Discrete - Time Signal

- Definition
- Notation
- Classification
- Manipulation

Discrete Time Signal

- As discussed before a signal is a parameter that depends on an independent parameter

- Definition : A Discrete-time Signal is a function of an independent variable that is an integer and is formally denoted by $x = \{x(n)\}$, $-\infty < n < \infty$

- This means that the independent parameter of the Discrete-time signal has to be represented by integers.
- For eg: if we sample an analog signal $X(t)$ at time instants 0, T_s, $2T_s$, ….

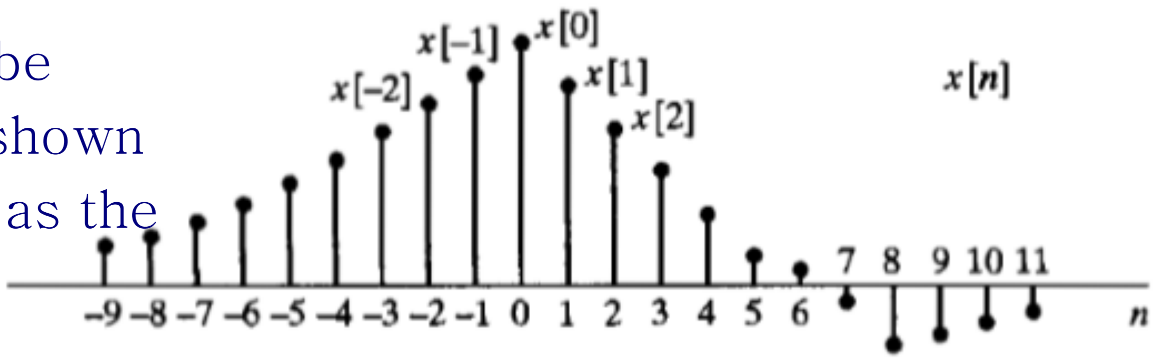
i.e at $t=nT_s$, then the discrete time signal obtained is denoted by $x(n)$ and $x(n) = X(nT_s)$

- So here the independent parameter is time and n=0,1,2,3,… represent t=0, T_s, $2T_s$, ….
- A discrete time signal is basically just a sequence of numbers, this will become clear as we discuss representation of discrete time signals

Representation of Discrete Time Signals

- Graphical Representation

A Discrete time signal may be represented graphically as shown in the figure. This is known as the lollipop representation.



Note that the signal is represented only at integers, −9,−8,⋯.,0,⋯. This is because discrete
time signal domain is only integers. In between integers the signal is not equal to zero , In fact
at these points (for eg between 1 and 2) , the signal does not exist !!!

- Functional Representation

Another kind of representation is by denoting the discrete time signal as a function

Eg: 1) $x(n) = A\sin(\omega_0 n + \Theta)$

$3n-5$, if $-9 \le n < 0$

2) $x(n) = 4n+2$, if $0 \le n < 5$

0, otherwise

Representation of Discrete Time Signals contd.

- Sequential Representation

A discrete time signal may also be represented as a sequence of numbers.

For example x = {2,3,4,9,3} is a signal. What this means is that n=0 is at the

arrow, i.e. x(0)=3, and therefore x(−1)=2; x(1)=4; x(2)=9; x(3)=3.

Therefore we see that any discrete time signal is basically just a sequence of

numbers. It is for this reason we often refer to discrete time signals as

sequences .

- Finite Duration Sequence: If a sequence is non-zero only over a finite period of time, e.g. x={1,4,1}; This sequence has 3 samples, so it is

  called a 3 point sequence.

- Infinite Duration Sequence: If a sequence is non-zero over an infinite duration of time, so it may range over ($-\infty < n < \infty$) or ($-\infty < n < a$) or

Some Basic/Important Sequences/Signals

Finite duration sequences:

- Unit Sample Sequence:

$\delta(n) = 1$, $n=0$

0 , otherwise

Infinite duration sequences:

- Unit Step Sequence:

$u(n) = 1$, $n=0$

0 , otherwise

- Unit Ramp Sequence:

$u_r(n) = n$, $n \geq 0$

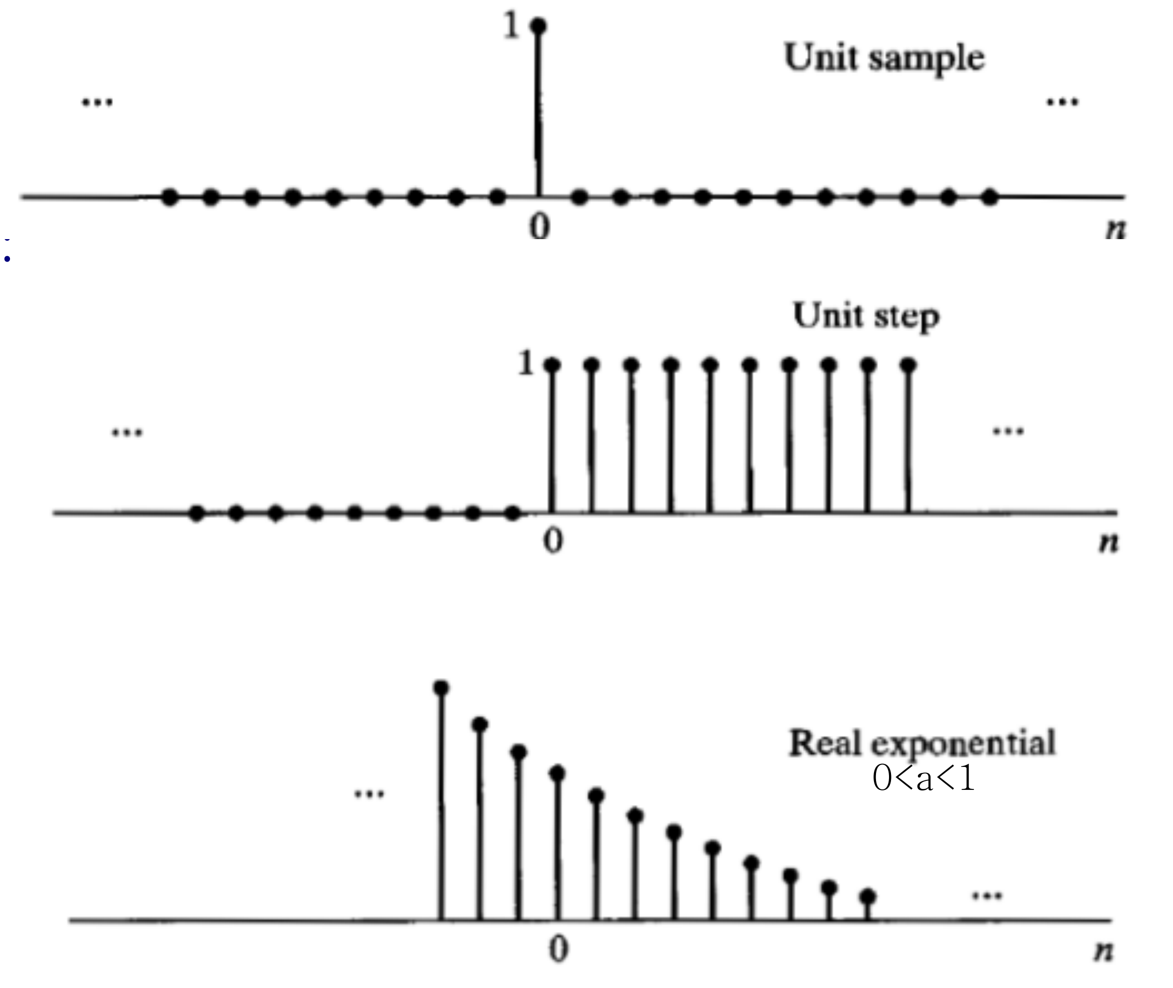
0 , otherwise

- Exponential Sequence:

$x(n)= a_n$, for all n

- Sinusoidal Sequence:

$x(n)= A\sin(\omega_0 n+ \Theta)$



Classification of Discrete Time Signals

Discrete Time signals

Deterministic	Non-Deterministic
All past, present, future values of the signal are known. Can predict the value of the signal	Cannot predict value of the signal, varies randomly, hence also called random signal

