

## 3.4 Analysis of Members under Flexure (Part III)

This section covers the following topics

- Analysis for Ultimate Strength
- Analysis of a Rectangular Section

### 3.4.1 Analysis for Ultimate Strength

#### Introduction

A prestressed member usually remains uncracked under service loads. The analysis under service loads assumes the material to be linear elastic. After cracking, the behaviour of a prestressed member is similar to a non-prestressed reinforced concrete member. With increasing load, the stress versus strain behaviour of concrete becomes non-linear. Close to the yielding of the prestressing steel, the stress versus strain behaviour of steel also becomes non-linear.

The analysis of a prestressed member for ultimate strength is similar to that of a reinforced concrete member. The analysis aims to calculate the ultimate moment capacity (ultimate moment of resistance). The capacity is compared with the demand at ultimate loads.

There is an inconsistency in the traditional analysis at the ultimate state. The force demand is calculated based on elastic analysis, with superposition for the different load cases using the load factors. But the capacity is calculated based on the non-linear limit state analysis. The inconsistency is justified by the following arguments.

- 1) The moment versus curvature relationship is almost linear till the yielding of the steel. The moment versus curvature relationship is also referred to as the behaviour and is explained in Section 3.6, Analysis of Member under Flexure (Part V).
- 2) The moment at yield is only slightly lower than the ultimate moment capacity. Hence the behaviour is practically linear for most of the range of the moment.
- 3) The calculated moment demand for a load case based on elastic analysis is well within the moment at yield. Hence, superposition for the load cases is applied to find out the moment demand under combined loads.

Of course, superposition cannot be used to calculate the deflection under combined loads.

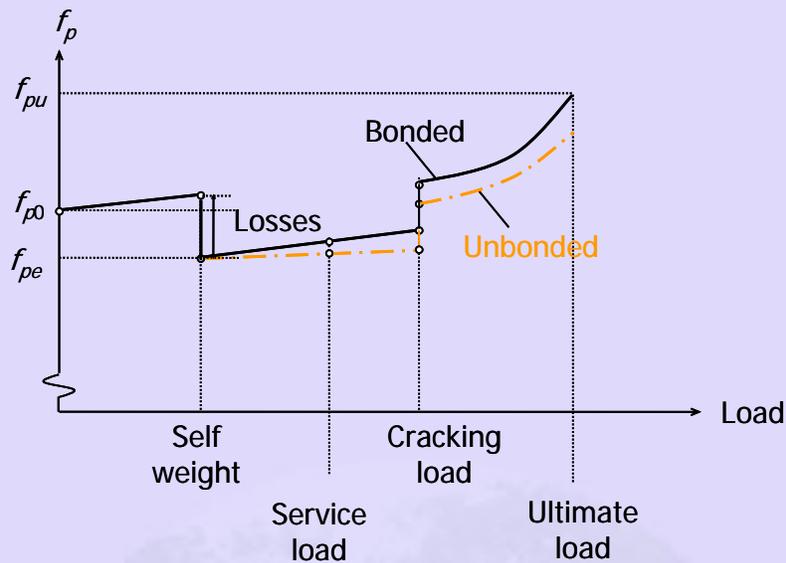
### Variation of Stress in Prestressing Steel

In non-prestressed reinforced concrete members, the tension and consequently the stress in steel increase almost proportionately with increasing moment till yielding. The lever arm between the resultant compression and tension remains almost constant. In prestressed concrete members, the tension and consequently the stress in prestressing steel increase slightly with increasing moment till cracking of concrete. The increase in moment changes the lever arm significantly. This is explained in Section 3.2, Analysis of Member under Flexure (Part I). After cracking, the stress in prestressing steel increases rapidly with moment.

The following sketch explains the variations of the stress in prestressing steel ( $f_p$ ) with increasing load. The variations are shown for bonded and unbonded tendons. After the prestress is transferred while the member is supported at the ends, the stress will tend to increase from the value after losses ( $f_{p0}$ ) due to the moment under self weight. Simultaneously the stress will tend to drop due to the time dependent losses such as from creep, shrinkage and relaxation. The losses of prestress are covered in Section 2.3, Losses in Prestress (Part III). The effective prestress after time dependent losses is denoted as  $f_{pe}$ .

