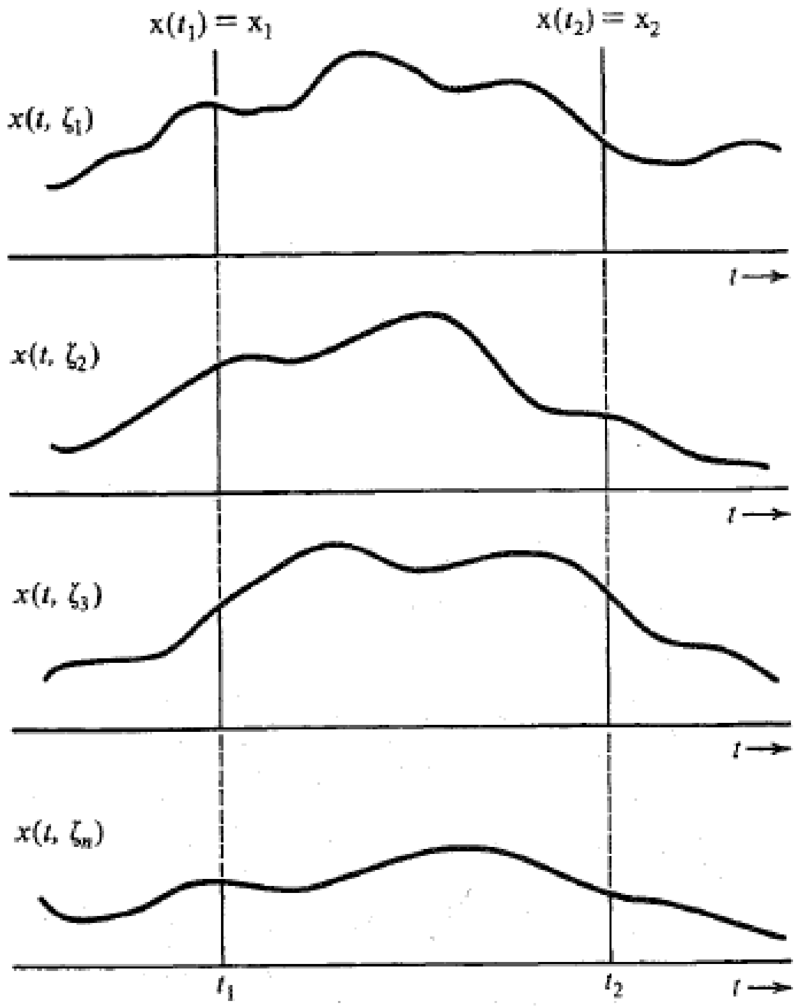
Random Processes

Random Process

Definition of Random Process

- Consider we want to determine the probability of temperature at 12 pm in certain city
  - The RV is X = "temperature at 12 pm in city A"
  - We have to record the data for many days to get $p_X(x)$
- However, the temperature is a function of time
  - The temperature varies between time to time and days to days
  - The RV = "temperature in city A" is function of time
- An RV that is a function of time (or any other variable) is called a random process or stochastic process