Energy Signal	Power Signal
Energy of the signal is finite, $0<E<\infty$, All finite duration signals are energy signals	Energy of the signal is infinite, power is finite, $E \to \infty$ and $0<P<\infty$ Eg: All periodic signals

Periodic	Aperiodic
If signal satisfies the condition, $x(n+N)=x(n)$, for all n for some period N. The smallest N for which the condition is satisfied is called fundamental period	Does not have any N for which condition is satisfied

Symmetric/Even	Antisymmetric/Odd
Satisfies the condition $x(n)=x(-n)$ Symmetric about origin	Satisfies the condition $x(n)= -x(-n)$ AntiSymmetric about origin

Note:

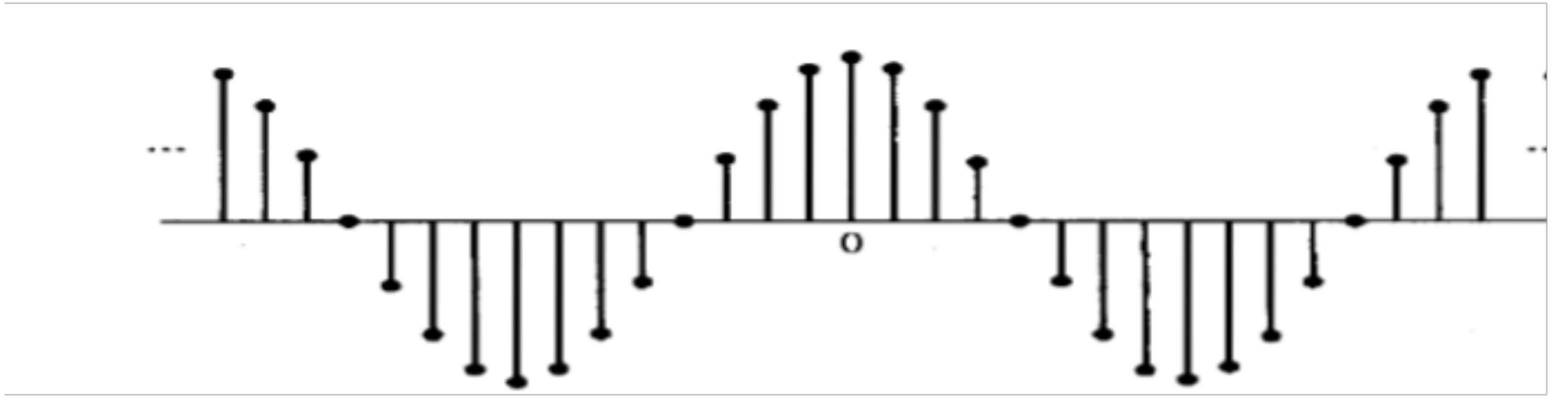
- Energy of Signal is given by: $$\sum_{n=-\infty}^{\infty} |x(n)|^2$$
- Power of Signal is given by: $$\lim_{n\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2$$

Sinusoidal Discrete-Time Signals

- A sinusoidal discrete-time signal is of the form $x(n) = A\sin(\omega_0 n + \Theta)$
- Note that all sinusoidal discrete-time signals are not periodic
- Periodicity occurs if and only if there is an N such that $x(n)=x(n+N)$ for all n, i.e $A\sin(\omega_0 n+\Theta) = A\sin(\omega_0 (n+N)+\Theta) \Rightarrow \omega_0 N=2\pi k$

$\Rightarrow (2\pi/\omega_0)=(k/N)$

$\Rightarrow (2\pi/\omega_0)$ is a rational number



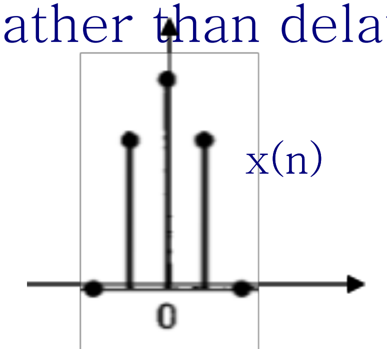
Note: Discrete time sinusoids whose frequencies are separated by $2\pi k$, i.e. an integer multiple of 2π are identical/indistinguishable. This is the crucial reason for aliasing