Due to the moment under service loads, the stress in the prestressing steel will slightly increase from  $f_{pe}$ . The increase is more at the section of maximum moment in a bonded tendon as compared to the increase in average stress for an unbonded tendon. The stress in a bonded tendon is not uniform along the length. Usually the increase in stress is neglected in the calculations under service loads. If the loads are further increased, the stress increases slightly till cracking.

After cracking, there is a jump of the stress in the prestressing steel. Beyond that, the stress increases rapidly with moment till the ultimate load. At ultimate, the stress is represented as  $f_{pu}$ . Similar to the observation for pre-cracking, the average stress in an unbonded tendon is less than the stress at the section of maximum moment for a bonded tendon.



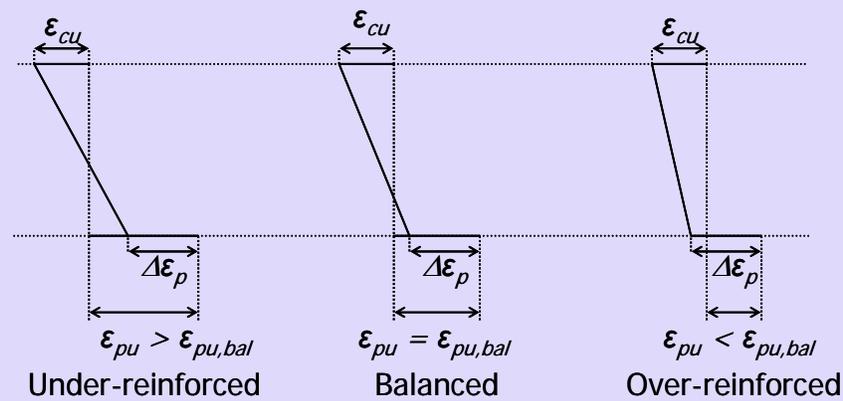
**Figure 3-4.1** Variation of stress in prestressing steel

The above sketch assumes that the section is failing in flexure. Other types of failure are not considered.

### Conditions at Ultimate Limit State

In the limit states method of analysis, the limit state of collapse (ultimate state) of a member under flexure is defined as the state when the extreme concrete compressive strain reaches a value of 0.0035. At ultimate, let the extreme concrete compressive strain be denoted as  $\epsilon_{cu}$ . Thus,  $\epsilon_{cu} = 0.0035$ .

Depending on the amount of prestressing steel, a section can be under-reinforced or over-reinforced. For an under-reinforced section, the amount of prestressing steel is less and the steel yields before the extreme concrete strain reaches 0.0035. For an over-reinforced section, the amount of steel is high and the steel does not yield at ultimate. The transition situation is called a balanced condition. The strain profiles across the depths of prestressed flexural members (up to the depth of CGS) for the three situations are shown below.



**Figure 3-4.2** Strain profiles along the depths of three prestressed members

In the above sketch,

$\epsilon_{pu}$  = strain in the prestressing steel at the level of CGS at ultimate condition

$\epsilon_{pu,bal}$  = strain in the steel for a balanced section.

The strain difference ( $\Delta\epsilon_p$ ) is the strain in the prestressed tendons when the adjacent concrete has zero strain ( $\epsilon_c = 0$ ). The strain difference gets locked during the transfer of prestress. The value can be determined as follows.

For pre-tensioned members, the strain difference gets locked when the tendons are cut. The strain difference at that instant is given as follows.

$$\Delta\epsilon_p = \epsilon_{pi} - 0 \quad (3-4.1a)$$

Here,

$\epsilon_{pi}$  = strain in tendons just before transfer

$\epsilon_c$  = strain in concrete is zero.