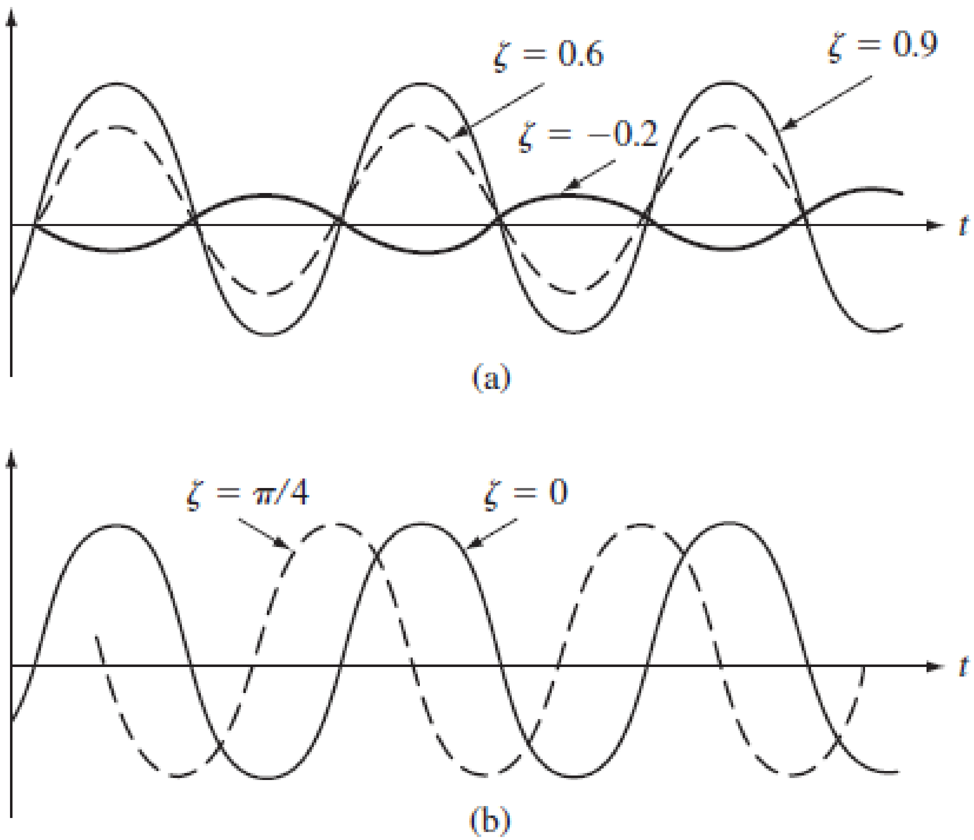
Definition of Random Process



$$X(t, \zeta) \qquad t \in I$$

- The collection of all possible waveform is known as ensemble of the random process x(t)
- A waveform in the collection is called a sample function
- Sample-function amplitudes at some instant are the values taken by the RV in various trials

Definition of Random Process



(a) Sinusoid with random amplitude, (b) Sinusoid with random phase.

Definition of Random Process

- A discrete-time stochastic process is one when the index set I is a countable set
- When the index set is continuous, it is called continuous-time stochastic process
- Example:

Let ζ be a number selected at random from the interval $S = [0, 1]$, and let $b_1 b_2 \ldots$ be the binary expansion of ζ:

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i} \qquad \text{where } b_i \in \{0, 1\}.$$

Define the discrete-time random process $X(n, \zeta)$ by

$$X(n, \zeta) = b_n \qquad n = 1, 2, \ldots.$$

The resulting process is sequence of binary numbers, with $X(n, \zeta)$ equal to the nth number in the binary expansion of ζ.

Definition of Random Process

- The number of waveforms in an ensemble may be finite or infinite
- Example of infinite waveform: temperature
- Example of finite waveform: output of binary signal generator
  - Consider that we will examine the output over the period 0 to 2T
  - We will have only $2^2 = 4$ waveforms

Definition of Random Process

- Note that the randomness occurs to the selection of waveforms
- The waveforms (in the ensemble) themselves are deterministic
- Example: In the experiment of tossing a coin four times, there are 16 possible outcomes. The randomness is which of the 16 outcomes will occur in a given trial

