{3}[3n(\theta+\phi+sc) + 3m - m^3 - 2c^3] + 2k_2[s(\theta + \phi - 2sc + nm - 4cn) - cn^2]\right\}$$

$$LF_M = \frac{wR}{4}[(1-2m^2)(\theta+\phi) + nm + sc + 2k_2(\theta + \phi + nm - sc - 2cn)]$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = \frac{wR}{2}\left[\theta c(1-2s^2) - sc^2 - \frac{k_1 4s^3}{3} + k_2 2(s^3 + s - c\theta)\right]$$

$$LF_V = wR\left[\frac{s}{2}(\theta s^2 - \theta c^2 + sc) + k_1 s(\theta + sc) + k_2 s(\theta - 3sc)\right]$$

$$LF_M = wR[\tfrac{1}{2}(\theta s^2 - \theta c^2 + sc) + k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0012	0.0055	0.0315	0.1471	0.1667	0.8291
$\frac{LF_V}{wR}$	0.0199	0.0635	0.2371	0.7987	0.7260	2.6808
$\frac{LF_M}{wR}$	0.0226	0.0778	0.1535	0.5556	0.3927	1.5531

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

5f. Uniform horizontal load on left side only

$$LF_H = \frac{wR}{4}\left[\theta c(4s^2-1) + sc^2 + 2k_1\left(\theta - 2\theta c - sc + 2sc^2 + \frac{2s^2}{3}\right) + 2k_2(sc^2 - s^3 - \theta c)\right]$$

$$LF_V = \frac{wR}{4}\left\{-\theta s(4s^2-1) - s^2c - \frac{2k_1}{3}(1 - 3c^2 + 2c^3) + 2k_2[\theta s + 2(1-c)(1-2c^2)]\right\}$$

$$LF_M = \frac{wR}{4}[-\theta(4s^2-1) - sc + 2k_2(2s - 3sc + \theta)]$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0005	0.0541	0.6187
$\frac{LF_V}{wR}$	0.0016	0.0729	0.4406
$\frac{LF_M}{wR}$	0.0040	0.1083	0.6073

5g. Uniform horizontal load on right side only

$$LF_H = \frac{wR}{4}\left[sc^2 - \theta c + \frac{2k_1}{3}(2s^3 + 3sc - 3\theta) + 2k_2(s - \theta c)\right]$$

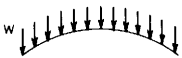
$$LF_V = \frac{wR}{4}\left[\theta s - s^2c + \frac{2k_1}{3}(1 - 3c^2 + 2c^3) - 2k_2(2 - 4c^2 + 2c^3 - \theta s)\right]$$

$$LF_M = \frac{wR}{4}[\theta - sc + 2k_2(\theta - 2s + sc)]$$

For $\alpha = \beta = 0$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0000	0.0039	0.0479
$\frac{LF_V}{wR}$	0.0009	0.0448	0.3448
$\frac{LF_M}{wR}$	0.0010	0.0276	0.1781

5h. Vertical loading uniformly distributed along the circumference (by gravity or linear acceleration)



(*Note:* The full span is loaded)

$$LF_H = wR\left[\frac{k_1}{2}(2\theta c^2 - \theta - sc) + k_2\left(\frac{R_{cg}}{R} + 1\right)(\theta - sc) + \frac{R_{cg}}{R}2c(\theta c - s)\right]$$

$$LF_V = wR\left[k_1\theta(\theta + sc) + 2k_2 s(s - 2\theta c) - \frac{R_{cg}}{R}2s(\theta c - s)\right]$$

$$LF_M = 2wR\left(\frac{R_{cg}}{R} + k_2\right)(s - \theta c)$$

See case 1h for a definition of the radius R_{cg}

For $\alpha = \beta = 0$ and $R_{cg} = R$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0149	0.4076	2.3562
$\frac{LF_V}{wR}$	0.1405	1.8294	6.4674
$\frac{LF_M}{wR}$	0.1862	1.3697	4.0000

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

5i. Horizontal loading uniformly distributed along the circumference (by gravity or linear acceleration)

(*Note:* The full span is loaded)

$$LF_H = wR\left[k_1\theta(\theta - sc) + \frac{R_{cg}}{R}2s(\theta c - k_2 s)\right]$$

$$LF_V = wR\left[\frac{k_1}{2}(2\theta c^2 - \theta - sc) + \left(\frac{R_{cg}}{R}+1\right)k_2(sc-\theta) + \left(2k_2 - \frac{R_{cg}}{R}\right)2\theta s^2\right]$$

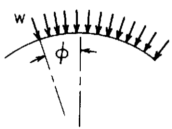
$$LF_M = wR\left(k_2 - \frac{R_{cg}}{R}\right)2\theta s$$

See case 1h for a definition of the radius R_{cg}

For $\alpha = \beta = 0$ and $R_{cg} = R$

θ	30°	60°	90°
$\frac{LF_H}{wR}$	0.0009	0.0501	0.4674
$\frac{LF_V}{wR}$	−0.0050	−0.1359	−0.7854
$\frac{LF_M}{wR}$	0.0000	0.0000	0.0000

5j. Partial uniformly distributed radial loading



$$LF_H = wR\left[\frac{k_1}{2}(scm + c^2 n - \theta m - \phi m) + k_2(s + n - \theta c - \phi c)\right]$$

$$LF_V = wR\left[\frac{k_1}{2}(\theta n + \phi n + scn + s^2 m) + k_2(\theta s + \phi s - 2scn + 2c^2 m - c - m)\right]$$

$$LF_M = wR[k_2(\theta + \phi - sm - cn)]$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = wR[k_1 c(sc - \theta) + 2k_2(s - \theta c)]$$

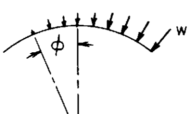
$$LF_V = wR[k_1 s(\theta + sc) + 2k_2 s(\theta - 2sc)]$$

$$LF_M = wR[2k_2(\theta - sc)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0013	0.0055	0.0353	0.1505	0.2146	0.8741
$\frac{LF_V}{wR}$	0.0208	0.0639	0.2819	0.8200	1.0708	2.8357
$\frac{LF_M}{wR}$	0.0236	0.0783	0.1812	0.5708	0.5708	1.6491

5k. Partial uniformly increasing distributed radial loading



$$LF_H = \frac{wR}{\theta+\phi}\left\{\frac{k_1}{2}[scn - (\theta+\phi)n + 2c - m - c^2 m] + \frac{k_2}{2}[(\theta+\phi)(2s - \theta c - \phi c) + 2c - 2m]\right\}$$

$$LF_V = \frac{wR}{\theta+\phi}\left\{\frac{k_1}{2}[2s + 2n - (\theta+\phi)m - smc - c^2 n] + \frac{k_2}{2}[(\theta+\phi)(\theta s + \phi s - 2c) - 2s - 2n + 4smc + 4c^2 n]\right\}$$

$$LF_M = \frac{wR}{\theta+\phi}\left\{\frac{k_2}{2}[(\theta+\phi)^2 + 2(cm - sn - 1)]\right\}$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = \frac{wR}{2\theta}[k_1 s(sc - \theta) + 2k_2\theta(s - \theta c)]$$

$$LF_V = \frac{wR}{2\theta}[k_1(2s - sc^2 - \theta c) + 2k_2(2sc^2 + s\theta^2 - s - \theta c)]$$

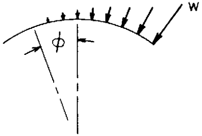
$$LF_M = \frac{wR}{2\theta}[k_2(\theta^2 - s^2)]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0003	0.0012	0.0074	0.0330	0.0451	0.1963
$\frac{LF_V}{wR}$	0.0054	0.0169	0.0737	0.2246	0.2854	0.8245
$\frac{LF_M}{wR}$	0.0059	0.0198	0.0461	0.1488	0.1488	0.4536

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

5l. Partial second-order increase in distributed radial loading



$$LF_H = \frac{wR}{(\theta+\phi)^2}\left\{k_1[(\theta+\phi)(2c+m) - 2s - 2n - c^2n - scm] + \frac{k_2}{3}[3(\theta+\phi)(\theta s + \phi s + 2c) - 6s - 6n - c(\theta+\phi)^3]\right\}$$

$$LF_V = \frac{wR}{(\theta+\phi)^2}\left\{k_1[(\theta+\phi)(2s-n) + mc^2 - 3m - scn + 2c] - \frac{k_2}{3}[3(\theta+\phi)(\theta c + \phi c + 2s) - 6c - 6m + 12c(mc - sn) - s(\theta+\phi)^3]\right\}$$

$$LF_M = \frac{wR}{(\theta+\phi)^2}\left\{\frac{k_2}{3}[6(sm + cn - \theta - \phi) + (\theta+\phi)^3]\right\}$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = \frac{wR}{2\theta^2}\left[k_1(3\theta c - 3s + s^3) + 2k_2\left(\theta c - s + s\theta^2 - \frac{2c\theta^3}{3}\right)\right]$$

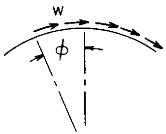
$$LF_V = \frac{wR}{2\theta^2}\left[k_1 s(\theta - sc) + 2k_2\left(2s^2c - \theta s - c\theta^2 + \frac{2s\theta^3}{3}\right)\right]$$

$$LF_M = \frac{wR}{\theta^2}\left[k_2\left(sc - \theta + \frac{2\theta^3}{3}\right)\right]$$

For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	0.0001	0.0004	0.0025	0.0116	0.0155	0.0701
$\frac{LF_V}{wR}$	0.0022	0.0070	0.0303	0.0947	0.1183	0.3579
$\frac{LF_M}{wR}$	0.0024	0.0080	0.0186	0.0609	0.0609	0.1913

5m. Partial uniformly distributed tangential loading



$$LF_H = \frac{wR}{2}[(\theta+\phi)^2c + k_1(\theta n + \phi n - scn - s^2m + 2m - 2c) + 2k_2(m - c - \theta s - \phi s)]$$

$$LF_V = \frac{wR}{2}[-(\theta+\phi)^2s + k_1(\theta m + \phi m + c^2n + scm - 2s - 2n) + 2k_2(\theta c + \phi c + 2s^2n - n - 2scm + s)]$$

$$LF_M = \frac{wR}{2}[-(\theta+\phi)^2 + 2k_2(1 + sn - cm)]$$

If $\phi = \theta$ (the full span is loaded)

$$LF_H = wR[2\theta^2c + k_1s(\theta - sc) - k_2 2\theta s]$$

$$LF_V = wR[-2\theta^2s + k_1(\theta c - s - s^3) + 2k_2(\theta c + s - 2sc^2)]$$

$$LF_M = wR(-2\theta^2 + k_2 2s^2)$$

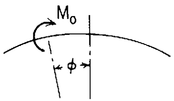
For $\alpha = \beta = 0$

θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H}{wR}$	−0.0001	−0.0009	−0.0077	−0.0518	−0.0708	−0.4624
$\frac{LF_V}{wR}$	−0.0028	−0.0133	−0.0772	−0.3528	−0.4483	−1.9428
$\frac{LF_M}{wR}$	−0.0031	−0.0155	−0.0483	−0.2337	−0.2337	−1.0687

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

Reference no., loading	Loading terms and some selected numerical values

5n. Concentrated couple

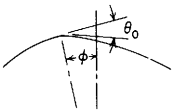


$$LF_H = \frac{M_o}{R}[(\theta + \phi)c - k_2(s + n)]$$
$$LF_V = \frac{M_o}{R}[-(\theta + \phi)s + k_2(c - m)]$$
$$LF_M = \frac{M_o}{R}(-\theta - \phi)$$

For $\alpha = \beta = 0$

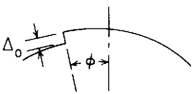
θ	30°		60°		90°	
ϕ	0°	15°	0°	30°	0°	45°
$\frac{LF_H R}{M_o}$	−0.0466	−0.0786	−0.3424	−0.5806	−1.0000	−1.7071
$\frac{LF_V R}{M_o}$	−0.3958	−0.4926	−1.4069	−1.7264	−2.5708	−3.0633
$\frac{LF_M R}{M_o}$	−0.5236	−0.7854	−1.0472	−1.5708	−1.5708	−2.3562

5p. Concentrated angular displacement



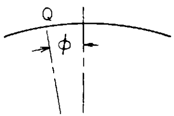
$$LF_H = \frac{\theta_o EI}{R^2}(m - c)$$
$$LF_V = \frac{\theta_o EI}{R^2}(s - n)$$
$$LF_M = \frac{\theta_o EI}{R^2}(1)$$

5q. Concentrated radial displacement



$$LF_H = \frac{\Delta_o EI}{R^3}(n)$$
$$LF_V = \frac{\Delta_o EI}{R^3}(m)$$
$$LF_M = 0$$

5r. Uniform temperature rise over that span to the right of point Q



$$LF_H = -(T - T_o)\frac{\gamma EI}{R^2}(n + s)$$
$$LF_V = (T - T_o)\frac{\gamma EI}{R^2}(c - m)$$
$$LF_M = 0$$