Simple Manipulations of Discrete Time Signals

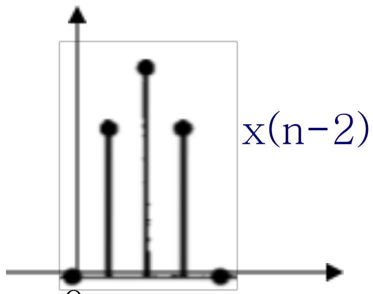
Manipulations of time/independent variable:

- Time Delay:

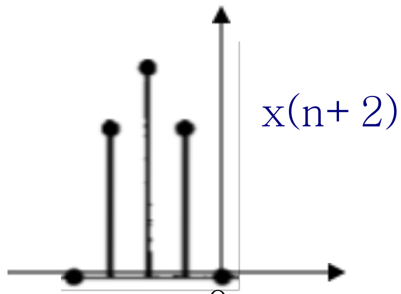
Change n → (n−k), i.e y(n)=x(n−k), if k is −ve, it means the signal is advanced
rather than delayed.



Original Signal



Delayed Signal



Advanced Signal

- Folding/Reflection:

y(n)=x(−n), i.e. a reflection or folding about the y-axis

- Time Scaling;

y(n)=x(kn), where k is an integer. This means your choosing to keep only every kth member of your sequence. This corresponds to increasing your sampling period or decreasing your sampling frequency i.e. downsampling

Simple Manipulations of Discrete Time Signals

Manipulations of the Amplitude/Dependent variable:

- Amplitude Scaling:

$y(n)= Ax(n) \; ; -\infty < n < \infty$

- Addition of Sequences:

$y(n)= x_1(n) + x_2(n); -\infty < n < \infty$

- Multiplication of Sequences:

$y(n)= x_1(n)\, x_2(n); -\infty < n < \infty$

- Convolution of Sequences:

$$y(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

Any signal x(n) can be written as $x(n)=x(n)*\delta(n)$

Discrete - Time System

- Definition
- Classification/Properties
- Linear Time-Invariant System
- Linear Convolution
- Linear Constant Co-efficient Difference Equations

Discrete-Time System

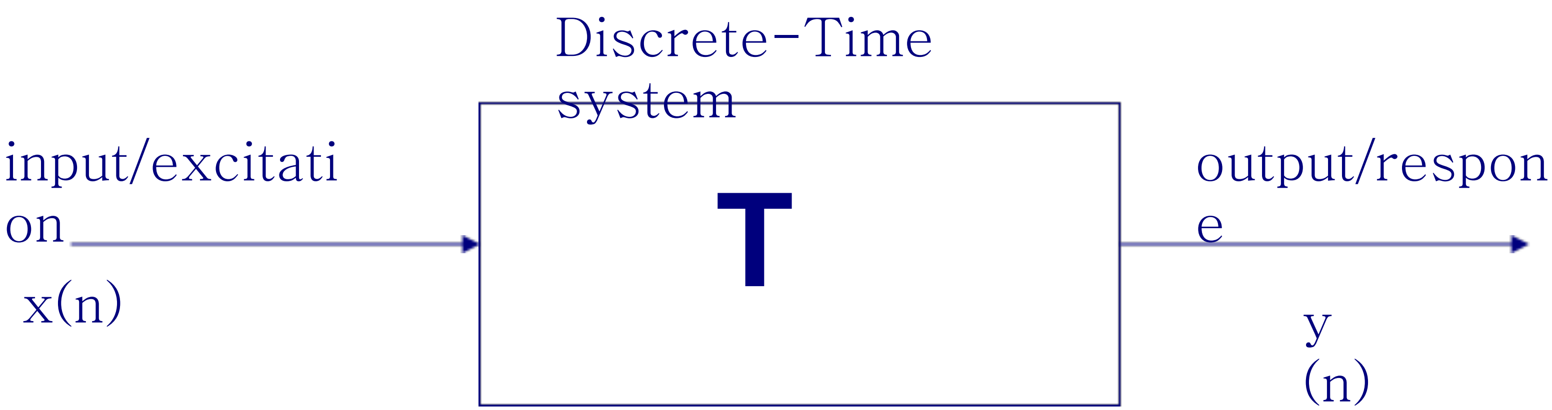
- A system is something that forces a change in the signal.