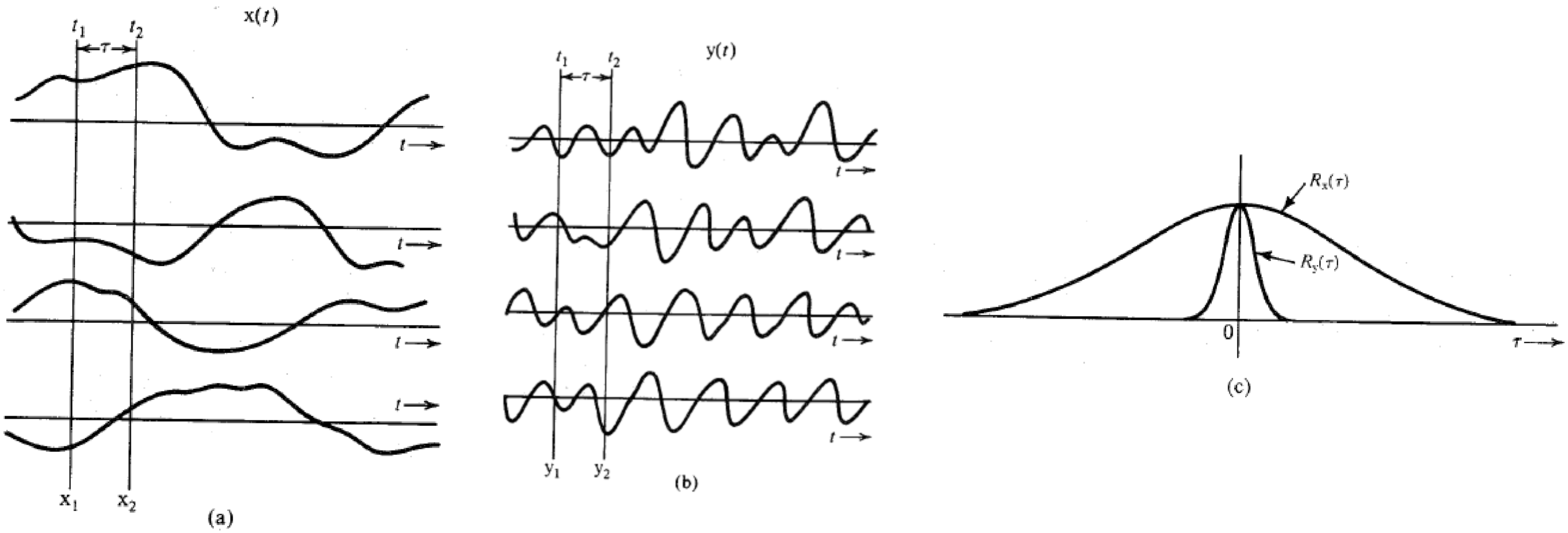
Specification of A Random Process

- How to specify random process:
  - Analytical
  - Experimental
- Analytical: mathematical expression
- Experimental:
  - We need to find some quantitatibe measure
  - Random process can be seen as collection of infinite, independent number of RV
  - We need pdf

Autocorrelation

- The autocorrelation function is defined as

$$R_X(t_1,t_2) = \overline{x(t_1)\,x(t_2)} = \overline{x_1 x_2}$$



Example

A random process X(t) is defined by

$$X(t) = 2\cos(2\pi t + Y)$$

where Y is a discrete random variable with P(Y = 0) = ½ and P(Y = π/2) = ½. Find $\mu_X(1)$ and $R_{XX}(0,1)$

Solution:

X(1) is a random variables with values $2\cos(2\pi)$ and $2\cos(2\pi + \pi/2)$

With probability ½ each

$$P\left[X(1) = 2\right] = \frac{1}{2} \qquad P\left[X(1) = 0\right] = \frac{1}{2}$$

Example (Cont'd)

Thus,

$$\mu_X(1) = E\{X(1)\} = 2\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = 1$$

The autocorrelation will be searched for t = 0 and t = 1

The values of P[X(0) = 0, X(1) = 0] = ½

P[X(0) = 2, X(1) = 2] = ½

Thus, the autocorrelation is

$$R_{XX}(0,1) = E\{X(0)X(1)\}$$

$$= (2)(2)\frac{1}{2} + (0)(0)\frac{1}{2} = 2$$

Classification of Random Process

- Stationary and Nonstationary random process
  - A random process whose statistical characteristics do not change with time
  - Random process $X(t)$ and $X(t+\varepsilon)$ have the same statistical properties
  - In other words, time translation of a sample function results in another sample function of the random process having the same probability
  - Example of stationary random process: noise
  - If the statistical characteristics depend on time, it is called nonstationary random process
  - Example of nonstationary random process: temperature

Classification of Random Process

- Wide-sense stationary process (WSS)
  - The mean and autocorrelation function do not vary with a time shift

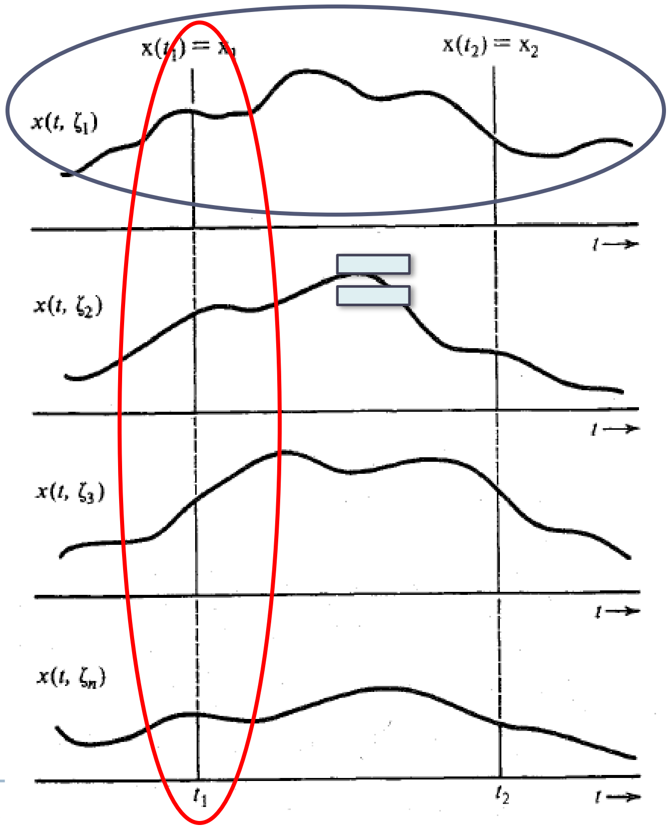
$$E\left\{X(t)\right\} = K = \text{constant}$$