T = uniform temperature
T_o = unloaded temperature

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

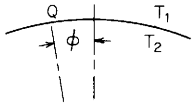
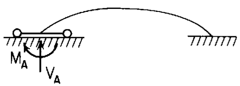
5s. Linear temperature differential through the thickness t for that span to the right of point Q 	$LF_H = (T_1 - T_2)\dfrac{\gamma EI}{Rt}(n + s - \theta c - \phi c)$ $LF_V = (T_1 - T_2)\dfrac{\gamma EI}{Rt}(m - c + \theta s + \phi s)$ $LF_M = (T_1 - T_2)\dfrac{\gamma EI}{Rt}(\theta + \phi)$ *Note:* The temperature at the centroidal axis is the initial unloaded temperature
6. Left end pinned, right end fixed 	Since $\delta_{HA} = 0, \delta_{VA} = 0,$ and $M_A = 0,$ $H_A = \dfrac{B_{VV}LF_H - B_{HV}LF_V}{B_{HH}B_{VV} - B_{HV}^2}$ $V_A = \dfrac{B_{HH}LF_V - B_{HV}LF_H}{B_{HH}B_{VV} - B_{HV}^2}$ $\psi_A = \dfrac{R^2}{EI}(B_{MH}H_A + B_{MV}V_A - LF_M)$ Use load terms given above for cases 5a–5s
7. Left end guided in horizontal direction, right end fixed 	Since $\delta_{VA} = 0, \psi_A = 0,$ and $H_A = 0,$ $V_A = \dfrac{B_{MM}LF_V - B_{MV}LF_M}{B_{VV}B_{MM} - B_{MV}^2}$ $\dfrac{M_A}{R} = \dfrac{B_{VV}LF_M - B_{MV}LF_V}{B_{VV}B_{MM} - B_{MV}^2}$ $\delta_{HA} = \dfrac{R^3}{EI}\left(B_{HV}V_A + B_{HM}\dfrac{M_A}{R} - LF_H\right)$ Use load terms given above for cases 5a–5s
8. Left end guided in vertical direction, right end fixed 	Since $\delta_{HA} = 0, \psi_A = 0,$ and $V_A = 0,$ $H_A = \dfrac{B_{MM}LF_H - B_{HM}LF_M}{B_{HH}B_{MM} - B_{HM}^2}$ $\dfrac{M_A}{R} = \dfrac{B_{HH}LF_M - B_{HM}LF_H}{B_{HH}B_{MM} - B_{HM}^2}$ $\delta_{VA} = \dfrac{R^3}{EI}\left(B_{VH}H_A + B_{VM}\dfrac{M_A}{R} - LF_V\right)$ Use load terms given above for cases 5a–5s

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

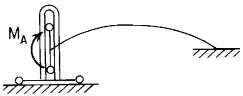
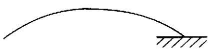
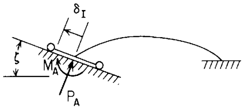
9. Left end roller supported in vertical direction only, right end fixed 	Since $\delta_{VA} = 0, H_A = 0,$ and $M_A = 0,$ Use load terms given above for cases 5a–5s	$V_A = \dfrac{LF_V}{B_{VV}}, \qquad \delta_{HA} = \dfrac{R^3}{EI}(B_{HV}V_A - LF_H)$ $\psi_A = \dfrac{R^2}{EI}(B_{MV}V_A - LF_M)$
10. Left end roller supported in horizontal direction only, right end fixed 	Since $\delta_{HA} = 0, V_A = 0,$ and $M_A = 0,$ Use load terms given above for cases 5a–5s	$H_A = \dfrac{LF_H}{B_{HH}}, \qquad \delta_{VA} = \dfrac{R^3}{EI}(B_{VH}H_A - LF_V)$ $\psi_A = \dfrac{R^2}{EI}(B_{MH}H_A - LF_M)$
11. Left end restrained against rotation only, right end fixed 	Since $\psi_A = 0, H_A = 0,$ and $V_A = 0,$ Use load terms given above for cases 5a–5s	$\dfrac{M_A}{R} = \dfrac{LF_M}{B_{MM}}, \qquad \delta_{HA} = \dfrac{R^3}{EI}\left(B_{HM}\dfrac{M_A}{R} - LF_H\right)$ $\delta_{VA} = \dfrac{R^3}{EI}\left(B_{VM}\dfrac{M_A}{R} - LF_V\right)$
12. Left end free, right end fixed 	Since $H_A = 0, V_A = 0,$ and $M_A = 0,$ this is a statically determinate problem. The deflections at the free end are given by Use load terms given above for cases 5a–5s	$\delta_{HA} = \dfrac{-R^3}{EI}LF_H, \qquad \delta_{VA} = \dfrac{-R^3}{EI}LF_V$ $\psi_A = \dfrac{-R^2}{EI}LF_M$

TABLE 9.3 Reaction and deformation formulas for circular arches (*Continued*)

13. Left end guided along an inclined surface at angle ζ, right end fixed

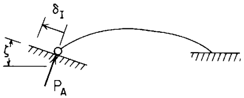


Since there is no deflection perpendicular to the incline, the following three equations must be solved for M_A, P_A, and δ_I:

$$\delta_I \frac{EI\cos\zeta}{R^3} = P_A(B_{HV}\cos\zeta - B_{HH}\sin\zeta) + B_{HM}\frac{M_A}{R} - LF_H$$
$$\delta_I \frac{EI\sin\zeta}{R^3} = P_A(B_{VV}\cos\zeta - B_{VH}\sin\zeta) + B_{VM}\frac{M_A}{R} - LF_V$$
$$0 = P_A(B_{MV}\cos\zeta - B_{MH}\sin\zeta) + B_{MM}\frac{M_A}{R} - LF_M$$

Use load terms given above for cases 5a–5s

14. Left end roller supported along an inclined surface at angle ζ, right end fixed



Since there is no deflection perpendicular to the incline and $M_A = 0$, the following equations give P_A, δ_I, and ψ_A:

$$P_A = \frac{LF_V\cos\zeta - LF_H\sin\zeta}{B_{HH}\sin^2\zeta - 2B_{HV}\sin\zeta\cos\zeta + B_{VV}\cos^2\zeta}$$
$$\delta_I = \frac{R^3}{EI}\{P_A[B_{HV}(\cos^2\zeta - \sin^2\zeta) + (B_{VV} - B_{HH})\sin\zeta\cos\zeta] - LF_H\cos\zeta - LF_V\sin\zeta\}$$
$$\psi_A = \frac{R^2}{EI}[P_A(B_{MV}\cos\zeta - B_{MH}\sin\zeta) - LF_M]$$

Use load terms given above for cases 5a–5s

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature

NOTATION: W = applied load normal to the plane of curvature (force); M_o = applied bending moment in a plane tangent to the curved axis of the beam (force-length); T_o = applied twisting moment in a plane normal to the curved axis of the beam (force-length); w = distributed load (force per unit length); t_o = distributed twisting moment (force-length per unit length); V_A = reaction force, M_A = reaction bending moment, T_A = reaction twisting moment, y_A = deflection normal to the plane of curvature, Θ_A = slope of the beam axis in the plane of the moment M_A, and ψ_A = roll of the beam cross section in the plane of the twisting moment T_A, all at the left end of the beam. Similarly, V_B, M_B, T_B, y_B, Θ_B, and ψ_B are the reactions and displacements at the right end: V, M, T, y, Θ, and ψ are internal shear forces, moments, and displacements at an angular position x rad from the left end. All loads and reactions are positive as shown in the diagram; y is positive upward; Θ is positive when y increases as x increases; and ψ is positive in the direction of T.

R = radius of curvature of the beam axis (length); E = modulus of elasticity (force per unit area); I = area moment of inertia about the bending axis (length to the fourth power) (note that this must be a principal axis of the beam cross section); G = modulus of rigidity (force per unit area); ν = Poisson's ratio; K = torsional stiffness constant of the cross section (length to the fourth power) (see page 383); θ = angle in radians from the left end to the position of the loading; ϕ = angle (radians) subtended by the entire span of the curved beam. See page 131 For a definition of the term $\langle x-\theta\rangle^n$.

The following constants and functions are hereby defined to permit condensing the tabulated formulas which follow. $\beta = EI/GK$.

$$F_1 = \frac{1+\beta}{2} x \sin x - \beta(1-\cos x)$$

$$F_2 = \frac{1+\beta}{2}(x\cos x - \sin x)$$

$$F_3 = -\beta(x-\sin x) - \frac{1+\beta}{2}(x\cos x - \sin x)$$

$$F_4 = \frac{1+\beta}{2} x\cos x + \frac{1-\beta}{2}\sin x$$

$$F_5 = -\frac{1+\beta}{2} x \sin x$$

$$F_6 = F_1$$

$$F_7 = F_5$$

$$F_8 = \frac{1-\beta}{2}\sin x - \frac{1+\beta}{2} x\cos x$$

$$F_9 = F_2$$

$$F_{a1} = \left\{\frac{1+\beta}{2}(x-\theta)\sin(x-\theta) - \beta[1-\cos(x-\theta)]\right\}\langle x-\theta\rangle^0$$

$$F_{a2} = \frac{1+\beta}{2}[(x-\theta)\cos(x-\theta) - \sin(x-\theta)]\langle x-\theta\rangle^0$$

$$F_{a3} = \{-\beta[x-\theta-\sin(x-\theta)] - F_{a2}\}\langle x-\theta\rangle^0$$

$$F_{a4} = \left[\frac{1+\beta}{2}(x-\theta)\cos(x-\theta) + \frac{1-\beta}{2}\sin(x-\theta)\right]\langle x-\theta\rangle^0$$

$$C_1 = \frac{1+\beta}{2}\phi\sin\phi - \beta(1-\cos\phi)$$

$$C_2 = \frac{1+\beta}{2}(\phi\cos\phi - \sin\phi)$$

$$C_3 = -\beta(\phi-\sin\phi) - \frac{1+\beta}{2}(\phi\cos\phi - \sin\phi)$$

$$C_4 = \frac{1+\beta}{2}\phi\cos\phi + \frac{1-\beta}{2}\sin\phi$$

$$C_5 = -\frac{1+\beta}{2}\phi\sin\phi$$

$$C_6 = C_1$$

$$C_7 = C_5$$

$$C_8 = \frac{1-\beta}{2}\sin\phi - \frac{1+\beta}{2}\phi\cos\phi$$

$$C_9 = C_2$$

$$C_{a1} = \frac{1+\beta}{2}(\phi-\theta)\sin(\phi-\theta) - \beta[1-\cos(\phi-\theta)]$$

$$C_{a2} = \frac{1+\beta}{2}[(\phi-\theta)\cos(\phi-\theta) - \sin(\phi-\theta)]$$

$$C_{a3} = -\beta[\phi-\theta-\sin(\phi-\theta)] - C_{a2}$$

$$C_{a4} = \frac{1+\beta}{2}(\phi-\theta)\cos(\phi-\theta) + \frac{1-\beta}{2}\sin(\phi-\theta)$$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

$$F_{a5} = -\frac{1+\beta}{2}(x-\theta)\sin(x-\theta)\langle x-\theta\rangle^0$$

$$F_{a6} = F_{a1}$$

$$F_{a7} = F_{a5}$$

$$F_{a8} = \left[\frac{1-\beta}{2}\sin(x-\theta) - \frac{1+\beta}{2}(x-\theta)\cos(x-\theta)\right]\langle x-\theta\rangle^0$$

$$F_{a9} = F_{a2}$$

$$F_{a12} = \frac{1+\beta}{2}[(x-\theta)\sin(x-\theta) - 2 + 2\cos(x-\theta)]\langle x-\theta\rangle^0$$

$$F_{a13} = \left[\beta\left\{1-\cos(x-\theta) - \frac{(x-\theta)^2}{2}\right\} - F_{a12}\right]\langle x-\theta\rangle^0$$

$$F_{a15} = F_{a2}$$

$$F_{a16} = F_{a3}$$

$$F_{a18} = \left[1-\cos(x-\theta) - \frac{1+\beta}{2}(x-\theta)\sin(x-\theta)\right]\langle x-\theta\rangle^0$$

$$F_{a19} = F_{a12}$$

$$C_{a5} = -\frac{1+\beta}{2}(\phi-\theta)\sin(\phi-\theta)$$

$$C_{a6} = C_{a1}$$

$$C_{a7} = C_{a5}$$

$$C_{a8} = \frac{1-\beta}{2}\sin(\phi-\theta) - \frac{1+\beta}{2}(\phi-\theta)\cos(\phi-\theta)$$

$$C_{a9} = C_{a2}$$

$$C_{a12} = \frac{1+\beta}{2}[(\phi-\theta)\sin(\phi-\theta) - 2 + 2\cos(\phi-\theta)]$$

$$C_{a13} = \beta\left[1-\cos(\phi-\theta) - \frac{(\phi-\theta)^2}{2}\right] - C_{a12}$$

$$C_{a15} = C_{a2}$$

$$C_{a16} = C_{a3}$$

$$C_{a18} = 1-\cos(\phi-\theta) - \frac{1+\beta}{2}(\phi-\theta)\sin(\phi-\theta)$$

$$C_{a19} = C_{a12}$$

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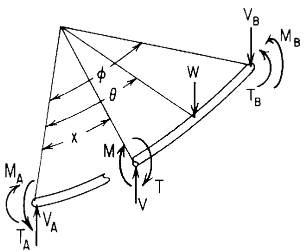
1. Concentrated intermediate lateral load



Transverse shear $= V = V_A - W\langle x-\theta\rangle^0$

Bending moment $= M = V_A R\sin x + M_A\cos x - T_A\sin x - WR\sin(x-\theta)\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A\sin x + T_A\cos x - WR[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

$$\text{Deflection} = y = y_A + \Theta_A R\sin x + \psi_A R(1-\cos x) + \frac{M_A R^2}{EI}F_1 + \frac{T_A R^2}{EI}F_2 + \frac{V_A R^3}{EI}F_3 - \frac{WR^3}{EI}F_{a3}$$

$$\text{Bending slope} = \Theta = \Theta_A\cos x + \psi_A\sin x + \frac{M_A R}{EI}F_4 + \frac{T_A R}{EI}F_5 + \frac{V_A R^2}{EI}F_6 - \frac{WR^2}{EI}F_{a6}$$

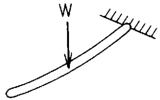
$$\text{Roll slope} = \psi = \psi_A\cos x - \Theta_A\sin x + \frac{M_A R}{EI}F_7 + \frac{T_A R}{EI}F_8 + \frac{V_A R^2}{EI}F_9 - \frac{WR^2}{EI}F_{a9}$$

For tabulated values: $V = K_V W, \quad M = K_M WR, \quad T = K_T WR, \quad y = K_y\frac{WR^3}{EI}, \quad \Theta = K_\Theta\frac{WR^2}{EI}, \quad \psi = K_\psi\frac{WR^2}{EI}$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