- Definition ：A Discrete-Time system is a device or algorithm that performs a

certain operation on a discrete-time signal called the input/excitation and

produces another discrete-time signal called output/response.

$y(n) \equiv \mathbf{T}\,[x(n)]$



Classification/Properties of Discrete-Time Systems

- Static/Memoryless and Dynamic Sytems:

If the output of the system depends only on the present input, then the system is Static or Memoryless

Eg: $y(n) = 2x(n)$; $y(n)=e^{x(n)}$

If the output of the system depends on past and/or future inputs then the system is called Dynamic or a System with Memory

Eg: $y(n)= x(n-2)$; $y(n)= 0.5[x(n) + x(n-1)]$

- Time-Invariant and Time-Variant Systems:

A system is called time-invariant if a delay/advance in the input just causes a time - delay/advance in the output, but the form of the output does not change.

i.e. if $y(n) \equiv \mathbf{T}[x(n)]$; $y(n-k) \equiv \mathbf{T}[x(n-k)]$ Eg: $y(n)=2x(n)$

A system is called time - variant if a delay/advance in the input changes the shape/form of the output

$y(n-k) \neq \mathbf{T}[x(n-k)]$ Eg: $y(n)=nx(n)$

Classification/Properties of Discrete-Time Systems contd

- Linear and Non-Linear Systems

If the weighted sum of signals gives the corresponding weighted sum of their responses (this is also known as the principle of superposition), then a system is linear i.e if

$x(n) = ax_1(n) + bx_2(n)$

$y(n) = ay_1(n) + by_2(n)$

Eg: $y(n)=nx(n)$; $y(n)=Ax(n)+B$; $y(n)=x(n^2)$

If a system does not obey the principle of superposition, it is non-linear

Eg: $y(n)=x^2(n)$; $y(n)=e^{x(n)}$

- Causal and Non-Causal Systems

If the response of a system depends only on past and present inputs but not on future inputs it is called Causal, i.e. it is non-anticipatory

$y(n) = F[x(n), x(n-1), x(n-2), \cdots\cdots]$

Eg: $y(n)=2x(n-3)+x(n)$; $y(n)= nx(n-2)+x(n)-y(n-1)$

A system is called non-causal if its response depends upon future outputs

Eg: $y(n)= [x(n-1)+x(n)+x(n+1)]/3$

Classification/Properties of Discrete-Time Systems contd

- Stable and Unstable Systems:

A system is called stable if a bounded input produces a bounded output

Such a system is said to be bounded input bounded output (BIBO) stable.

Mathematically this means that

if for $|x(n)|<M$, where M is some finite integer, then

we get $|y(n)|<N$, where N is some finite integer, the system is stable.

Examples of Stable systems:

$y(n)=2x(n)$; if $x(n)<10$, $y(n)<20$

$y(n)=y(n)=e^{x(n)}$; if $x(n)<5$; $y(n)< e^{5}$

If a bounded input does not give a bounded output the system is unstable

Example:

If $x(n)=u(n)$; and $y(n)=2^{n}x(n)$. Now if $x(n)=1$ for all n, then $x(n)<1$

but $y(n)=2^{n} \rightarrow \infty$

Linear Time-Invariant or Shift-Invariant Systems
LTI or LSI systems

- Linear Time Invariant Systems are characterized by their response to the unit sample sequence
- This means that if we know the response of a system to the unit sample sequence i.e. $\delta(n)$ we can tell the response to any arbitrary signal $x(n)$

- Impulse Response : The response of an LTI system to unit sample sequence is called impulse response and is denoted by $h(n)$

$h(n) = \mathbf{T}\,[\delta(n)\,]$

Response of LTI system to arbitrary input

- We have seen before that x(n)=x(n)∗δ(n)= $\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$
- Now the response y(n)= **T** [x(n)] can be found as follows

y(n)= **T** [x(n)]

= **T** [x(n)∗δ(n)]

= **T** [$\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$]

= $\sum_{k=-\infty}^{\infty} x(k)$ **T** [$\delta(n-k)$] ;from linearity

= $\sum_{k=-\infty}^{\infty} x(k)h(n-k)$ from time-invariance

= x(n)∗h(n)