$$R_X(t_1, t_2) = R_X(\tau), \qquad \tau = t_2 - t_1$$

  - All stationary processes are WSS, but not vice versa

Classification of Random Process

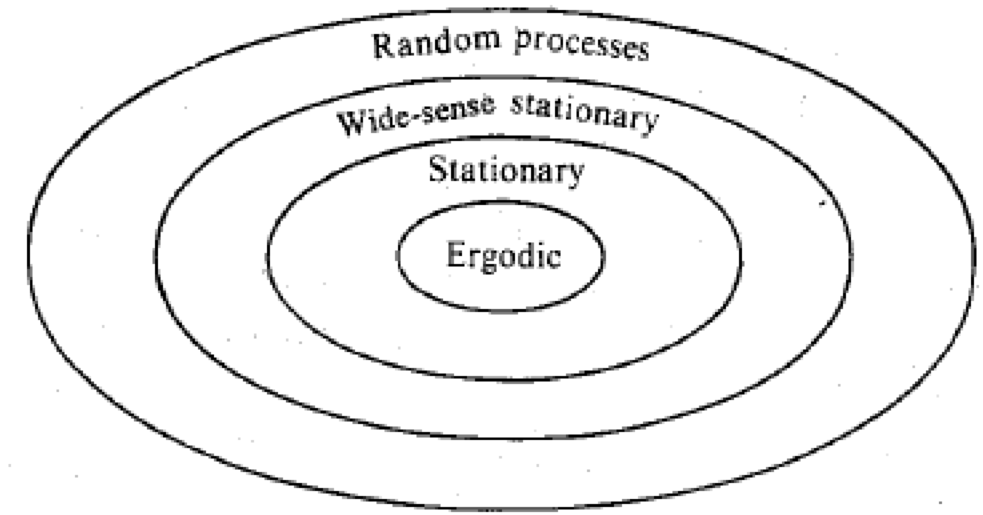
- Ergodic
  - Ensemble averages are equal to time averages of any sample function



$$\overline{x(t)} = \widetilde{x(t)}$$

$$R_X(\tau) = \Re_X(\tau)$$

Classification of Random Process



Example

- Show that $x(t) = A\cos(\omega_0 t + \theta)$, where A and ω_0 are constants and θ is a random variable that is uniformly distributed over (0, 2π), is ergodic

Solution:

The ensemble average is

$$\bar{x} = \int_{-\infty}^{\infty} \left[x(\theta) \right] f_\theta(\theta)\, d\theta = \int_{0}^{2\pi} \left[A\cos(\omega_0 t + \theta) \right] \frac{1}{2\pi}\, d\theta = 0$$

The time average is

$$\langle x(t) \rangle = \frac{1}{T_0} \int_{0}^{T_0} A\cos(\omega_0 t + \theta) = 0$$

Power Spectral Density of A Random Process

- PSD depicts the power spread in frequency domain
- It is easy to get the frequency domain for deterministic signal by using Fourier transform
- How to compute the PSD for random signal?
- Random signals are power signals
- Several questions of determining the PSD for random process arise:
  - We may not be able to describe the sample function analytically
  - Every sample function may be different from another one
- PSD is defined for stationary (or WSS)

Power Spectral Density of A Random Process

- We have to define the PSD of random process as the ensemble average of the PSDs of all sample functions
- Suppose that $x(t,\xi_i)$ represents a sample function of a random process $x(t)$. The truncated version of this sample function is obtained by multiplying the signals with rectangular function $rect(t/T)$

$$x_T(t,\xi_i) = \begin{cases} x(t,\xi_i), & |t| < 0.5T \\ 0, & elsewhere \end{cases}$$

Power Spectral Density of A Random Process

- The Fourier transform is

$$X_T(f,\xi_i) = \int_{-\infty}^{\infty} x_T(t,\xi_i) e^{-j2\pi ft} dt$$
$$= \int_{-T/2}^{T/2} x_T(t,\xi_i) e^{-j2\pi ft} dt$$