For post-tensioned members, the strain difference gets locked when the tendons are anchored. The strain difference at that instant is given as follows.

$$\Delta\epsilon_p = \epsilon_{p0} - \epsilon_{c0} \quad (3-4.1b)$$

Here,

$\epsilon_{p0}$  = strain in tendons due to  $P_0$ , the prestress after transfer

$\epsilon_{c0}$  = strain in concrete due to  $P_0$ .

In general at any load stage,

$$\Delta\epsilon_p = \epsilon_{pe} - \epsilon_{ce} \quad (3-4.1c)$$

Here,

$\epsilon_{pe}$  = strain in tendons due to  $P_e$ , the prestress at service

$\epsilon_{ce}$  = strain in concrete due to  $P_e$ .

As mentioned under material properties, the prestressing steel does not have a definite yield point. The 0.2% proof stress is defined when the steel reaches an inelastic strain of 0.2%. Hence, unlike reinforced concrete, the transition from under-reinforced to over-reinforced section is gradual and there is no definite balanced condition. **IS:1343 - 1980** does not explicitly enforce an under-reinforced section. But the **IRS Concrete Bridge Code** requires that the strain in the outermost tendon is not less than the following.

$$\frac{0.87f_{pk}}{E_p} + 0.005$$

The above value can be considered to be the strain in the steel at balanced condition.

### Assumptions for Analysis

The analysis of members under flexure for ultimate strength considers the following.

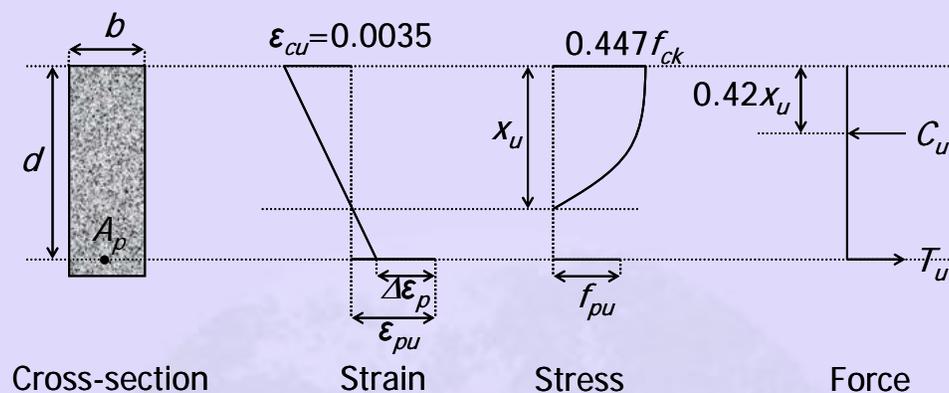
- 1) Plane sections perpendicular to the axis of the member remain plane till the ultimate state.
- 2) Perfect bond is retained between concrete and prestressing steel for bonded tendons.
- 3) Tension in concrete is neglected.
- 4) The design stress versus strain curves of concrete and steel are considered.

The methods of analysis will be presented for three types of sections.

- 1) Rectangular section: A rectangular section is easy to cast, but it is not an efficient section.
- 2) Flanged section: A precast flanged section, with flanges either at top or bottom needs costlier formwork. But the section is efficient in flexure. A flanged section can also be made of precast web and cast-in-place slab.
- 3) Partially prestressed section: A section in a member containing both prestressed and non-prestressed reinforcement.

### 3.4.4 Analysis of a Rectangular Section

The following sketch shows the beam cross section, strain profile, stress diagram and force couple at the ultimate state.



**Figure 3-4.3** Sketches for analysis of a rectangular section

The variables in the above figure are explained.

- $b$  = breadth of the section
- $d$  = depth of the centroid of prestressing steel (CGS)
- $A_p$  = area of the prestressing steel
- $\Delta\epsilon_p$  = strain difference
- $x_u$  = depth of the neutral axis at ultimate
- $\epsilon_{pu}$  = strain in prestressing steel at the level of CGS at ultimate
- $f_{pu}$  = stress in prestressing steel at ultimate

The stress block in concrete is derived from the constitutive relationship for concrete. The relationship is explained in Section 1.6, Concrete (Part II). The compressive force in concrete can be calculated by integrating the stress block along the depth. The stress in the tendon is calculated from the constitutive relationship for prestressing steel. The relationship is explained in Section 1.7, Prestressing Steel.