Linear Convolution

- We have seen previously that the response of an LTI system to any signal is the convolution of the signal with its impulse response. This type of convolution is called linear convolution.
- Properties of linear convolution:

Commutative law: $x(n)*h(n)=h(n)*x(n)$;

Associative law: $[x(n)*h(n)]*g(n)=x(n)*[h(n)*g(n)]$

Distributive law: $x(n)*(h(n)+g(n))=x(n)*h(n)+x(n)*g(n)$

- Step Response : The response of an LTI system to unit step sequences

$$s(n)=u(n)*h(n)=\sum_{k=-\infty}^{\infty}u(k)h(n-k)$$

$$=\sum_{k=0}^{\infty}h(n-k)$$

Also $h(n)=s(n)-s(n-1)$

Performing Linear Convolution

- Analytical Evaluation of the Convolution Sum

To evaluate the Convolution sum analytically simply perform the summation

Eg: h(n)=u(n)−u(n−5)=1 for $0 \le n \le 4$

$x(n)=a^n u(n)$

$y(n)=x(n)*h(n) = h(n)*x(n)= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

$= \sum_{k=0}^{4} a^{n-k}$

$= a^n + a^{n-1} + a^{n-2} + a^{n-3} + a^{n-4}$

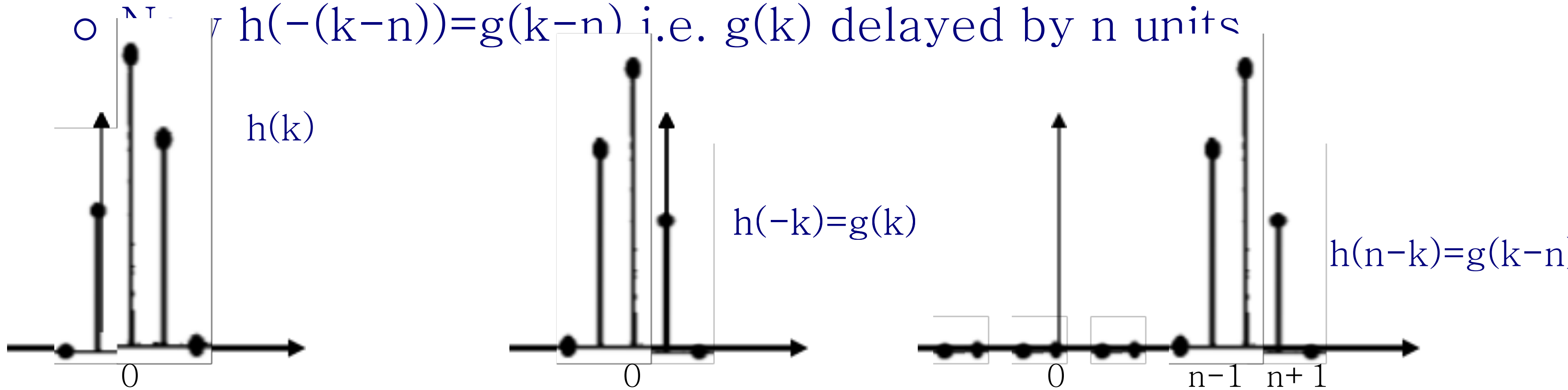
$= a^{n-4} (a^5-1)/(a-1)$

Performing Linear Convolution

- Graphical Evaluation of the Convolution sum

To perform graphical evaluation, for each n,

- Represent x(k) and h(n-k) graphically
- Multiply x(k) and h(n-k)
- Sum up all values of the product
- Representing h(n-k):
  - Observe that h(n-k) =h(-(k-n))
  - Draw h(k) and perform a reflection about y-axis to get h(-k) i.e. g(k)=h(-k)
  - Now h(-(k-n))=g(k-n) i.e. g(k) delayed by n units