1a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{-WR^3}{EI}[C_{a6}\sin\phi - C_{a9}(1-\cos\phi) - C_{a3}], \qquad \Theta_A = \frac{WR^2}{EI}(C_{a6}\cos\phi - C_{a9}\sin\phi)$$

$$\psi_A = \frac{WR^2}{EI}(C_{a9}\cos\phi + C_{a6}\sin\phi)$$

$$V_B = -W$$

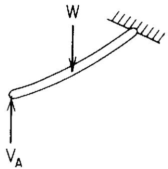
$$M_B = -WR\sin(\phi-\theta)$$

$$T_B = -WR[1-\cos(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	−0.1607	−1.2485	−0.6285	−0.1576	−7.6969	−3.7971	−0.6293
$K_{\Theta A}$	0.3058	1.1500	0.3938	0.0535	2.6000	−0.1359	−0.3929
$K_{\psi A}$	0.0590	0.5064	0.3929	0.1269	3.6128	2.2002	0.3938
K_{VB}	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000	−1.0000
K_{MB}	−0.7071	−1.0000	−0.8660	−0.5000	−0.0000	−0.8660	−0.8660
K_{TB}	−0.2929	−1.0000	−0.5000	−0.1340	−2.0000	−1.5000	−0.5000

1b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{C_{a9}(1-\cos\phi) - C_{a6}\sin\phi + C_{a3}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = \frac{WR^2}{EI}\frac{(C_{a3}C_9 - C_{a9}C_3)\sin\phi + (C_{a9}C_6 - C_{a6}C_9)(1-\cos\phi) + (C_{a6}C_3 - C_{a3}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = \frac{WR^2}{EI}\frac{[C_{a6}(C_3+C_9) - C_6(C_{a3}+C_{a9})]\sin\phi + (C_{a9}C_3 - C_{a3}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$V_B = V_A - W$$

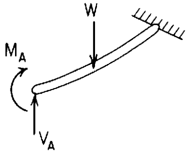
$$M_B = V_A R\sin\phi - WR\sin(\phi-\theta)$$

$$T_B = V_A R(1-\cos\phi) - WR[1-\cos(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.5136	0.1420	0.5034	0.1262	0.4933	0.0818
$K_{\Theta A}$	−0.0294	−0.0148	−0.1851	−0.0916	−1.4185	−0.6055
$K_{\psi A}$	0.0216	0.0106	0.1380	0.0630	0.4179	0.0984
K_{MB}	−0.1368	−0.1584	−0.3626	−0.3738	−0.8660	−0.8660
K_{TB}	0.0165	0.0075	0.0034	−0.0078	−0.5133	−0.3365
$K_{M\theta}$	0.1329	0.0710	0.2517	0.1093	0.4272	0.0708

1c. Right end fixed, left end supported and slope guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{(C_{a9}C_4 - C_{a6}C_7)(1-\cos\phi) + (C_{a6}C_1 - C_{a3}C_4)\cos\phi + (C_{a3}C_7 - C_{a9}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = WR\frac{(C_{a6}C_9 - C_{a9}C_6)(1-\cos\phi) + (C_{a3}C_6 - C_{a6}C_3)\cos\phi + (C_{a9}C_3) - C_{a3}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = \frac{WR^2}{EI}\frac{C_{a3}(C_4C_9 - C_6C_7) + C_{a6}(C_3C_7 - C_1C_9) + C_{a9}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A - W$$

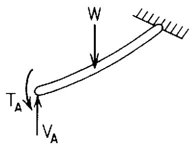
$$M_B = V_AR\sin\phi + M_A\cos\phi - WR\sin(\phi - \theta)$$

$$T_B = V_AR(1-\cos\phi) + M_A\sin\phi - WR[1 - \cos(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.7407	0.2561	0.7316	0.2392	0.6686	0.1566
K_{MA}	−0.1194	−0.0600	−0.2478	−0.1226	−0.5187	−0.2214
$K_{\psi A}$	−0.0008	−0.0007	−0.0147	−0.0126	−0.2152	−0.1718
K_{MB}	−0.0607	−0.1201	−0.1344	−0.2608	−0.3473	−0.6446
K_{TB}	−0.0015	−0.0015	−0.0161	−0.0174	−0.1629	−0.1869
$K_{M\theta}$	0.0764	0.0761	0.1512	0.1458	0.3196	0.2463

1d. Right end fixed, left end supported and roll guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{[(C_{a3} + C_{a9})C_5 - C_{a6}(C_2 + C_8)]\sin\phi + (C_{a3}C_8 - C_{a9}C_2)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = WR\frac{[C_{a6}(C_3 + C_9) - C_6(C_{a3} + C_{a9})]\sin\phi + (C_{a9}C_3 - C_{a3}C_9)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\frac{C_{a3}(C_5C_9 - C_6C_8) + C_{a6}(C_3C_8 - C_2C_9) + C_{a9}(C_2C_6 - C_3C_5)}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A - W$$

$$M_B = V_AR\sin\phi - T_A\sin\phi - WR\sin(\phi - \theta)$$

$$T_B = V_AR(1-\cos\phi) + T_A\cos\phi - WR[1 - \cos(\phi - \theta)]$$

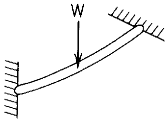
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.5053	0.1379	0.4684	0.1103	0.3910	0.0577
K_{TA}	−0.0226	−0.0111	−0.0862	−0.0393	−0.2180	−0.0513
$K_{\Theta A}$	−0.0252	−0.0127	−0.1320	−0.0674	−1.1525	−0.5429
K_{MB}	−0.1267	−0.1535	−0.3114	−0.3504	−0.8660	−0.8660
K_{TB}	−0.0019	−0.0015	−0.0316	−0.0237	−0.5000	−0.3333
$K_{M\theta}$	0.1366	0.0745	0.2773	0.1296	0.5274	0.0944

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

1e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{C_{a3}(C_4C_8 - C_5C_7) + C_{a6}(C_2C_7 - C_1C_8) + C_{a9}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = WR\frac{C_{a3}(C_5C_9 - C_6C_8) + C_{a6}(C_3C_8 - C_2C_9) + C_{a9}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = WR\frac{C_{a3}(C_6C_7 - C_4C_9) + C_{a6}(C_1C_9 - C_3C_7) + C_{a9}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A - W$$

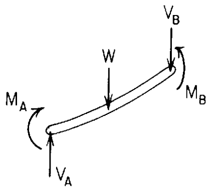
$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - WR\sin(\phi - \theta)$$

$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi - WR[1 - \cos(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°	360°	
θ	15°	30°	60°	90°	90°	180°
K_{VA}	0.7424	0.7473	0.7658	0.7902	0.9092	0.5000
K_{MA}	−0.1201	−0.2589	−0.5887	−0.8488	−0.9299	−0.3598
K_{TA}	0.0009	0.0135	0.1568	0.5235	0.7500	1.0000
K_{MB}	−0.0606	−0.1322	−0.2773	−0.2667	0.0701	−0.3598
K_{TB}	−0.0008	−0.0116	−0.1252	−0.3610	−0.2500	−1.0000
$K_{M\theta}$	0.0759	0.1427	0.2331	0.2667	0.1592	0.3598

1f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = W\frac{[-C_1\sin\phi + C_4(1 - \cos\phi)][1 - \cos(\phi - \theta)] + C_{a3}\sin^2\phi - C_{a6}\sin\phi(1 - \cos\phi)}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$M_A = +WR\frac{C_{a6}(1 - \cos\phi)^2 - C_{a3}(1 - \cos\phi)\sin\phi + [C_3\sin\phi - C_6(1 - \cos\phi)][1 - \cos(\phi - \theta)]}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$\psi_A = \frac{WR^2}{EI}\frac{(C_{a3}C_4 - C_{a6}C_1)(1 - \cos\phi) - (C_{a3}C_6 - C_{a6}C_3)\sin\phi - (C_3C_4 - C_1C_6)[1 - \cos(\phi - \theta)]}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$V_B = V_A - W$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - WR\sin(\phi - \theta)$$

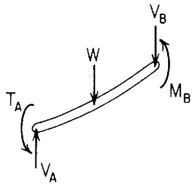
$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 - \frac{WR^2}{EI}C_{a9}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°
θ	15°	30°	60°	90°
K_{VA}	0.7423	0.7457	0.7500	0.7414
K_{MA}	−0.1180	−0.2457	−0.5774	−1.2586
$K_{\psi A}$	−0.0024	−0.0215	−0.2722	−2.5702
K_{MB}	−0.0586	−0.1204	−0.2887	−0.7414
$K_{\psi B}$	−0.0023	−0.0200	−0.2372	−2.3554
$K_{M\theta}$	0.0781	0.1601	0.3608	0.7414

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

1g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$

$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = W\frac{(C_5 \sin\phi - C_2\cos\phi)[1-\cos(\phi-\theta)] + C_{a3}\cos^2\phi - C_{a6}\sin\phi\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = WR\frac{(C_3\cos\phi - C_6\sin\phi)[1-\cos(\phi-\theta)] - (C_{a3}\cos\phi - C_{a6}\sin\phi)(1-\cos\phi)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\frac{(C_2C_6 - C_3C_5)[1-\cos(\phi-\theta)] + (C_{a3}C_5 - C_{a6}C_2)(1-\cos\phi) + (C_{a6}C_3 - C_{a3}C_6)\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A - W$$

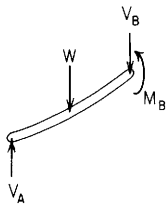
$$M_B = V_AR\sin\phi - T_A\sin\phi - WR\sin(\phi-\theta)$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_AR}{EI}C_8 + \frac{V_AR^2}{EI}C_9 - \frac{WR^2}{EI}C_{a9}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.5087	0.1405	0.5000	0.1340	0.6257	0.2141
K_{TA}	−0.0212	−0.0100	−0.0774	−0.0327	−0.2486	−0.0717
$K_{\Theta A}$	−0.0252	−0.0127	−0.1347	−0.0694	−1.7627	−0.9497
K_{MB}	−0.1253	−0.1524	−0.2887	−0.3333	−0.8660	−0.8660
$K_{\psi B}$	−0.0016	−0.0012	−0.0349	−0.0262	−0.9585	−0.6390
$K_{M\theta}$	0.1372	0.0753	0.2887	0.1443	0.7572	0.2476

1h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = W\frac{1-\cos(\phi-\theta)}{1-\cos\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\left\{\frac{C_{a3}\sin\phi + C_6[1-\cos(\phi-\theta)]}{1-\cos\phi} - \frac{C_3\sin\phi[1-\cos(\phi-\theta)]}{(1-\cos\phi)^2} - C_{a6}\right\}$$

$$\psi_A = \frac{WR^2}{EI}\left\{\frac{C_{a6}\sin\phi - C_{a3}\cos\phi}{1-\cos\phi} - (C_6\sin\phi - C_3\cos\phi)\frac{1-\cos(\phi-\theta)}{(1-\cos\phi)^2}\right\}$$

$$V_B = V_A - W$$

$$M_B = V_AR\sin\phi - WR\sin(\phi-\theta)$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_AR^2}{EI}C_9 - \frac{WR^2}{EI}C_{a9}$$

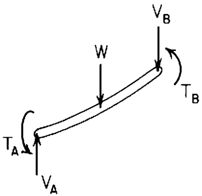
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	0.4574	0.1163	0.5000	0.1340	0.7500	0.2500
$K_{\Theta A}$	−0.0341	−0.0169	−0.1854	−0.0909	−2.0859	−1.0429
$K_{\psi A}$	0.0467	0.0220	0.1397	0.0591	0.4784	0.1380
K_{MB}	−0.1766	−0.1766	−0.3660	−0.3660	−0.8660	−0.8660
$K_{\psi B}$	0.0308	0.0141	0.0042	−0.0097	−0.9878	−0.6475
$K_{M\theta}$	0.1184	0.0582	0.2500	0.1160	0.6495	0.2165

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

1i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$

$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{(C_{a3}+C_{a9})\sin\phi + (C_2+C_8)\sin(\phi-\theta)}{(C_2+C_3+C_8+C_9)\sin\phi}$$

$$T_A = WR\frac{(C_{a3}+C_{a9})\sin\phi - (C_3+C_9)\sin(\phi-\theta)}{(C_2+C_3+C_8+C_9)\sin\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\,\frac{C_{a3}(C_8+C_9) - C_{a9}(C_2+C_3) + (C_2C_9 - C_3C_8)\sin(\phi-\theta)/\sin\phi}{(C_2+C_3+C_8+C_9)\sin\phi}$$

$$V_B = V_A - W$$

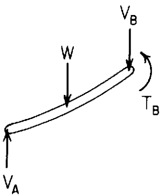
$$T_B = V_A R(1-\cos\phi) + T_A\cos\phi - WR[1-\cos(\phi-\theta)]$$

$$\Theta_B = \Theta_A\cos\phi + \frac{V_AR^2}{EI}C_6 + \frac{T_AR}{EI}C_5 - \frac{WR^2}{EI}C_{a6}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	270°
θ	15°	30°	90°
K_{VA}	0.6667	0.6667	0.6667
K_{TA}	−0.0404	−0.1994	0.0667
$K_{\Theta A}$	−0.0462	−0.3430	−4.4795
K_{TB}	0.0327	0.1667	−1.3333
$K_{\Theta B}$	0.0382	0.3048	1.7333
$K_{M\theta}$	0.1830	0.4330	0.0000

1j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = W\frac{\sin(\phi-\theta)}{\sin\phi}$$

$$\Theta_A = \frac{WR^2}{EI}\left\{\frac{C_{a3}\cos\phi - C_{a9}(1-\cos\phi)}{\sin\phi} - [C_3\cos\phi - C_9(1-\cos\phi)]\frac{\sin(\phi-\theta)}{\sin^2\phi}\right\}$$

$$\psi_A = \frac{WR^2}{EI}\left[C_{a3} + C_{a9} - (C_3+C_9)\frac{\sin(\phi-\theta)}{\sin\phi}\right]$$

$$V_B = V_A - W$$

$$T_B = V_A R(1-\cos\phi) - WR[1-\cos(\phi-\theta)]$$

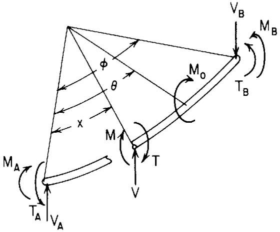
$$\Theta_B = \Theta_A\cos\phi + \psi_A\sin\phi + \frac{V_AR^2}{EI}C_6 - \frac{WR^2}{EI}C_{a6}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		270°	
θ	15°	30°	30°	60°	90°	180°
K_{VA}	0.7071	0.3660	0.8660	0.5000	0.0000	−1.0000
$K_{\Theta A}$	−0.0575	−0.0473	−0.6021	−0.5215	−3.6128	0.0000
$K_{\psi A}$	0.0413	0.0334	0.4071	0.3403	−4.0841	−8.1681
K_{TB}	0.0731	0.0731	0.3660	0.3660	−2.0000	−2.0000
$K_{\Theta B}$	0.0440	0.0509	0.4527	0.4666	6.6841	14.3810
$K_{M\theta}$	0.1830	0.1830	0.4330	0.4330	0.0000	0.0000