In the force diagram,

$$C_u = 0.36f_{ck}x_ub \quad (3-4.2)$$

$$T_u = A_p f_{pu} \quad (3-4.3)$$

The strengths of the materials are denoted by the following symbols.

- $f_{ck}$  = characteristic compressive strength of concrete
- $f_{pk}$  = characteristic tensile strength of prestressing steel

For analysis of a prestressed section, three principles of mechanics are used. First, the equilibrium relates the external applied forces with the internal forces. Second, the compatibility condition relates the strain in the prestressing steel with the strain in concrete at the level of CGS. This also considers the first two assumptions given in the previous section. The third principle involves the constitutive relationships of the materials.

Based on the above principles of mechanics, the following equations are derived.

### 1) Equations of equilibrium

The first equation states that the resultant axial force is zero. This means that the compression and the tension in the force couple balance each other.

$$\begin{aligned}\sum F &= 0 \\ \Rightarrow T_u &= C_u \\ \Rightarrow A_p f_{pu} &= 0.36 f_{ck} x_u b\end{aligned}\quad (3-4.4)$$

The second equation relates the ultimate moment capacity ( $M_{uR}$ ) with the internal couple in the force diagram.

$$\begin{aligned}M_{uR} &= T_u (d - 0.42x_u) \\ &= A_p f_{pu} (d - 0.42x_u)\end{aligned}\quad (3-4.5)$$

### 2) Equation of compatibility

The depth of the neutral axis is related to the depth of CGS by the similarity of the triangles in the strain diagram.

$$\frac{x_u}{d} = \frac{0.0035}{0.0035 + \varepsilon_{pu} - \Delta\varepsilon_p}\quad (3-4.6)$$

### 3) Constitutive relationships

#### a) Concrete

The constitutive relationship for concrete is considered in the expression  $C_u = 0.36 f_{ck} x_u b$ . This is based on the area under the design stress-strain curve for concrete under compression.

## b) Prestressing steel

$$f_{pu} = F(\varepsilon_{pu}) \quad (3-4.7)$$

The function  $F(\varepsilon_{pu})$  represents the design stress-strain curve for prestressing steel under tension.

The known variables in an analysis are:  $b$ ,  $d$ ,  $A_p$ ,  $\Delta\varepsilon_p$ ,  $f_{ck}$ ,  $f_{pk}$ .

The unknown quantities are:  $x_u$ ,  $M_{uR}$ ,  $\varepsilon_{pu}$ ,  $f_{pu}$ .

The objective of the analysis is to find out  $M_{uR}$ , the ultimate moment capacity. The simultaneous equations 3-4.1 to 3-4.7 can be solved iteratively. This procedure of analysis is called the **strain compatibility method**. The steps are as follows.

- 1) Assume  $x_u$ .
- 2) Calculate  $\varepsilon_{pu}$  by rearranging the terms of Eqn. 3-4.6.
- 3) Calculate  $f_{pu}$  from Eqn. 3-4.7.
- 4) Calculate  $T_u$  from Eqn. 3-4.3.
- 5) Calculate  $C_u$  from Eqn. 3-4.2.

If Eqn. 3-4.4 ( $T_u = C_u$ ) is not satisfied, change  $x_u$ .

If  $T_u < C_u$  decrease  $x_u$ . If  $T_u > C_u$  increase  $x_u$ .

- 6) Calculate  $M_{uR}$  from Eqn. 3-4.5.

The capacity  $M_{uR}$  can be compared with the demand under ultimate loads.