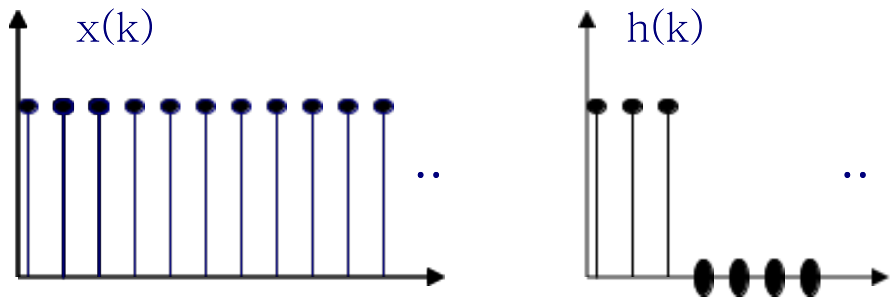
Graphical Evaluation Contd

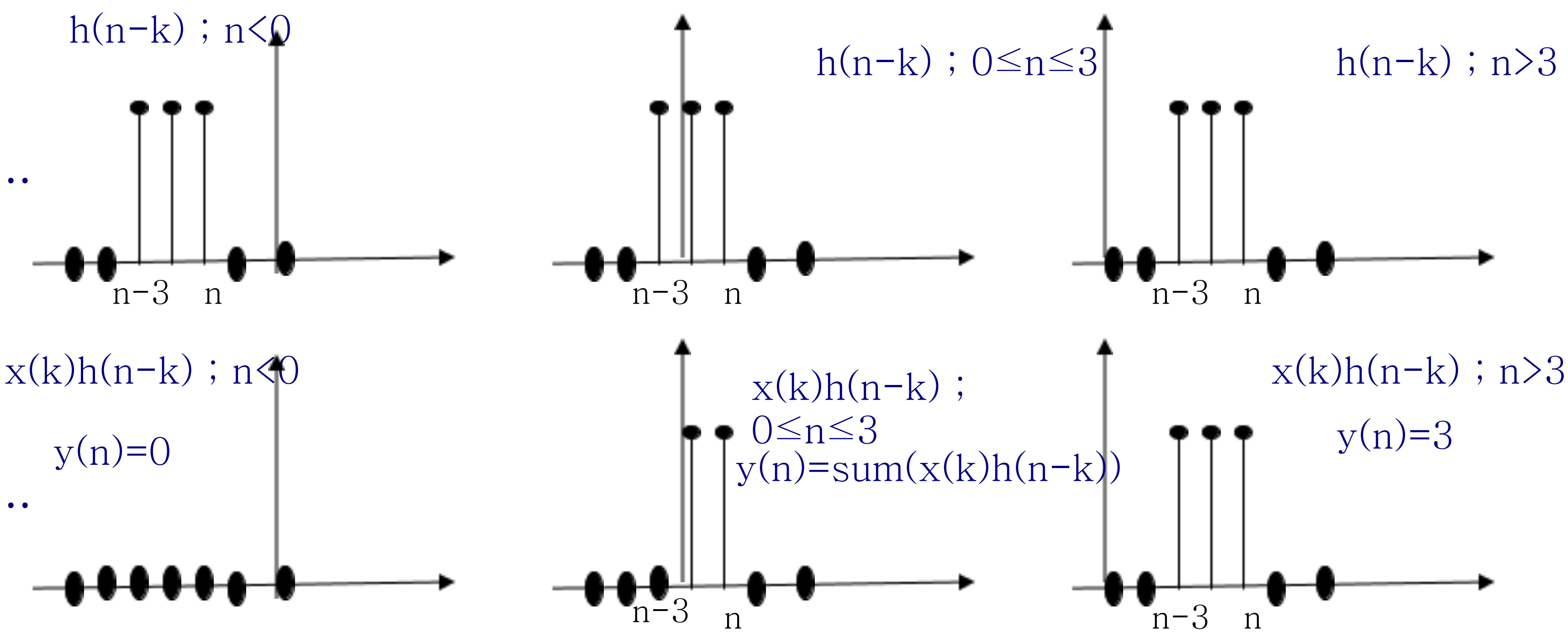
- Multiplying x(k) and h(n-k) and summing the product

The product is non zero only when there is an overlap between x(k) and h (n-k)

Eg: $x(n)=u(n)$

$h(n)=2[u(n)-u(n-3)]$





Properties of LTI systems

Causality of LTI systems:

A causal system satisfies

$y(n)= F[x(n),x(n-1),x(n-2),\cdots\cdots]$

For LTI systems F is a linear function

Restating the condition:

Now if $x(n)=0$ for $n<n0$, then for $n<n0$, $y(n)= F[x(n),x(n-1),x(n-2),\cdots\cdots]$ $=F[0,0,0,\cdots]=0$. Therefore if $x(n)=0$ for $n<n0$, then $y(n)= 0$ for $n<n0$

Condition on impulse response:

For a causal LTI system

$y(n)=x(n)*h(n) = h(n)*x(n)=\sum_{k=-\infty}^{\infty} h(k)x(n-k)$

In the above the term multiplying $x(n-k)$ for $k<0$, must be 0, i.e

$$h(n)=0 \text{ for } n<0$$

Convention: A sequence $x(n)$ which is 0 for $n<0$, is called a causal sequence

Anticausal System: A system for which $h(n)=0$ for $n>=0$

Properties of LTI systems contd

- Stability of LTI systems

For an LTI system to be BIBO stable, if |x(n)|≤M for all n, where M is a
finite integer then
|y(n)|= $| \sum_{k=-\infty}^{\infty} x(k)h(n-k) |$

≤ $\sum_{k=-\infty}^{\infty} | x(k) || h(n-k) |$

≤ $M \sum_{k=-\infty}^{\infty} | h(n-k) |$

= $M \sum_{k=-\infty}^{\infty} | h(k) |$ which is finite iff $\sum_{k=-\infty}^{\infty} | h(k) | < \infty$

Hence an LTI system is stable iff h(n) is absolutely summable i.e. $\sum_{k=-\infty}^{\infty} | h(k) | < \infty$

Linear Constant Co-efficient Difference equations (LCCDEs)

- An equation of the form shown below is called a LCCDE

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Examples: 1) 2y(n)+ 3y(n-2)=4x(n)+ 2x(n-5); N=2, M=5

2) y(n) = 2x(n)+ 3x(n-1)+ 7x(n-2) ;N=0,M=2

3) Accumulator: y(n)= $\sum_{k=-\infty}^{N} x(k)$

This can be written as y(n)=y(n-1)+ x(n)

Also, y(n)-y(n-1)=x(n)

Solving LCCDEs

- To solve an LCCDE we require two things
  - Initial/Auxiliary conditions: These give us values y(n) or x(n) at particular times. Initial rest conditions /Zero initial conditions imply that y(n)= 0 for n<0 The response for these conditions is called the zero-state response/forced response
  - Input: We also need to know what the input x(n) is. If the input is zero then the solution is zero-input response / natural response

- Solution of LCCDE is of the form

$y(n) = y_p(n) + y_h(n)$

Here $y_h(n)$ is the homogeneous solution, and $y_p(n)$ is the particular solution

Solving LCCDEs- Homogeneous solution

- Homogeneous solution $y_h(n)$ is the solution to the equation $\sum_{k=0}^{N} a_k y(n-k) = 0$

Also gives zero-input response

- To solve this equation write the characteristic polynomial of the equation

(4.1) $$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$ ....

If the roots of this equation are unique and $\lambda 1, \lambda 2, \lambda 3 \cdots .., \lambda n$, then the homogeneous solution $y_h(n) = C_1\lambda_{1n} + C_2\lambda_{2n} + C_3\lambda_{3n} + \cdots .. + C_N\lambda_{Nn}$

where $C_1, C_2, C_3, \cdots, C_N$ are constants that depend on the input and the initial

conditions

Example: y(n)-4y(n-2)=0

Characteristic polynomial: $\lambda_n - 4\lambda_{n-2} = 0 \Rightarrow \lambda_2 - 4 = 0 \Rightarrow \lambda = +2$ or -2

Homogeneous Solution: $y_h(n) = C_1 (2_n) + C_1 (-2)_n$

Solving LCCDEs- Particular solution

- The particular solution depends on the input and is any solution that satisfies

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad \text{(4.2)}$$

Usually the particular solution has the same form as the input

Final Solution

For the final solution put $y(n) = y_p(n) + y_h(n)$

and solve for the constants C_i using the initial /auxiliary conditions

To find the impulse response, $y(n)=y_h(n)$ only, and solve for constants