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

2. Concentated intermediate bending moment



Transverse shear $= V = V_A$

Bending moment $= M = V_A R \sin x + M_A \cos x - T_A \sin x + M_o \cos(x-\theta)\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A \sin x + T_A \cos x + M_o \sin(x-\theta)\langle x-\theta\rangle^0$

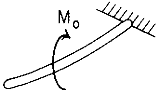
Vertical deflection $= y = y_A + \Theta_A R \sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI} F_1 + \dfrac{T_A R^2}{EI} F_2 + \dfrac{V_A R^3}{EI} F_3 + \dfrac{M_o R^2}{EI} F_{a1}$

Bending slope $= \Theta = \Theta_A \cos x + \psi_A \sin x + \dfrac{M_A R}{EI} F_4 + \dfrac{T_A R}{EI} F_5 + \dfrac{V_A R^2}{EI} F_6 + \dfrac{M_o R}{EI} F_{a4}$

Roll slope $= \psi = \psi_A \cos x - \Theta_A \sin x + \dfrac{M_A R}{EI} F_7 + \dfrac{T_A R}{EI} F_8 + \dfrac{V_A R^2}{EI} F_9 + \dfrac{M_o R}{EI} F_{a7}$

For tabulated values $V = K_V \dfrac{M_o}{R}$, $M = K_M M_o$, $T = K_T M_o$ $y = K_y \dfrac{M_o R^2}{EI}$, $\Theta = K_\Theta \dfrac{M_o R}{EI}$, $\psi = K_\psi \dfrac{M_o R}{EI}$

End restraints,reference no.	Formulas for boundary values and selected numerical values
2a. Right end fixed, left end free	$y_A = \dfrac{M_o R^2}{EI}[C_{a4}\sin\phi - C_{a7}(1-\cos\phi) - C_{a1}]$



$V_A = 0$ $M_A = 0$ $T_A = 0$
$y_B = 0$ $\Theta_B = 0$ $\psi_B = 0$

$\Theta_A = \dfrac{M_o R}{EI}(C_{a7}\sin\phi - C_{a4}\cos\phi)$

$\psi_A = -\dfrac{M_o R}{EI}(C_{a4}\sin\phi + C_{a7}\cos\phi)$

$V_B = 0$, $M_B = M_o\cos(\phi-\theta)$

$T_B = M_o \sin(\phi-\theta)$

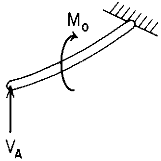
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	0.3058	1.1500	1.1222	0.6206	2.6000	4.0359	1.6929
$K_{\Theta A}$	−0.8282	−1.8064	−1.0429	−0.3011	−3.6128	−1.3342	0.4722
$K_{\psi A}$	0.0750	0.1500	−0.4722	−0.4465	0.0000	−2.0859	−1.0429
K_{MB}	0.7071	0.0000	0.5000	0.8660	−1.0000	−0.5000	0.5000
K_{TB}	0.7071	1.0000	0.8660	0.5000	0.0000	0.8660	0.8660

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

2b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = \frac{-M_o}{R}\frac{C_{a7}(1-\cos\phi) - C_{a4}\sin\phi + C_{a1}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = -\frac{M_oR}{EI}\frac{(C_{a1}C_9 - C_{a7}C_3)\sin\phi + (C_{a7}C_6 - C_{a4}C_9)(1-\cos\phi) + (C_{a4}C_3 - C_{a1}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = -\frac{M_oR}{EI}\frac{[(C_{a4}(C_9 + C_3) - C_6(C_{a1} + C_{a7})]\sin\phi + (C_{a7}C_3 - C_{a1}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$V_B = V_A$

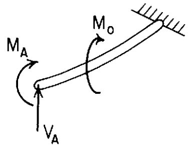
$M_B = V_A R\sin\phi + M_o\cos(\phi - \theta)$

$T_B = V_A R(1-\cos\phi) + M_o\sin(\phi - \theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−1.9021	−0.9211	−0.8989	−0.4971	−0.3378	−0.5244	−0.2200
$K_{\Theta A}$	−0.2466	−0.7471	−0.0092	0.2706	−2.7346	0.0291	1.0441
$K_{\psi A}$	0.1872	0.6165	−0.0170	−0.1947	1.2204	−0.1915	−0.2483
K_{MB}	−0.6379	−0.9211	−0.3989	0.3689	−1.0000	−0.5000	0.5000
K_{TB}	0.1500	0.0789	−0.0329	0.0029	−0.6756	−0.1827	0.4261

2c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{(C_{a7}C_4 - C_{a4}C_7)(1-\cos\phi) + (C_{a4}C_1 - C_{a1}C_4)\cos\phi + (C_{a1}C_7 - C_{a7}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = -M_o\frac{(C_{a4}C_9 - C_{a7}C_6)(1-\cos\phi) + (C_{a1}C_6 - C_{a4}C_3)\cos\phi + (C_{a7}C_3 - C_{a1}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = -\frac{M_oR}{EI}\frac{C_{a1}(C_4C_9 - C_6C_7) + C_{a4}(C_3C_7 - C_1C_9) + C_{a7}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$V_B = V_A$

$M_B = V_A R\sin\phi + M_A\cos\phi + M_o\cos(\phi - \theta)$

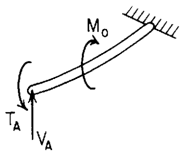
$T_B = V_A R(1-\cos\phi) + M_A\sin\phi + M_o\sin(\phi - \theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	−1.7096	−1.6976	−0.8876	−0.8308	−0.5279	−0.3489
K_{MA}	−0.0071	0.3450	−0.0123	0.3622	0.0107	0.3818
$K_{\psi A}$	−0.0025	0.0029	−0.0246	0.0286	−0.1785	0.2177
K_{MB}	−0.3478	0.0095	−0.3876	0.0352	−0.5107	0.1182
K_{TB}	−0.0057	0.0056	−0.0338	0.0314	−0.1899	0.1682

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

2d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{[(C_{a1}+C_{a7})C_5 - C_{a4}(C_2+C_8)]\sin\phi + (C_{a1}C_8 - C_{a7}C_2)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = -M_o\frac{[C_{a4}(C_3+C_9) - (C_{a1}+C_{a7})C_6]\sin\phi + (C_{a7}C_3 - C_{a1}C_9)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = -\frac{M_oR}{EI}\frac{C_{a1}(C_5C_9 - C_6C_8) + C_{a4}(C_3C_8 - C_2C_9) + C_{a7}(C_2C_6 - C_3C_5)}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A$$

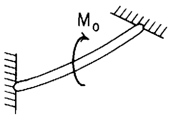
$$M_B = V_AR\sin\phi - T_A\sin\phi + M_o\cos(\phi-\theta)$$

$$T_B = V_AR(1-\cos\phi) + T_A\cos\phi + M_o\sin(\phi-\theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	30°	60°
K_{VA}	−1.9739	−1.0773	−0.8946	−0.4478	−0.6366	−0.4775	−0.1592
K_{TA}	−0.1957	−0.3851	0.0106	0.1216	−0.6366	0.0999	0.1295
$K_{\Theta A}$	−0.2100	−0.5097	−0.0158	0.1956	−1.9576	−0.0928	0.8860
K_{MB}	−0.5503	−0.6923	−0.4052	0.2966	−1.0000	−0.5000	0.5000
K_{TB}	−0.0094	−0.0773	−0.0286	0.0522	−0.6366	−0.1888	0.4182

2e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{C_{a1}(C_4C_8 - C_5C_7) + C_{a4}(C_2C_7 - C_1C_8) + C_{a7}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = -M_o\frac{C_{a1}(C_5C_9 - C_6C_8) + C_{a4}(C_3C_8 - C_2C_9 + C_{a7}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = -M_o\frac{C_{a1}(C_6C_7 - C_4C_9) + C_{a4}(C_1C_9 - C_3C_7) + C_{a7}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi + M_o\cos(\phi-\theta)$$

$$T_B = V_AR(1-\cos\phi) + M_A\sin\phi + T_A\cos\phi + M_o\sin(\phi-\theta)$$

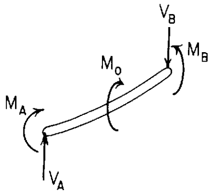
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°	360°	
θ	15°	30°	60°	90°	90°	180°
K_{VA}	−1.7040	−0.8613	−0.4473	−0.3115	−0.1592	−0.3183
K_{MA}	−0.0094	−0.0309	−0.0474	0.0584	−0.0208	0.5000
K_{TA}	0.0031	0.0225	0.1301	0.2788	0.5908	−0.3183
K_{MB}	−0.3477	−0.3838	−0.4526	−0.4097	−0.0208	−0.5000
K_{TB}	−0.0036	−0.0262	−0.1586	−0.3699	−0.4092	−0.3183

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

2f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = +\frac{M_o}{R}\,\frac{[C_1 \sin\phi - C_4(1-\cos\phi)]\sin(\phi-\theta) - C_{a1}\sin^2\phi + C_{a4}\sin\phi(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$M_A = -M_o\,\frac{[C_3\sin\phi - C_6(1-\cos\phi)]\sin(\phi-\theta) - C_{a1}(1-\cos\phi)\sin\phi + C_{a4}(1-\cos\phi)^2}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$\psi_A = \frac{MoR}{EI}\,\frac{(C_3C_4 - C_1C_6)\sin(\phi-\theta) + (C_{a1}C_6 - C_{a4}C_3)\sin\phi - (C_{a1}C_4 - C_{a4}C_1)(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

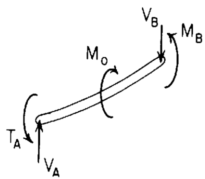
$$V_B = V_A$$

$$M_B = V_A R\sin\phi + M_A\cos\phi + M_o\cos(\phi-\theta)$$

$$\psi_B = \psi_A\cos\phi + \frac{M_A R}{EI}C_7 + \frac{V_A R^2}{EI}C_9 + \frac{M_o R}{EI}C_{a7}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)				
ϕ	45°	90°	180°	270°
θ	15°	30°	60°	90°
K_{VA}	−1.7035	−0.8582	−0.4330	−0.2842
K_{MA}	−0.0015	−0.0079	−0.0577	−0.2842
$K_{\psi A}$	−0.0090	−0.0388	−0.2449	−1.7462
K_{MB}	−0.3396	−0.3581	−0.4423	−0.7159
$K_{\psi B}$	−0.0092	−0.0418	−0.2765	−1.8667

2g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{M_o}{R}\,\frac{C_{a1}\cos^2\phi - C_{a4}\sin\phi\cos\phi + (C_5\sin\phi - C_2\cos\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = -M_o\,\frac{(C_{a4}\sin\phi - C_{a1}\cos\phi)(1-\cos\phi) + (C_3\cos\phi - C_6\sin\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = \frac{-M_o R}{EI}\,\frac{(C_{a1}C_5 - C_{a4}C_2)(1-\cos\phi) + (C_{a4}C_3 - C_{a1}C_6)\cos\phi + (C_2C_6 - C_3C_5)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\cos\phi}$$

$$V_B = V_A$$

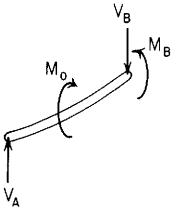
$$M_B = V_A R\sin\phi - T_A\sin\phi + M_o\cos(\phi-\theta)$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_A R}{EI}C_8 + \frac{V_A R^2}{EI}C_9 + \frac{M_o R}{EI}C_{a7}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)							
ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−1.9576	−1.0000	−0.8660	−0.5000	−0.3378	−0.3888	−0.3555
K_{TA}	−0.1891	−0.3634	0.0186	0.1070	−0.6756	0.0883	0.1551
$K_{\Theta A}$	−0.2101	−0.5163	−0.0182	0.2001	−2.7346	−0.3232	1.3964
K_{MB}	−0.5434	−0.6366	−0.3847	0.2590	−1.0000	−0.5000	0.5000
$K_{\psi B}$	−0.0076	−0.0856	−0.0316	0.0578	−1.2204	−0.3619	0.8017

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

2h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{M_o}{R}\frac{\sin(\phi-\theta)}{1-\cos\phi}$$

$$\Theta_A = -\frac{M_oR}{EI}\left[\frac{C_{a1}\sin\phi + C_6\sin(\phi-\theta)}{1-\cos\phi} - \frac{C_3\sin\phi\sin(\phi-\theta)}{(1-\cos\phi)^2} - C_{a4}\right]$$

$$\psi_A = -\frac{M_oR}{EI}\left[\frac{C_{a4}\sin\phi - C_{a1}\cos\phi}{1-\cos\phi} + \frac{(C_3\cos\phi - C_6\sin\phi)\sin(\phi-\theta)}{(1-\cos\phi)^2}\right]$$

$$V_B = V_A$$

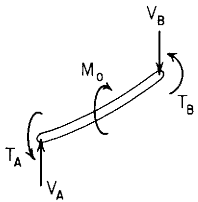
$$M_B = V_AR\sin\phi + M_o\cos(\phi-\theta)$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_AR^2}{EI}C_9 + \frac{M_oR}{EI}C_{a7}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−2.4142	−1.0000	−0.8660	−0.5000	0.0000	−0.4330	−0.4330
$K_{\Theta A}$	−0.2888	−0.7549	−0.0060	0.2703	−3.6128	−0.2083	1.5981
$K_{\psi A}$	0.4161	0.6564	−0.0337	−0.1933	1.3000	−0.1700	−0.2985
K_{MB}	−1.0000	−1.0000	−0.3660	0.3660	−1.0000	−0.5000	0.5000
$K_{\psi B}$	0.2811	0.0985	−0.0410	0.0036	−1.3000	−0.3515	0.8200