- The normalized energy in time interval (-T/2,T/2)

$$E_T = \int_{-\infty}^{\infty} x_T^2(t) dt = \int_{-\infty}^{\infty} \left| X_T(f) \right|^2 df$$

Power Spectral Density of A Random Process

- The mean normalized energy is obtained by taking the ensemble average

$$\overline{E}_T = \int_{-T/2}^{T/2} \overline{x^2(t)}dt = \int_{-\infty}^{\infty} \overline{x_T^2(t)}dt = \int_{-\infty}^{\infty} \overline{|X_T(f)|^2} df$$

- The normalized average power is the energy expended per unit time

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \overline{x^2(t)}dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \overline{x_T^2(t)}dt$$

$$= \int_{-\infty}^{\infty} \left[\lim_{T\to\infty} \frac{1}{T} \overline{|X_T(f)|^2} \right] df = \overline{\langle x^2(t) \rangle}$$

Power Spectral Density of A Random Process

- The definition of PSD is

$$P = \int_{-\infty}^{\infty} S_x(f)\,df$$

- So that

$$S_x(f) = \lim_{T \to \infty} \left(\frac{\overline{\left| X_T(f) \right|^2}}{T} \right)$$

Power Spectral Density of A Random Process

- Wiener-Khintchine Theorem

When $x(t)$ *is a wide-sense stationary process, the PSD can be obtained from the Fourier transform of the autocorrelation function*

$$R_x(\tau) \Leftrightarrow S_x(f)$$

$$S_x(f) = F\left[R_x(f)\right] = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi ft} d\tau$$

$$R_x(\tau) = F^{-1}\left[S_x(f)\right] = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi ft} df$$

Power Spectral Density of A Random Process

- Properties of PSD

PSD is always real

$$R_x(\tau) = \overline{x(t)x(t+\tau)} \qquad R_x(-\tau) = \overline{x(t)x(t-\tau)}$$

Let $t - \tau = \sigma$, we have

$$\boxed{R_x(-\tau) = \overline{x(\sigma)x(\sigma+\tau)} = R_x(\tau)}$$

Therefore, the PSD is even function when $x(t)$ is real

Power Spectral Density of A Random Process

- Properties of PSD (Cont'd)

When $x(t)$ is wide-sense stationary

$$\int_{-\infty}^{\infty} S_x(f)\,df = P = \overline{x^2} = R_x(0)$$

White Noise Processes

- A random process $x(t)$ is said to be a white noise process if the PSD is constant over all frequencies

$$S_x(f) = \frac{N_0}{2}$$

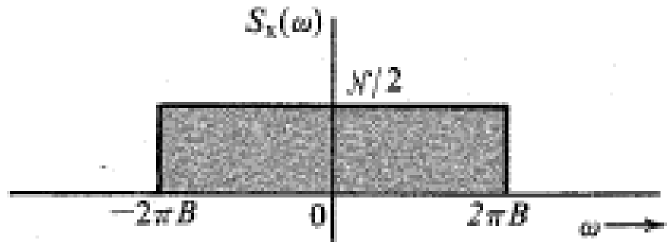
where N_0 is a positive constant

- The autocorrelation function for the white noise process is obtained by taking the inverse Fourier transform

$$R_x(\tau) = \frac{N_0}{2}\delta(\tau)$$

Example

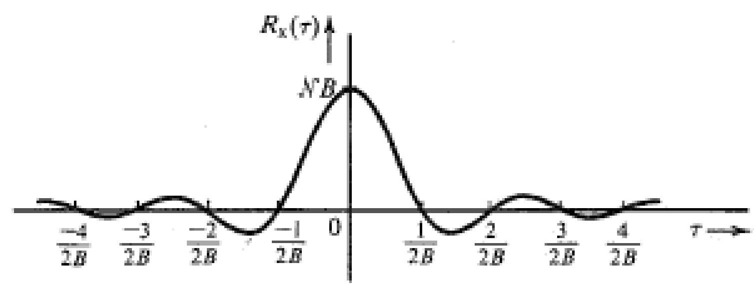
- Determine the autocorrelation and the power of a low-pass random process with white noise PSD $S_X(\omega) = \frac{N}{2}$



Solution:

From the figure we have $S_x(f) = \frac{N}{2} rect\left(\frac{\omega}{4\pi B}\right)$

$$R_x(\tau) = NB\text{sinc}(2\pi B\tau)$$



Example (Cont'd)