In the strain compatibility method, the difficult step is to calculate  $x_u$  and  $f_{pu}$ . **IS:1343 - 1980** allows to calculate these variables approximately from Table 11, Appendix B, based on the amount of prestressing steel. The latter is expressed as a prestressed reinforcement index  $\omega_p$ .

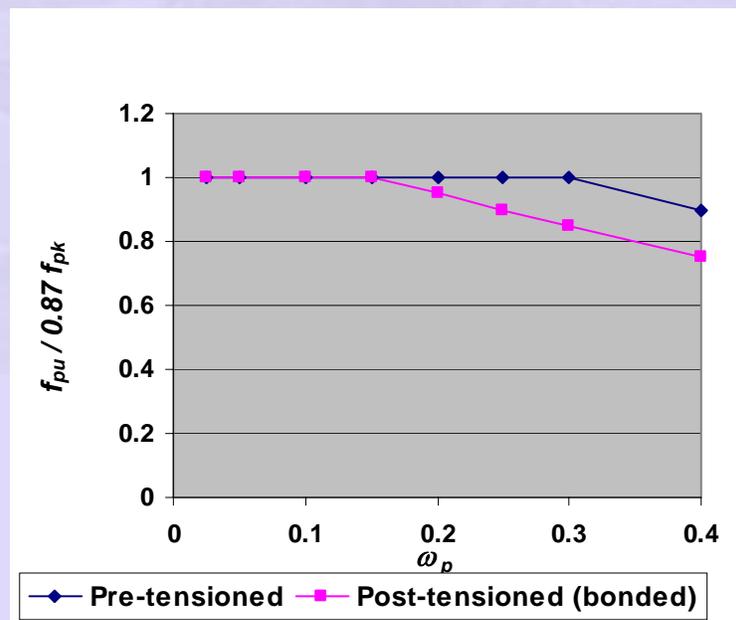
$$\omega_p = \frac{A_p f_{pk}}{b d f_{ck}} \quad (3-4.8)$$

Table 11 is reproduced as Table 3-4.1 which is applicable for pre-tensioned and bonded post-tensioned beams. The values of  $f_{pu}$  and  $x_u$  are given as  $f_{pu}/(0.87f_{pk})$  and  $x_u/d$ , respectively.

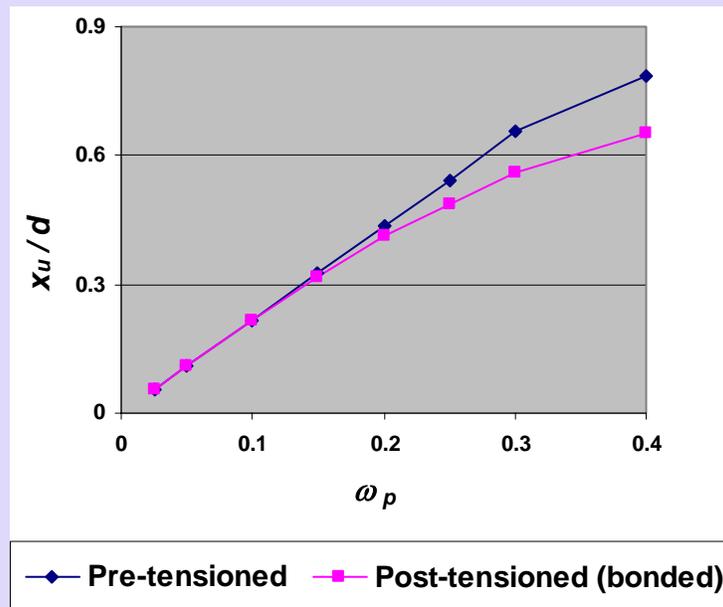
**Table 3-4.1** Values of  $x_u$  and  $f_{pu}$  for pre-tensioned and bonded post-tensioned rectangular beams (Table 11, IS:1343 - 1980)