2i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o}{R}\frac{(C_{a1}+C_{a7})\sin^2\phi + (C_2+C_8)\cos(\phi-\theta)\sin\phi}{(C_2+C_3+C_8+C_9)\sin^2\phi}$$

$$T_A = -M_o\frac{(C_{a1}+C_{a7})\sin^2\phi - (C_3+C_9)\cos(\phi-\theta)\sin\phi}{(C_2+C_3+C_8+C_9)\sin^2\phi}$$

$$\Theta_A = -\frac{M_oR}{EI}\frac{[C_{a1}(C_8+C_9) - C_{a7}(C_2+C_3)]\sin\phi + (C_2C_9 - C_3C_8)\cos(\phi-\theta)}{(C_2+C_3+C_8+C_9)\sin^2\phi}$$

$$V_B = V_A$$

$$T_B = V_AR(1-\cos\phi) + T_A\cos\phi + M_o\sin(\phi-\theta)$$

$$\Theta_B = \Theta_A\cos\phi + \frac{T_AR}{EI}C_5 + \frac{V_AR^2}{EI}C_6 + \frac{M_oR}{EI}C_{a4}$$

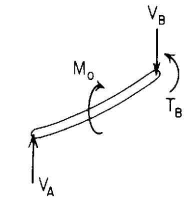
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		270°	
θ	0°	15°	0°	30°	0°	90°
K_{VA}	−1.2732	−1.2732	−0.6366	−0.6366	−0.2122	−0.2122
K_{TA}	−0.2732	−0.0485	−0.6366	−0.1366	−0.2122	0.7878
$K_{\Theta A}$	−0.3012	−0.0605	−0.9788	−0.2903	−5.1434	0.1259
K_{TB}	0.1410	0.0928	0.3634	0.2294	−1.2122	−0.2122
$K_{\Theta B}$	0.1658	0.1063	0.6776	0.3966	0.4259	2.0823

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

2j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{M_o \cos(\phi - \theta)}{R \sin\phi}$$

$$\Theta_A = -\frac{M_o R}{EI}\left\{\frac{C_{a1}\cos\phi - C_{a7}(1-\cos\phi)}{\sin\phi} - \frac{[C_3\cos\phi - C_9(1-\cos\phi)]\cos(\phi-\theta)}{\sin^2\phi}\right\}$$

$$\psi_A = -\frac{M_o R}{EI}\left[C_{a1} + C_{a7} - \frac{(C_3 + C_9)\cos(\phi-\theta)}{\sin\phi}\right]$$

$$V_B = V_A$$

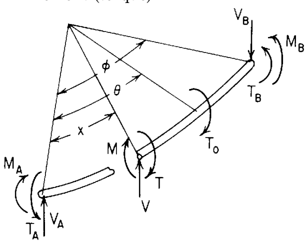
$$T_B = V_A R(1-\cos\phi) + M_o \sin(\phi-\theta)$$

$$\Theta_B = \Theta_A \cos\phi + \psi_A \sin\phi + \frac{V_A R^2}{EI} C_6 + \frac{M_o R}{EI} C_{a4}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	−1.0000	−1.2247	−1.3660	0.0000	−0.5000	−0.8660
$K_{\Theta A}$	−0.3774	−0.0740	0.1322	−1.8064	−0.4679	0.6949
$K_{\psi A}$	0.2790	0.0495	−0.0947	1.3000	0.2790	−0.4684
K_{TB}	0.4142	0.1413	−0.1413	1.0000	0.3660	−0.3660
$K_{\Theta B}$	0.2051	0.1133	−0.0738	1.1500	0.4980	−0.4606

3. Concentrated intermediate twisting moment (torque)



Transverse shear $= V = V_A$

Bending moment $= M = V_A R \sin x + M_A \cos x - T_A \sin x - T_0 \sin(x-\theta)\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A \sin x + T_A \cos x + T_0 \cos(x-\theta)\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R \sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI}F_1 + \dfrac{T_A R^2}{EI}F_2 + \dfrac{V_A R^3}{EI}F_3 + \dfrac{T_o R^2}{EI}F_{a2}$

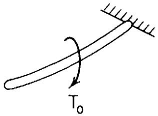
Bending slope $= \Theta = \Theta_A \cos x + \psi_A \sin x + \dfrac{M_A R}{EI}F_4 + \dfrac{T_A R}{EI}F_5 + \dfrac{V_A R^2}{EI}F_6 + \dfrac{T_o R}{EI}F_{a5}$

Roll slope $= \psi = \psi_A \cos x - \Theta_A \sin x + \dfrac{M_A R}{EI}F_7 + \dfrac{T_A R}{EI}F_8 + \dfrac{V_A R^2}{EI}F_9 + \dfrac{T_o R}{EI}F_{a8}$

For tabulated values: $V = K_V \dfrac{T_o}{R}$, $\quad M = K_M T_o$, $\quad T = K_T T_o$, $\quad y = K_y \dfrac{T_o R^2}{EI}$, $\quad \Theta = K_\Theta \dfrac{T_o R}{EI}$, $\quad \psi = K_\psi \dfrac{T_o R}{EI}$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

3a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{T_o R^2}{EI}[C_{a5}\sin\phi - C_{a8}(1-\cos\phi) - C_{a2}]$$

$$\Theta_A = -\frac{T_o R}{EI}(C_{a5}\cos\phi - C_{a8}\sin\phi)$$

$$\psi_A = -\frac{T_o R}{EI}(C_{a8}\cos\phi + C_{a5}\sin\phi)$$

$$V_B = 0$$

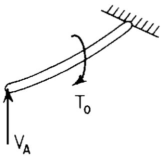
$$M_B = -T_o \sin(\phi - \theta)$$

$$T_B = T_o \cos(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	−0.0590	−0.5064	0.0829	0.3489	−3.6128	0.0515	1.8579
$K_{\Theta A}$	−0.0750	−0.1500	−0.7320	−0.5965	0.0000	−2.0859	−1.0429
$K_{\psi A}$	0.9782	1.8064	1.0429	0.3011	3.6128	1.0744	−0.7320
K_{MB}	−0.7071	−1.0000	−0.8660	−0.5000	0.0000	−0.8660	−0.8660
K_{TB}	0.7071	0.0000	0.5000	0.8660	−1.0000	−0.5000	0.5000

3b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{C_{a8}(1-\cos\phi) - C_{a5}\sin\phi + C_{a2}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = -\frac{T_o R}{EI}\frac{(C_{a2}C_9 - C_{a8}C_3)\sin\phi + (C_{a8}C_6 - C_{a5}C_9)(1-\cos\phi) + (C_{a5}C_3 - C_{a2}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = -\frac{T_o R}{EI}\frac{[C_{a5}(C_9 + C_3) - C_6(C_{a2} + C_{a8}]\sin\phi + (C_{a8}C_3 - C_{a2}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$V_B = V_A$$

$$M_B = V_A R\sin\phi - T_o\sin(\phi - \theta)$$

$$T_B = V_A R(1-\cos\phi) + T_o\cos(\phi - \theta)$$

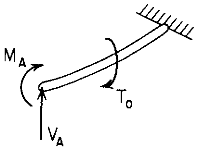
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.3668	0.4056	−0.0664	−0.2795	0.4694	−0.0067	−0.2414
$K_{\Theta A}$	−0.1872	−0.6165	−0.6557	−0.2751	−1.2204	−2.0685	−0.4153
$K_{\psi A}$	0.9566	1.6010	1.0766	0.4426	1.9170	1.0985	0.1400
K_{MB}	−0.4477	−0.5944	−0.9324	−0.7795	0.0000	−0.8660	−0.8660
K_{TB}	0.8146	0.4056	0.4336	0.5865	−0.0612	−0.5134	0.0172

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

3c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{(C_{a8}C_4 - C_{a5}C_7)(1-\cos\phi) + (C_{a5}C_1 - C_{a2}C_4)\cos\phi + (C_{a2}C_7 - C_{a8}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = -T_o\frac{(C_{a5}C_9 - C_{a8}C_6)(1-\cos\phi) + (C_{a2}C_6 - C_{a5}C_3)\cos\phi + (C_{a8}C_3 - C_{a2}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = -\frac{T_oR}{EI}\frac{C_{a2}(C_4C_9 - C_6C_7) + C_{a5}(C_3C_7 - C_1C_9) + C_{a8}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$V_B = V_A$

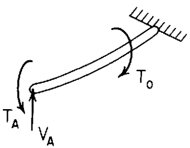
$M_B = V_AR\sin\phi + M_A\cos\phi - T_o\sin(\phi - \theta)$

$T_B = V_AR(1-\cos\phi) + M_A\sin\phi + T_o\cos(\phi - \theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	1.8104	1.1657	0.7420	0.0596	0.6201	0.2488	−0.1901
K_{MA}	−0.7589	−0.8252	−0.8776	−0.3682	−0.4463	−0.7564	−0.1519
$K_{\psi A}$	0.8145	1.0923	0.5355	0.2156	1.3724	0.1754	−0.0453
K_{MB}	0.0364	0.1657	−0.1240	−0.4404	0.4463	−0.1096	−0.7141
K_{TB}	0.7007	0.3406	0.3644	0.5575	0.2403	−0.0023	0.1199

3d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\frac{[(C_{a2} + C_{a8})C_5 - C_{a5}(C_2 + C_8)]\sin\phi + (C_{a2}C_8 - C_{a8}C_2)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = -T_o\frac{[C_{a5}(C_3 + C_9) - C_6(C_{a2} + C_{a8})]\sin\phi + (C_{a8}C_3 - C_{a2}C_9)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = -\frac{T_oR}{EI}\frac{C_{a2}(C_5C_9 - C_6C_8) + C_{a5}(C_3C_8 - C_2C_9) + C_{a8}(C_2C_6 - C_3C_5)}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$V_B = V_A$

$M_B = V_AR\sin\phi - T_A\sin\phi - T_o\sin(\phi - \theta)$

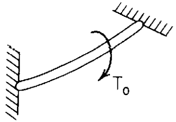
$T_B = V_AR(1-\cos\phi) + T_A\cos\phi + T_o\cos(\phi - \theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	−0.3410	−0.4177	−0.3392	−0.3916	−0.2757	−0.2757
K_{TA}	−0.6694	−0.3198	−0.6724	−0.2765	−0.5730	−0.0730
$K_{\Theta A}$	−0.0544	−0.0263	−0.2411	−0.1046	−1.3691	−0.3262
K_{MB}	−0.2678	−0.3280	−0.5328	−0.6152	−0.8660	−0.8660
K_{TB}	0.2928	0.6175	0.1608	0.4744	−0.4783	0.0217

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

3e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -\frac{T_o}{R}\,\frac{C_{a2}(C_4C_8 - C_5C_7) + C_{a5}(C_2C_7 - C_1C_8) + C_{a8}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = -T_o\,\frac{C_{a2}(C_5C_9 - C_6C_8) + C_{a5}(C_3C_8 - C_2C_9) + C_{a8}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = -T_o\,\frac{C_{a2}(C_6C_7 - C_4C_9) + C_{a5}(C_1C_9 - C_3C_7) + C_{a8}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A$$

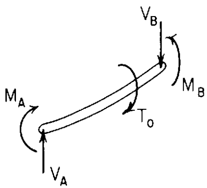
$$M_B = V_A R \sin\phi + M_A \cos\phi - T_A \sin\phi - T_o \sin(\phi - \theta)$$

$$T_B = V_A R(1 - \cos\phi) + M_A \sin\phi + T_A \cos\phi + T_o \cos(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°	180°	270°	360°	
θ	15°	30°	60°	90°	90°	180°
K_{VA}	0.1704	0.1705	0.1696	0.1625	0.1592	0.0000
K_{MA}	−0.2591	−0.4731	−0.6994	−0.7073	−0.7500	0.0000
K_{TA}	−0.6187	−0.4903	−0.1278	0.2211	0.1799	0.5000
K_{MB}	−0.1252	−0.2053	−0.1666	0.0586	0.2500	0.0000
K_{TB}	0.2953	0.1974	−0.0330	−0.1302	0.1799	−0.5000

3f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = \frac{T_o}{R}\,\frac{[C_1 \sin\phi - C_4(1 - \cos\phi)]\cos(\phi - \theta) - C_{a2}\sin^2\phi + C_{a5}(1 - \cos\phi)\sin\phi}{C_4(1 - \cos\phi)^2 + C_3 \sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$M_A = -T_o\,\frac{[C_3 \sin\phi - C_6(1 - \cos\phi)]\cos(\phi - \theta) - C_{a2}(1 - \cos\phi)\sin\phi + C_{a5}(1 - \cos\phi)^2}{C_4(1 - \cos\phi)^2 + C_3 \sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$\psi_A = \frac{T_o R}{EI}\,\frac{(C_3C_4 - C_1C_6)\cos(\phi - \theta) + (C_{a2}C_6 - C_{a5}C_3)\sin\phi - (C_{a2}C_4 - C_{a5}C_1)(1 - \cos\phi)}{C_4(1 - \cos\phi)^2 + C_3 \sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$V_B = V_A$$

$$M_B = V_A R \sin\phi + M_A \cos\phi - T_o \sin(\phi - \theta)$$

$$\psi_B = \psi_A \cos\phi + \frac{M_A R}{EI} C_7 + \frac{V_A R^2}{EI} C_9 + \frac{T_o R}{EI} C_{a8}$$

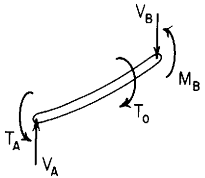
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	1.0645	0.5147	0.8696	0.4252	0.5000	0.2500
K_{MA}	−1.4409	−1.4379	−0.8696	−0.9252	−0.3598	−0.7573
$K_{\psi A}$	1.6003	1.3211	1.2356	0.6889	1.4564	0.1746
K_{MB}	−0.9733	−1.1528	−0.1304	−0.4409	0.3598	−0.1088
$K_{\psi B}$	1.1213	1.1662	0.4208	0.4502	0.3500	−0.0034

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

3g. Right end supported and slope-guided, left end supported and roll-guided.