We calculate the power

$$P_x = \overline{x^2} = R_x(0) = NB$$

We can also calculate the power by using integral

$$P_x = 2\int_0^{\infty} S_x(\omega)\,df$$

$$= 2\int_0^{B} \frac{N}{2}\,df$$

$$= NB$$

The Gaussian Random Process

- A random process $x(t)$ is said to be Gaussian if the random variables $x_1 = x(t_1), x_2 = x(t_2), \ldots, x_N = x(t_N)$ have an N-dimensional Gaussian PDF for any N and any $t_1, t_2, \ldots, t_N$
- The N-dimensional Gaussian PDF can be written compactly by using matrix notation
- Let **x** be the column vector denoting the N random variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{bmatrix}$$

The Gaussian Random Process

- The N-dimensional Gaussian PDF is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} |Det\mathbf{C}|^{1/2}} e^{-(1/2)\left[(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right]}$$

where the mean vector is

$$\mathbf{m} = \overline{\mathbf{x}} = \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_N} \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{bmatrix}$$

The Gaussian Random Process

- The covariance matrix C is defined by

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{bmatrix}$$

where

$$c_{ij} = \overline{(x_i - m_i)(x_j - m_j)} = \overline{\left[x(t_i) - m_i\right]\left[x(t_j) - m_j\right]}$$

- For WSS random process

$$m_i = \overline{x(t_i)} = m_j = \overline{x(t_j)} = m$$

The Gaussian Random Process

- The elements of the covariance matrix is then

$$c_{ij} = R_x\left(t_j - t_i\right) - m^2$$

- If x_i happen to be uncorrelated $\overline{x_i x_j} = \overline{x_i}\,\overline{x_j}$ for $i \neq j$ the covariance matrix becomes

$$\mathbf{C} = \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

where $\sigma^2 = \overline{x^2} - m^2 = R_x\left(0\right) - m^2$

The Gaussian Random Process

- Some properties of Gaussian processes:
  - $\mathbf{f_x}(\mathbf{x})$ depends only on **C** and on **m**, which is another way of saying that the N-dimensional Gaussian PDF is completely specified by the first- and second-order moments
  - Since the $[x_i = x(t_i)]$ are jointly Gaussian, the $x_i = x(t_i)$ are individually Gaussian
  - When **C** is a diagonal matrix, the random variables are uncorrelated
  - A linear transformation of a set of Gaussian random variables produces another set of Gaussian random variables
  - A WSS Gaussian process is also strict-sense stationary

The Gaussian Random Process

- Theorem:

  If the input to a linear system is a Gaussian random process, the system output is also a Gaussian process

Bandpass Random Process

- Bandpass waveform can be represented by

$$v(t) = \mathrm{Re}\left\{ g(t) e^{j\omega_c t} \right\}$$

or

$$v(t) = x(t)\cos\omega_c t - y(t)\sin\omega_c t$$

and

$$v(t) = R(t)\cos\left[\omega_c t + \theta(t)\right]$$

where

$$g(t) = \left|g(t)\right| e^{j\angle g(t)} = x(t) + jy(t)$$

Bandpass Random Process

- The spectrum of bandpass waveform is

$$V(f) = \frac{1}{2}\left[G(f - f_c) + G^*(-f - f_c)\right]$$

- In communication systems, the random processes may be random signals, noise, or signals corrupted by noise
- If $v(t)$ is Gaussian process, $g(t)$, $x(t)$, and $y(t)$ are Gaussian processes since they are linear functions of $v(t)$
- But, $R(t)$ and $\angle g(t)$ are not Gaussian because they are nonlinear functions of $v(t)$

Bandpass Random Process

- Theorem:

  If $x(t)$ and $y(t)$ are jointly WSS processes, the real bandpass process $v(t) = \mathrm{Re}\left\{g(t)e^{j\omega_c t}\right\} = x(t)\cos\omega_c t - y(t)\sin\omega_c t$ will be WSS if and only if

$$\overline{x(t)} = \overline{y(t)} = 0$$

$$R_{xy}(\tau) = -R_{yx}(\tau)$$

$$R_x(\tau) = R_y(\tau)$$

Bandpass Random Process

- Theorem:

  If $x(t)$ and $y(t)$ are jointly WSS processes, the real bandpass process $v(t) = \mathrm{Re}\left\{g(t)e^{j(\omega_c t+\theta_c)}\right\} = x(t)\cos(\omega_c t + \theta_c) - y(t)\sin(\omega_c t + \theta_c)$ will be WSS when θ_c is an independent random variable uniformly distributed over (0,2π)