$\omega_p$	$f_{pu}/(0.87f_{pk})$		$x_u/d$	
	Pre-tensioned	Bonded post-tensioned	Pre-tensioned	Bonded post-tensioned
0.025	1.0	1.0	0.054	0.054
0.05	1.0	1.0	0.109	0.109
0.10	1.0	1.0	0.217	0.217
0.15	1.0	1.0	0.326	0.316
0.20	1.0	0.95	0.435	0.414
0.25	1.0	0.90	0.542	0.488
0.30	1.0	0.85	0.655	0.558
0.40	0.9	0.75	0.783	0.653

The values of  $f_{pu}/(0.87f_{pk})$  and  $x_u/d$  from Table 3-4.1 are plotted in Figures 3-4.4 and 3-4.5, respectively. It is observed that with increase in  $\omega_p$ ,  $f_{pu}$  reduces (beyond certain values of  $\omega_p$ ) and  $x_u$  increases. This is expected because with increase in the amount and strength in the steel, the stress in steel drops and the depth of the neutral axis increases to maintain equilibrium.



**Figure 3-4.4** Variation of  $f_{pu}$  with respect to  $w_p$  (Table 3-4.1)



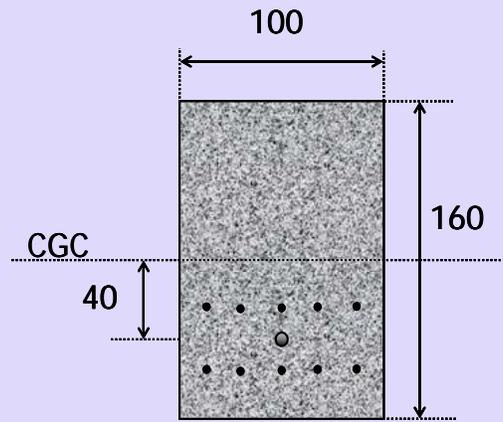
**Figure 3-4.5** Variation of  $x_u$  with respect to  $w_p$  (Table 3-4.1)

Thus given the value of  $\omega_p$  for a section, the values of  $f_{pu}$  and  $x_u$  can be approximately calculated from the above tables.

### Example 3-4.1

A prestressed concrete beam produced by pre-tensioning method has a rectangular cross-section of 100 mm  $\times$  160 mm ( $b \times h$ ). It is prestressed with 10 numbers of straight 2.5 mm diameter wires. Each wire is stressed up to a load of 6.8 kN. The design load versus strain curve for each wire is given in a tabular form. The grade of concrete is M 40. The value of  $\Delta\varepsilon_p$  is 0.0073.

Estimate the ultimate flexural strength of the member by the strain compatibility method.



Values in mm

Cross-section of member

Design load ( $P$ ) versus strain ( $\epsilon_p$ ) values for the prestressing wire are given for the range under consideration.

$\epsilon_p$	$P$ (kN)
0.006	5.4
0.008	7.6
0.010	9.0
0.012	10.0
0.014	10.7

## Solution

Strain difference

$$\Delta\epsilon_p = 0.0073$$

The effective depth of the CGS ( $d$ ) is 120 mm.

The strain compatibility method is shown in a tabular form. Here,

$P_u$  = load in a single wire obtained from the table

$T_u = 10 \times P_u$ , for the ten wires.

$x_u$ (mm)	$x_u/d$	$\epsilon_{pu} - \Delta\epsilon_p$ (3-4.6)	$\epsilon_{pu}$	$P_u$ (kN) (Table)	$T_u$ (kN)	$C_u$ (kN) (3-4.2)	Checking (3-4.4)
60	0.5	0.0035	0.0108	9.4	94.0	86.4	$T_u > C_u$
65	0.54	0.0030	0.0103	9.1	91.0	93.6	$T_u < C_u$
63.5	0.53	0.0031	0.0104	9.15	91.5	91.4	$T_u \approx C_u$

The ultimate flexural strength is given as follows.

$$\begin{aligned}
 M_{uR} &= T_u(d - 0.42x_u) \\
 &= 91.5(120.0 - 0.42 \times 63.5) \text{ kNmm} \\
 &= 8.5 \text{ kNm}
 \end{aligned}$$