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{T_o}{R}\frac{C_{a2}\cos^2\phi - C_{a5}\sin\phi\cos\phi + (C_5\sin\phi - C_2\cos\phi)\cos(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = -T_o\frac{(C_{a5}\sin\phi - C_{a2}\cos\phi)(1-\cos\phi) + (C_3\cos\phi - C_6\sin\phi)\cos(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = -\frac{T_oR}{EI}\frac{(C_{a2}C_5 - C_{a5}C_2)(1-\cos\phi) + (C_{a5}C_3 - C_{a2}C_6)\cos\phi + (C_2C_6 - C_3C_5)\cos(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A$$

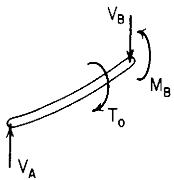
$$M_B = V_AR\sin\phi - T_A\sin\phi - T_o\sin(\phi-\theta)$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_AR}{EI}C_8 + \frac{V_AR^2}{EI}C_9 + \frac{T_oR}{EI}C_{a8}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	15°	30°	30°	60°	60°	120°
K_{VA}	−0.8503	−1.4915	−0.5000	−0.8660	−0.0512	−0.2859
K_{TA}	−0.8725	−0.7482	−0.7175	−0.4095	−0.6023	−0.0717
$K_{\Theta A}$	−0.0522	−0.0216	−0.2274	−0.0640	−1.9528	−0.2997
K_{MB}	−0.4843	−0.7844	−0.6485	−0.9566	−0.8660	−0.8660
$K_{\psi B}$	0.2386	0.5031	0.1780	0.5249	−0.9169	0.0416

3h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{T_o\cos(\phi-\theta)}{R(1-\cos\phi)}$$

$$\Theta_A = -\frac{T_oR}{EI}\left[\frac{C_{a2}\sin\phi + C_6\cos(\phi-\theta)}{1-\cos\phi} - \frac{C_3\sin\phi\cos(\phi-\theta)}{(1-\cos\phi)^2} - C_{a5}\right]$$

$$\psi_A = -\frac{T_oR}{EI}\left[\frac{C_{a5}\sin\phi - C_{a2}\cos\phi}{1-\cos\phi} + (C_3\cos\phi - C_6\sin\phi)\frac{\cos(\phi-\theta)}{(1-\cos\phi)^2}\right]$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi - T_o\sin(\phi-\theta)$$

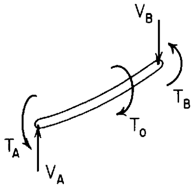
$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_AR^2}{EI}C_9 + \frac{T_oR}{EI}C_{a8}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−2.4142	0.0000	−0.5000	−0.8660	0.5000	0.2500	−0.2500
$K_{\Theta A}$	−0.4161	−0.6564	−0.6984	−0.3328	−1.3000	−2.7359	−0.3929
$K_{\psi A}$	2.1998	1.8064	1.2961	0.7396	1.9242	1.1590	0.1380
K_{MB}	−2.4142	−1.0000	−1.3660	−1.3660	0.0000	−0.8660	−0.8660
$K_{\psi B}$	1.5263	0.5064	0.5413	0.7323	−0.1178	−0.9878	0.0332

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

3i. Right end supported and roll-guided, left end supported and roll-guided.



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = 0$$

$$T_A = -T_o \frac{(C_{a2} + C_{a8})\sin^2\phi + (C_3 + C_9)\sin(\phi - \theta)\sin\phi}{(C_2 + C_3 + C_8 + C_9)\sin^2\phi}$$

$$\Theta_A = -\frac{T_o R}{EI} \frac{[C_{a2}(C_8 + C_9) - C_{a8}(C_2 + C_3)]\sin\phi - (C_2 C_9 - C_3 C_8)\sin(\phi - \theta)}{(C_2 + C_3 + C_8 + C_9)\sin^2\phi}$$

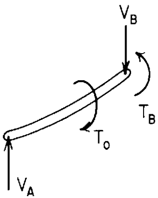
$$V_B = 0$$

$$T_B = V_A R(1 - \cos\phi) + T_A \cos\phi + T_o \cos(\phi - \theta)$$

$$\Theta_B = \Theta_A \cos\phi + \frac{T_A R}{EI} C_5 + \frac{V_A R^2}{EI} C_6 + \frac{T_o R}{EI} C_{a5}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)			
ϕ	45°	90°	270°
θ	15°	30°	90°
K_{VA}	0.0000	0.0000	0.0000
K_{TA}	−0.7071	−0.8660	0.0000
$K_{\Theta A}$	−0.0988	−0.6021	−3.6128
K_{TB}	0.3660	0.5000	−1.0000
$K_{\Theta B}$	0.0807	0.5215	0.0000

3j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = \frac{T_o \sin(\phi - \theta)}{R\sin\phi}$$

$$\Theta_A = -\frac{T_o R}{EI}\left\{\frac{C_{a2}\cos\phi - C_{a8}(1 - \cos\phi)}{\sin\phi} + [C_3\cos\phi - C_9(1 - \cos\phi)]\frac{\sin(\phi - \theta)}{\sin^2\phi}\right\}$$

$$\psi_A = -\frac{T_o R}{EI}\left[C_{a2} + C_{a8} + (C_3 + C_9)\frac{\sin(\phi - \theta)}{\sin\phi}\right]$$

$$V_B = V_A$$

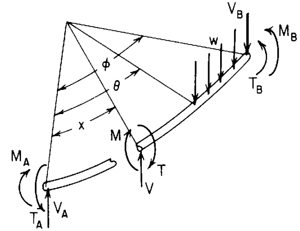
$$T_B = V_A R(1 - \cos\phi) + T_o\cos(\phi - \theta)$$

$$\Theta_B = \Theta_A\cos\phi + \psi_A\sin\phi + \frac{V_A R^2}{EI}C_6 + \frac{T_o R}{EI}C_{a5}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)						
ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	1.0000	0.7071	0.3660	1.0000	0.8660	0.5000
$K_{\Theta A}$	−0.2790	−0.2961	−0.1828	−1.3000	−1.7280	−1.1715
$K_{\psi A}$	1.0210	0.7220	0.3737	2.0420	1.7685	1.0210
K_{TB}	1.0000	1.0731	1.0731	1.0000	1.3660	1.3660
$K_{\Theta B}$	0.1439	0.1825	0.1515	0.7420	1.1641	0.9732

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4. Uniformly distributed lateral load



Transverse shear $= V = V_A - wR\langle x-\theta\rangle^1$

Bending moment $= M = V_A R\sin x + M_A\cos x - T_A\sin x - wR^2[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A\sin x + T_A\cos x - wR^2[x-\theta-\sin(x-\theta)]\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R\sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI}F_1 + \dfrac{T_A R^2}{EI}F_2 + \dfrac{V_A R^3}{EI}F_3 - \dfrac{wR^4}{EI}F_{a13}$

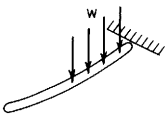
Bending slope $= \Theta = \Theta_A\cos x + \psi_A\sin x + \dfrac{M_A R}{EI}F_4 + \dfrac{T_A R}{EI}F_5 + \dfrac{V_A R^2}{EI}F_6 - \dfrac{wR^3}{EI}F_{a16}$

Roll slope $= \psi = \psi_A\cos x - \Theta_A\sin x + \dfrac{M_A R}{EI}F_7 + \dfrac{T_A R}{EI}F_8 + \dfrac{V_A R^2}{EI}F_9 - \dfrac{wR^3}{EI}F_{a19}$

For tabulated values: $V = K_V wR$, $M = K_M wR^2$, $T = K_T wR^2$, $y = K_y\dfrac{wR^4}{EI}$, $\Theta = K_\Theta\dfrac{wR^3}{EI}$, $\psi = K_\psi\dfrac{wR^3}{EI}$

End restraints, reference no.	Formulas for boundary values and selected numerical values

4a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = -\frac{wR^4}{EI}[C_{a16}\sin\phi - C_{a19}(1-\cos\phi) - C_{a13}]$$

$$\Theta_A = \frac{wR^3}{EI}(C_{a16}\cos\phi - C_{a19}\sin\phi)$$

$$\psi_A = \frac{wR^3}{EI}(C_{a19}\cos\phi + C_{a16}\sin\phi)$$

$$V_B = -wR(\phi-\theta)$$

$$M_B = -wR^2[1-\cos(\phi-\theta)]$$

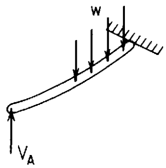
$$T_B = -wR^2[\phi-\theta-\sin(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	−0.0469	−0.7118	−0.2211	−0.0269	−8.4152	−2.2654	−0.1699
$K_{\Theta A}$	0.0762	0.4936	0.1071	0.0071	0.4712	−0.6033	−0.1583
$K_{\psi A}$	0.0267	0.4080	0.1583	0.0229	4.6000	1.3641	0.1071
K_{MB}	−0.2929	−1.0000	−0.5000	−0.1340	−2.0000	−1.5000	−0.5000
K_{TB}	−0.0783	−0.5708	−0.1812	−0.0236	−3.1416	−1.2284	−0.1812

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{C_{a19}(1-\cos\phi) - C_{a16}\sin\phi + C_{a13}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{(C_{a13}C_9 - C_{a19}C_3)\sin\phi + (C_{a19}C_6 - C_{a16}C_9)(1-\cos\phi) + (C_{a16}C_3 - C_{a13}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = \frac{wR^3}{EI}\frac{[C_{a16}(C_3+C_9) - C_6(C_{a13}+C_{a19})]\sin\phi + (C_{a19}C_3 - C_{a13}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$V_B = V_A - wR(\phi - \theta)$$

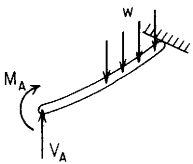
$$M_B = V_A R\sin\phi - wR^2[1-\cos(\phi-\theta)]$$

$$T_B = V_A R(1-\cos\phi) - wR^2[\phi - \theta - \sin(\phi-\theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2916	0.5701	0.1771	0.0215	1.0933	0.2943	0.0221
$K_{\Theta A}$	−0.1300	−0.1621	−0.0966	−0.0177	−2.3714	−1.3686	−0.2156
$K_{\psi A}$	0.0095	0.1192	0.0686	0.0119	0.6500	0.3008	0.0273
K_{MB}	−0.0867	−0.4299	−0.3229	−0.1124	−2.0000	−1.5000	−0.5000
K_{TB}	0.0071	−0.0007	−0.0041	−0.0021	−0.9549	−0.6397	−0.1370

4c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{(C_{a19}C_4 - C_{a16}C_7)(1-\cos\phi) + (C_{a16}C_1 - C_{a13}C_4)\cos\phi + (C_{a13}C_7 - C_{a19}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = wR^2\frac{(C_{a16}C_9 - C_{a19}C_6)(1-\cos\phi) + (C_{a13}C_6 - C_{a16}C_3)\cos\phi + (C_{a19}C_3 - C_{a13}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = \frac{wR^3}{EI}\frac{C_{a13}(C_4C_9 - C_6C_7) + C_{a16}(C_3C_7 - C_1C_9) + C_{a19}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

$$M_B = V_A R\sin\phi + M_A\cos\phi - wR^2[1-\cos(\phi-\theta)]$$

$$T_B = V_A R(1-\cos\phi) + M_A\sin\phi - wR^2[\phi - \theta - \sin(\phi-\theta)]$$

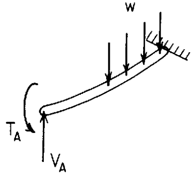
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.3919	0.7700	0.2961	0.0434	1.3863	0.4634	0.0487
K_{MA}	−0.0527	−0.2169	−0.1293	−0.0237	−0.8672	−0.5005	−0.0789
$K_{\psi A}$	−0.0004	−0.0145	−0.0111	−0.0027	−0.4084	−0.3100	−0.0689
K_{MB}	−0.0531	−0.2301	−0.2039	−0.0906	−1.1328	−0.9995	−0.4211
K_{TB}	−0.0008	−0.0178	−0.0143	−0.0039	−0.3691	−0.3016	−0.0838

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

4d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{[(C_{a13}+C_{a19})C_5 - C_{a16}(C_2+C_8)]\sin\phi + (C_{a13}C_8 - C_{a19}C_2)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = wR^2\frac{[C_{a16}(C_3+C_9) - C_6(C_{a13}+C_{a19})]\sin\phi + (C_{a19}C_3 - C_{a13}C_9)\cos\phi}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{C_{a13}(C_5C_9 - C_6C_8) + C_{a16}(C_3C_8 - C_2C_9) + C_{a19}(C_2C_6 - C_3C_5)}{[C_5(C_3+C_9) - C_6(C_2+C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

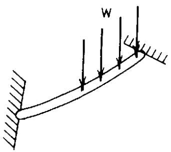
$$M_B = V_AR\sin\phi - T_A\sin\phi - wR^2[1 - \cos(\phi - \theta)]$$

$$T_B = V_AR(1 - \cos\phi) + T_A\cos\phi - wR^2[\phi - \theta - \sin(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2880	0.5399	0.1597	0.0185	0.9342	0.2207	0.0154
K_{TA}	−0.0099	−0.0745	−0.0428	−0.0075	−0.3391	−0.1569	−0.0143
$K_{\Theta A}$	−0.0111	−0.1161	−0.0702	−0.0131	−1.9576	−1.1171	−0.1983
K_{MB}	−0.0822	−0.3856	−0.2975	−0.1080	−2.0000	−1.5000	−0.5000
K_{TB}	−0.0010	−0.0309	−0.0215	−0.0051	−0.9342	−0.6301	−0.1362

4e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{C_{a13}(C_4C_8 - C_5C_7) + C_{a16}(C_2C_7 - C_1C_8) + C_{a19}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = wR^2\frac{C_{a13}(C_5C_9 - C_6C_8) + C_{a16}(C_3C_8 - C_2C_9) + C_{a19}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = wR^2\frac{C_{a13}(C_6C_7 - C_4C_9) + C_{a16}(C_1C_9 - C_3C_7) + C_{a19}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A - wR(\phi - \theta)$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - wR^2[1 - \cos(\phi - \theta)]$$

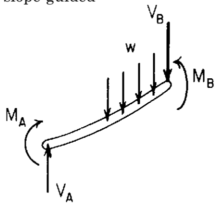
$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi - wR^2[\phi - \theta - \sin(\phi - \theta)]$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°		360°
θ	0°	15°	0°	30°	0°	60°	0°
K_{VA}	0.3927	0.1548	0.7854	0.3080	1.5708	0.6034	3.1416
K_{MA}	−0.0531	−0.0316	−0.2279	−0.1376	−1.0000	−0.6013	−2.1304
K_{TA}	0.0005	0.0004	0.0133	0.0102	0.2976	0.2259	3.1416
K_{MB}	−0.0531	−0.0471	−0.2279	−0.2022	−1.0000	−0.8987	−2.1304
K_{TB}	−0.0005	−0.0004	−0.0133	−0.0108	−0.2976	−0.2473	−3.1416

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = wR\frac{[C_4(1-\cos\phi) - C_1\sin\phi][\phi - \theta - \sin(\phi-\theta)] + C_{a13}\sin^2\phi - C_{a16}\sin\phi(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$M_A = wR^2\frac{[C_3\sin\phi - C_6(1-\cos\phi)][\phi-\theta-\sin(\phi-\theta)] + C_{a16}(1-\cos\phi)^2 - C_{a13}\sin\phi(1-\cos\phi)}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$\psi_A = \frac{wR^3}{EI}\frac{(C_{a13}C_4 - C_{a16}C_1)(1-\cos\phi) - (C_{a13}C_6 - C_{a16}C_3)\sin\phi - (C_3C_4 - C_1C_6)[\phi-\theta-\sin(\phi-\theta)]}{C_4(1-\cos\phi)^2 + C_3\sin^2\phi - (C_1+C_6)(1-\cos\phi)\sin\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

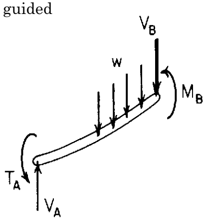
$$M_B = V_AR\sin\phi + M_A\cos\phi - wR^2[1-\cos(\phi-\theta)]$$

$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 - \frac{wR^3}{EI}C_{a19}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	0.3927	0.1549	0.7854	0.3086	1.5708	0.6142
K_{MA}	−0.0519	−0.0308	−0.2146	−0.1274	−1.0000	−0.6090
$K_{\psi A}$	−0.0013	−0.0010	−0.0220	−0.0171	−0.5375	−0.4155
K_{MB}	−0.0519	−0.0462	−0.2146	−0.1914	−1.0000	−0.8910
$K_{\psi B}$	−0.0013	−0.0010	−0.0220	−0.0177	−0.5375	−0.4393

4g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = wR\frac{(C_5\sin\phi - C_2\cos\phi)[\phi-\theta-\sin(\phi-\theta)] + C_{a13}\cos^2\phi - C_{a16}\sin\phi\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = wR^2\frac{(C_3\cos\phi - C_6\sin\phi)[\phi-\theta-\sin(\phi-\theta)] - (C_{a13}\cos\phi - C_{a16}\sin\phi)(1-\cos\phi)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{(C_2C_6 - C_3C_5)[\phi-\theta-\sin(\phi-\theta)] + (C_{a13}C_5 - C_{a16}C_2)(1-\cos\phi) + (C_{a16}C_3 - C_{a13}C_6)\cos\phi}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A - wR(\phi-\theta)$$

$$M_B = V_AR\sin\phi - T_A\sin\phi - wR^2[1-\cos(\phi-\theta)]$$

$$\psi = -\Theta_A\sin\phi + \frac{T_AR}{EI}C_8 + \frac{V_AR^2}{EI}C_9 - \frac{wR^3}{EI}C_{a19}$$

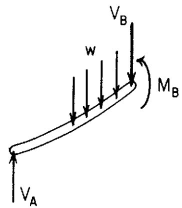
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2896	0.5708	0.1812	0.0236	1.3727	0.5164	0.0793
K_{TA}	−0.0093	−0.0658	−0.0368	−0.0060	−0.3963	−0.1955	−0.0226
$K_{\Theta A}$	−0.0111	−0.1188	−0.0720	−0.0135	−3.0977	−1.9461	−0.3644
K_{MB}	−0.0815	−0.3634	−0.2820	−0.1043	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	−0.0008	−0.0342	−0.0238	−0.0056	−1.7908	−1.2080	−0.2610

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

4h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = wR\frac{\phi - \theta - \sin(\phi - \theta)}{1 - \cos\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\left\{\frac{C_{a13}\sin\phi + C_6[\phi - \theta - \sin(\phi - \theta)]}{1 - \cos\phi} - \frac{C_3\sin\phi[\phi - \theta - \sin(\phi - \theta)]}{(1 - \cos\phi)^2} - C_{a16}\right\}$$

$$\psi_A = \frac{wR^3}{EI}\left\{\frac{C_{a16}\sin\phi - C_{a13}\cos\phi}{1 - \cos\phi} - (C_6\sin\phi - C_3\cos\phi)\frac{\phi - \theta - \sin(\phi - \theta)}{(1 - \cos\phi)^2}\right\}$$

$$V_B = V_A - wR(\phi - \theta)$$

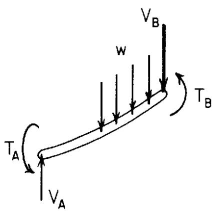
$$M_B = V_A R\sin\phi - wR^2[1 - \cos(\phi - \theta)]$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_A R^2}{EI}C_9 - \frac{wR^3}{EI}C_{a19}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.2673	0.5708	0.1812	0.0236	1.5708	0.6142	0.0906
$K_{\Theta A}$	−0.0150	−0.1620	−0.0962	−0.0175	−3.6128	−2.2002	−0.3938
$K_{\psi A}$	0.0204	0.1189	0.0665	0.0109	0.7625	0.3762	0.0435
K_{MB}	−0.1039	−0.4292	−0.3188	−0.1104	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	−0.0133	−0.0008	−0.0051	−0.0026	−1.8375	−1.2310	−0.2637

4i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$

$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{(C_{a13} + C_{a19})\sin\phi + (C_2 + C_8)[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$T_A = wR^2\frac{(C_{a13} + C_{a19})\sin\phi - (C_3 + C_9)[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\frac{C_{a13}(C_8 + C_9) - C_{a19}(C_2 + C_3) + (C_2C_9 - C_3C_8)[1 - \cos(\phi - \theta)]/\sin\phi}{(C_2 + C_3 + C_8 + C_9)\sin\phi}$$

$$V_B = V_A - wR(\phi - \theta)$$

$$T_B = V_A R(1 - \cos\phi) + T_A\cos\phi - wR^2[\phi - \theta - \sin(\phi - \theta)]$$

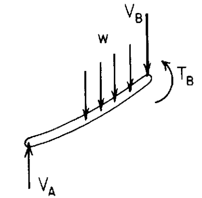
$$\Theta_B = \Theta_A\cos\phi + \frac{T_A R}{EI}C_5 + \frac{V_A R^2}{EI}C_6 - \frac{wR^3}{EI}C_{a16}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$),

ϕ	45°		90°		270°	
θ	0°	15°	0°	30°	0°	90°
K_{VA}	0.3927	0.1745	0.7854	0.3491	2.3562	1.0472
K_{TA}	−0.0215	−0.0149	−0.2146	−0.1509	3.3562	3.0472
$K_{\Theta A}$	−0.0248	−0.0173	−0.3774	−0.2717	−10.9323	−6.2614
K_{TB}	0.0215	0.0170	0.2146	0.1679	−3.3562	−2.0944
$K_{\Theta B}$	0.0248	0.0194	0.3774	0.2912	10.9323	9.9484

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

4j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = wR\frac{1-\cos(\phi-\theta)}{\sin\phi}$$

$$\Theta_A = \frac{wR^3}{EI}\left[\frac{C_{a13}\cos\phi - C_{a19}(1-\cos\phi)}{\sin\phi} - [C_3\cos\phi - C_9(1-\cos\phi)]\frac{1-\cos(\phi-\theta)}{\sin^2\phi}\right]$$

$$\psi_A = \frac{wR^3}{EI}\left[C_{a13} + C_{a19} - (C_3 + C_9)\frac{1-\cos(\phi-\theta)}{\sin\phi}\right]$$

$$V_B = V_A - wR(\phi-\theta)$$

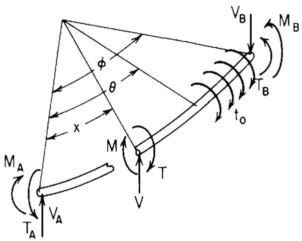
$$T_B = V_A R(1-\cos\phi) - wR^2[\phi - \theta - \sin(\phi-\theta)]$$

$$\Theta_B = \Theta_A\cos\phi + \psi_A\sin\phi + \frac{V_A R^2}{EI}C_6 - \frac{wR^3}{EI}C_{a16}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	0.4142	0.1895	0.0482	1.0000	0.5000	0.1340
$K_{\Theta A}$	−0.0308	−0.0215	−0.0066	−0.6564	−0.4679	−0.1479
$K_{\psi A}$	0.0220	0.0153	0.0047	0.4382	0.3082	0.0954
K_{TB}	0.0430	0.0319	0.0111	0.4292	0.3188	0.1104
$K_{\Theta B}$	0.0279	0.0216	0.0081	0.5367	0.4032	0.1404

5. Uniformly distributed torque



Transverse shear $= V = V_A$

Bending moment $= M = V_A R\sin x + M_A\cos x - T_A\sin x - t_o R[1-\cos(x-\theta)]\langle x-\theta\rangle^0$

Twisting moment $= T = V_A R(1-\cos x) + M_A\sin x + T_A\cos x + t_o R\sin(x-\theta)\langle x-\theta\rangle^0$

Vertical deflection $= y = y_A + \Theta_A R\sin x + \psi_A R(1-\cos x) + \dfrac{M_A R^2}{EI}F_1 + \dfrac{T_A R^2}{EI}F_2 + \dfrac{V_A R^3}{EI}F_3 + \dfrac{t_o R^3}{EI}F_{a12}$

Bending slope $= \Theta = \Theta_A\cos x + \psi_A\sin x + \dfrac{M_A R}{EI}F_4 + \dfrac{T_A R}{EI}F_5 + \dfrac{V_A R^2}{EI}F_6 + \dfrac{t_o R^2}{EI}F_{a15}$

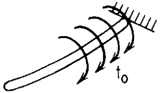
Roll slope $= \psi = \psi_A\cos x - \Theta_A\sin x + \dfrac{M_A R}{EI}F_7 + \dfrac{T_A R}{EI}F_8 + \dfrac{V_A R^2}{EI}F_9 + \dfrac{t_o R^2}{EI}F_{a18}$

For tabulated values: $V = K_V t_o, \quad M = K_M t_o R, \quad T = K_T t_o R, \quad y = K_y\dfrac{t_o R^3}{EI}, \quad \Theta = K_\Theta\dfrac{t_o R^2}{EI}, \quad \psi = K_\psi\dfrac{t_o R^2}{EI}$

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

5a. Right end fixed, left end free



$V_A = 0 \quad M_A = 0 \quad T_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$y_A = \frac{t_o R^3}{EI}[C_{a15}\sin\phi - C_{a18}(1-\cos\phi) - C_{a12}]$$

$$\Theta_A = -\frac{t_o R^2}{EI}(C_{a15}\cos\phi - C_{a18}\sin\phi)$$

$$\psi_A = -\frac{t_o R^2}{EI}(C_{a18}\cos\phi + C_{a15}\sin\phi)$$

$V_B = 0$

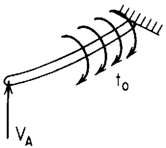
$M_B = -t_o R[1-\cos(\phi-\theta)]$

$T_B = t_o R\sin(\phi-\theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{yA}	0.0129	0.1500	0.2562	0.1206	0.6000	2.5359	1.1929
$K_{\Theta A}$	−0.1211	−0.8064	−0.5429	−0.1671	−3.6128	−2.2002	−0.3938
$K_{\psi A}$	0.3679	1.1500	0.3938	0.0535	2.0000	−0.5859	−0.5429
K_{MB}	−0.2929	−1.0000	−0.5000	−0.1340	−2.0000	−1.5000	−0.5000
K_{TB}	0.7071	1.0000	0.8660	0.5000	0.0000	0.8660	0.8660

5b. Right end fixed, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$

$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{C_{a18}(1-\cos\phi) - C_{a15}\sin\phi + C_{a12}}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\Theta_A = -\frac{t_o R^2}{EI}\frac{(C_{a12}C_9 - C_{a18}C_3)\sin\phi + (C_{a18}C_6 - C_{a15}C_9)(1-\cos\phi) + (C_{a15}C_3 - C_{a12}C_6)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$$\psi_A = -\frac{t_o R^2}{EI}\frac{[C_{a15}(C_9+C_3) - C_6(C_{a12}+C_{a18})]\sin\phi + (C_{a18}C_3 - C_{a12}C_9)\cos\phi}{C_9(1-\cos\phi) - C_6\sin\phi + C_3}$$

$V_B = V_A$

$M_B = V_A R\sin\phi - t_o R[1-\cos(\phi-\theta)]$

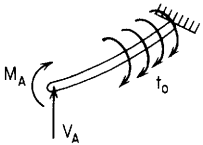
$T_B = V_A R(1-\cos\phi) + t_o R\sin(\phi-\theta)$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−0.0801	−0.1201	−0.2052	−0.0966	−0.0780	−0.3295	−0.1550
$K_{\Theta A}$	−0.0966	−0.6682	−0.3069	−0.0560	−3.4102	−1.3436	0.0092
$K_{\psi A}$	0.3726	1.2108	0.4977	0.1025	2.2816	0.6044	0.0170
K_{MB}	−0.3495	−1.1201	−0.7052	−0.2306	−2.0000	−1.5000	−0.5000
K_{TB}	0.6837	0.8799	0.6608	0.4034	−0.1559	0.2071	0.5560

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

5c. Right end fixed, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{(C_{a18}C_4 - C_{a15}C_7)(1-\cos\phi) + (C_{a15}C_1 - C_{a12}C_4)\cos\phi + (C_{a12}C_7 - C_{a18}C_1)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$M_A = -t_o R \frac{(C_{a15}C_9 - C_{a18}C_6)(1-\cos\phi) + (C_{a12}C_6 - C_{a15}C_3)\cos\phi + (C_{a18}C_3 - C_{a12}C_9)\sin\phi}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$\psi_A = -\frac{t_o R^2}{EI} \frac{C_{a12}(C_4C_9 - C_6C_7) + C_{a15}(C_3C_7 - C_1C_9) + C_{a18}(C_1C_6 - C_3C_4)}{(C_4C_9 - C_6C_7)(1-\cos\phi) + (C_1C_6 - C_3C_4)\cos\phi + (C_3C_7 - C_1C_9)\sin\phi}$$

$$V_B = V_A$$

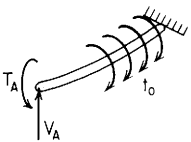
$$M_B = V_A R \sin\phi + M_A \cos\phi - t_o R[1 - \cos(\phi - \theta)]$$

$$T_B = V_A R(1 - \cos\phi) + M_A \sin\phi + t_o R \sin(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	0.6652	0.7038	0.1732	−0.0276	0.3433	−0.1635	−0.1561
K_{MA}	−0.3918	−0.8944	−0.4108	−0.0749	−1.2471	−0.4913	0.0034
$K_{\psi A}$	0.2993	0.6594	0.2445	0.0563	0.7597	0.0048	0.0211
K_{MB}	−0.0996	−0.2962	−0.3268	−0.1616	−0.7529	−1.0087	−0.5034
K_{TB}	0.6249	0.8093	0.6284	0.3975	0.6866	0.5390	0.5538

5d. Right end fixed, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{[(C_{a12} + C_{a18})C_5 - C_{a15}(C_2 + C_8)]\sin\phi + (C_{a12}C_8 - C_{a18}C_2)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$T_A = -t_o R \frac{[C_{a15}(C_3 + C_9) - C_6(C_{a12} + C_{a18})]\sin\phi + (C_{a18}C_3 - C_{a12}C_9)\cos\phi}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI} \frac{C_{a12}(C_5C_9 - C_6C_8) + C_{a15}(C_3C_8 - C_2C_9) + C_{a18}(C_2C_6 - C_3C_5)}{[C_5(C_3 + C_9) - C_6(C_2 + C_8)]\sin\phi + (C_3C_8 - C_2C_9)\cos\phi}$$

$$V_B = V_A$$

$$M_B = V_A R \sin\phi - T_A \sin\phi - t_o R[1 - \cos(\phi - \theta)]$$

$$T_B = V_A R(1 - \cos\phi) + T_A \cos\phi + t_o R \sin(\phi - \theta)$$

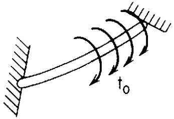
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−0.2229	−0.4269	−0.3313	−0.1226	−0.6366	−0.4775	−0.1592
K_{TA}	−0.3895	−0.7563	−0.3109	−0.0640	−1.1902	−0.3153	−0.0089
$K_{\Theta A}$	−0.0237	−0.2020	−0.1153	−0.0165	−1.9576	−0.9588	0.0200
K_{MB}	−0.1751	−0.6706	−0.5204	−0.1926	−2.0000	−1.5000	−0.5000
K_{TB}	0.3664	0.5731	0.5347	0.3774	−0.0830	0.2264	0.5566

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

5e. Right end fixed, left end fixed



$y_A = 0 \quad \Theta_A = 0 \quad \psi_A = 0$
$y_B = 0 \quad \Theta_B = 0 \quad \psi_B = 0$

$$V_A = -t_o \frac{C_{a12}(C_4C_8 - C_5C_7) + C_{a15}(C_2C_7 - C_1C_8) + C_{a18}(C_1C_5 - C_2C_4)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$M_A = -t_oR\frac{C_{a12}(C_5C_9 - C_6C_8) + C_{a15}(C_3C_8) - C_2C_9) + C_{a18}(C_2C_6 - C_3C_5)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$T_A = -t_oR\frac{C_{a12}(C_6C_7 - C_4C_9) + C_{a15}(C_1C_9 - C_3C_7) + C_{a18}(C_3C_4 - C_1C_6)}{C_1(C_5C_9 - C_6C_8) + C_4(C_3C_8 - C_2C_9) + C_7(C_2C_6 - C_3C_5)}$$

$$V_B = V_A$$

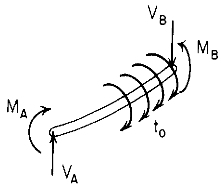
$$M_B = V_AR\sin\phi + M_A\cos\phi - T_A\sin\phi - t_oR[1 - \cos(\phi - \theta)]$$

$$T_B = V_AR(1 - \cos\phi) + M_A\sin\phi + T_A\cos\phi + t_oR\sin(\phi - \theta)$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	0.0000	−0.0444	0.0000	−0.0877	0.0000	−0.1657
K_{MA}	−0.1129	−0.0663	−0.3963	−0.2262	−1.0000	−0.4898
K_{TA}	−0.3674	−0.1571	−0.6037	−0.2238	−0.5536	−0.0035
K_{MB}	−0.1129	−0.1012	−0.3963	−0.3639	−1.0000	−1.0102
K_{TB}	0.3674	0.3290	0.6037	0.5522	0.5536	0.5382

5f. Right end supported and slope-guided, left end supported and slope-guided



$T_A = 0 \quad y_A = 0 \quad \Theta_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = t_o\frac{[C_1\sin\phi - C_4(1 - \cos\phi)]\sin(\phi - \theta) - C_{a12}\sin^2\phi + C_{a15}(1 - \cos\phi)\sin\phi}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$M_A = -t_oR\frac{[C_3\sin\phi - C_6(1 - \cos\phi)]\sin(\phi - \theta) - C_{a12}(1 - \cos\phi)\sin\phi + C_{a15}(1 - \cos\phi)^2}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$\psi_A = \frac{t_oR^2}{EI}\frac{(C_3C_4 - C_1C_6)\sin(\phi - \theta) + (C_{a12}C_6 - C_{a15}C_3)\sin\phi - (C_{a12}C_4 - C_{a15}C_1)(1 - \cos\phi)}{C_4(1 - \cos\phi)^2 + C_3\sin^2\phi - (C_1 + C_6)(1 - \cos\phi)\sin\phi}$$

$$V_B = V_A$$

$$M_B = V_AR\sin\phi + M_A\cos\phi - t_oR[1 - \cos(\phi - \theta)]$$

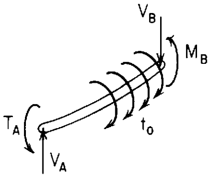
$$\psi_B = \psi_A\cos\phi + \frac{M_AR}{EI}C_7 + \frac{V_AR^2}{EI}C_9 + \frac{t_oR^2}{EI}C_{a18}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		180°	
θ	0°	15°	0°	30°	0°	60°
K_{VA}	0.0000	−0.2275	0.0000	−0.3732	0.0000	−0.4330
K_{MA}	−1.0000	−0.6129	−1.0000	−0.4928	−1.0000	−0.2974
$K_{\psi A}$	1.0000	0.6203	1.0000	0.5089	1.0000	0.1934
K_{MB}	−1.0000	−0.7282	−1.0000	−0.8732	−1.0000	−1.2026
$K_{\psi B}$	1.0000	0.7027	1.0000	0.7765	1.0000	0.7851

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

5g. Right end supported and slope-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -t_o \frac{C_{a12}\cos^2\phi - C_{a15}\sin\phi\cos\phi + (C_5\sin\phi - C_2\cos\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$T_A = -t_o R \frac{(C_{a15}\sin\phi - C_{a12}\cos\phi)(1-\cos\phi) + (C_3\cos\phi - C_6\sin\phi)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI}\frac{(C_{a12}C_5 - C_{a15}C_2)(1-\cos\phi) + (C_{a15}C_3 - C_{a12}C_6)\cos\phi + (C_2C_6 - C_3C_5)\sin(\phi-\theta)}{(C_5\sin\phi - C_2\cos\phi)(1-\cos\phi) + C_3\cos^2\phi - C_6\sin\phi\cos\phi}$$

$$V_B = V_A$$

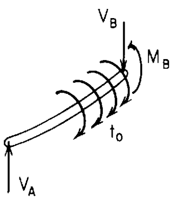
$$M_B = V_A R\sin\phi - T_A\sin\phi - t_o R[1-\cos(\phi-\theta)]$$

$$\psi_B = -\Theta_A\sin\phi + \frac{T_A R}{EI}C_8 + \frac{V_A R^2}{EI}C_9 + \frac{t_o R^2}{EI}C_{a18}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−0.8601	−1.0000	−0.8660	−0.5000	−0.5976	−0.5837	−0.4204
K_{TA}	−0.6437	−0.9170	−0.4608	−0.1698	−1.1953	−0.3014	0.0252
$K_{\Theta A}$	−0.0209	−0.1530	−0.0695	0.0158	−2.0590	−0.6825	0.6993
K_{MB}	−0.4459	−1.0830	−0.9052	−0.4642	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	0.2985	0.6341	0.5916	0.4716	−0.1592	0.4340	1.0670

5h. Right end supported and slope-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$T_B = 0 \quad y_B = 0 \quad \Theta_B = 0$

$$V_A = -\frac{t_o\sin(\phi-\theta)}{1-\cos\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI}\left[\frac{C_{a12}\sin\phi + C_6\sin(\phi-\theta)}{1-\cos\phi} - \frac{C_3\sin\phi\sin(\phi-\theta)}{(1-\cos\phi)^2} - C_{a15}\right]$$

$$\psi_A = -\frac{t_o R^2}{EI}\left[\frac{C_{a15}\sin\phi - C_{a12}\cos\phi}{1-\cos\phi} + (C_3\cos\phi - C_6\sin\phi)\frac{\sin(\phi-\theta)}{(1-\cos\phi)^2}\right]$$

$$V_B = V_A$$

$$M_B = V_A R\sin\phi - t_o R[1-\cos(\phi-\theta)]$$

$$\psi_B = \psi_A\cos\phi - \Theta_A\sin\phi + \frac{V_A R^2}{EI}C_9 + \frac{t_o R^2}{EI}C_{a18}$$

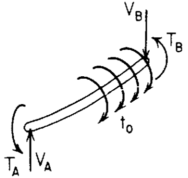
If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°	90°			180°		
θ	0°	0°	30°	60°	0°	60°	120°
K_{VA}	−2.4142	−1.0000	−0.8660	−0.5000	0.0000	−0.4330	−0.4330
$K_{\Theta A}$	−0.2888	−0.7549	−0.3720	−0.0957	−3.6128	−1.0744	0.7320
$K_{\psi A}$	1.4161	1.6564	0.8324	0.3067	2.3000	0.5800	−0.0485
K_{MB}	−2.0000	−2.0000	−1.3660	−0.6340	−2.0000	−1.5000	−0.5000
$K_{\psi B}$	1.2811	1.0985	0.8250	0.5036	−0.3000	0.3985	1.0700

TABLE 9.4 Formulas for curved beams of compact cross section loaded normal to the plane of curvature (*Continued*)

End restraints, reference no.	Formulas for boundary values and selected numerical values

5i. Right end supported and roll-guided, left end supported and roll-guided



$M_A = 0 \quad y_A = 0 \quad \psi_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$V_A = 0$

$$T_A = -t_o R \frac{(C_{a12} + C_{a18})\sin^2\phi + (C_3 + C_9)\sin\phi[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9)\sin^2\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI} \frac{[C_{a12}(C_8 + C_9) - C_{a18}(C_2 + C_3)]\sin\phi - (C_2 C_9 - C_3 C_8)[1 - \cos(\phi - \theta)]}{(C_2 + C_3 + C_8 + C_9)\sin^2\phi}$$

$V_B = 0$

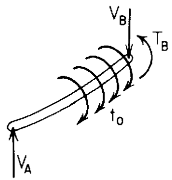
$T_B = T_A \cos\phi + t_o R \sin(\phi - \theta)$

$$\Theta_B = \Theta_A \cos\phi + \frac{T_A R}{EI} C_5 + \frac{t_o R^2}{EI} C_{a15}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°		90°		270°	
θ	0°	15°	0°	30°	0°	90°
K_{VA}	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
K_{TA}	−0.4142	−0.1895	−1.0000	−0.5000	1.0000	2.0000
$K_{\Theta A}$	−0.0527	−0.0368	−0.6564	−0.4679	−6.5692	−2.3000
K_{TB}	0.4142	0.3660	1.0000	0.8660	−1.0000	0.0000
$K_{\Theta B}$	0.0527	0.0415	0.6564	0.5094	6.5692	7.2257

5j. Right end supported and roll-guided, left end simply supported



$M_A = 0 \quad T_A = 0 \quad y_A = 0$
$M_B = 0 \quad y_B = 0 \quad \psi_B = 0$

$$V_A = \frac{t_o[1 - \cos(\phi - \theta)]}{\sin\phi}$$

$$\Theta_A = -\frac{t_o R^2}{EI}\left\{\frac{C_{a12}\cos\phi - C_{a18}(1 - \cos\phi)}{\sin\phi} + \frac{[C_3\cos\phi - C_9(1 - \cos\phi)][1 - \cos(\phi - \theta)]}{\sin^2\phi}\right\}$$

$$\psi_A = -\frac{t_o R^2}{EI}\left\{C_{a12} + C_{a18} + \frac{(C_3 + C_9)[1 - \cos(\phi - \theta)]}{\sin\phi}\right\}$$

$V_B = V_A$

$T_B = V_A R(1 - \cos\phi) + t_o R \sin(\phi - \theta)$

$$\Theta_B = \Theta_A \cos\phi + \phi_A \sin\phi + \frac{V_A R^2}{EI} C_6 + \frac{t_o R^2}{EI} C_{a15}$$

If $\beta = 1.3$ (solid or hollow round cross section, $\nu = 0.3$)

ϕ	45°			90°		
θ	0°	15°	30°	0°	30°	60°
K_{VA}	0.4142	0.1895	0.0482	1.0000	0.5000	0.1340
$K_{\Theta A}$	−0.1683	−0.0896	−0.0247	−1.9564	−1.1179	−0.3212
$K_{\psi A}$	0.4229	0.1935	0.0492	2.0420	1.0210	0.2736
K_{TB}	0.8284	0.5555	0.2729	2.0000	1.3660	0.6340
$K_{\Theta B}$	0.1124	0.0688	0.0229	1.3985	0.8804	0.2878

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