6.013 Electromagnetics and Applications
Spring 2009

MEDIA AND BOUNDARY CONDITIONS

Media: conductivity σ, permittivity ε, permeability μ

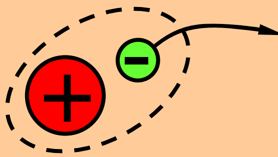
Media are the only tools we have to create or sense EM fields

Conductivity (σ):

$\bar{J}$ (current density, A/m²) = $nq\langle\bar{v}\rangle = \sigma\bar{E}$

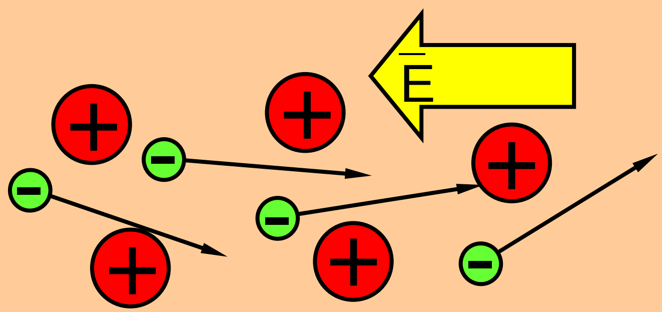
n = #q's/m³, q = charge (Coulombs), $\langle\bar{v}\rangle$ = average velocity (m/s)

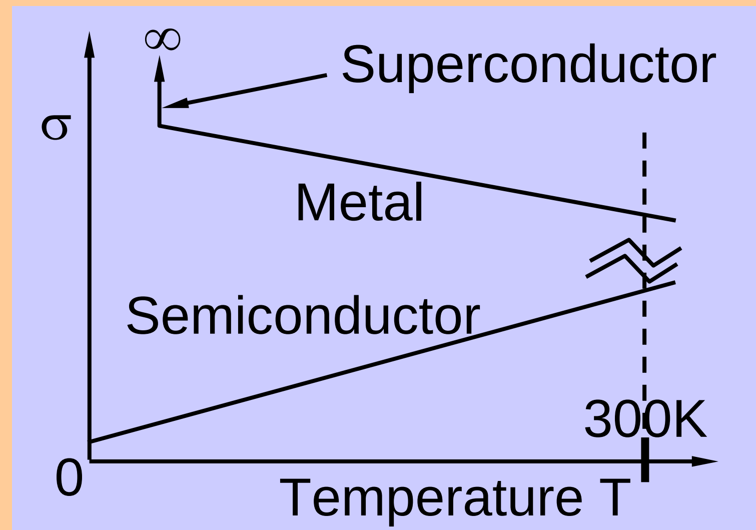
Semiconductors:



$P\{escape\} \propto e^{-W/kT}$

Metals:





$$J \propto \langle v\rangle = \langle at\rangle = \left\langle \frac{f}{m}t\right\rangle = \frac{qE}{m}\langle t\rangle, \quad \therefore \sigma \propto \frac{q}{m}\langle t\rangle; \quad \langle t\rangle = f(T_{emp})$$

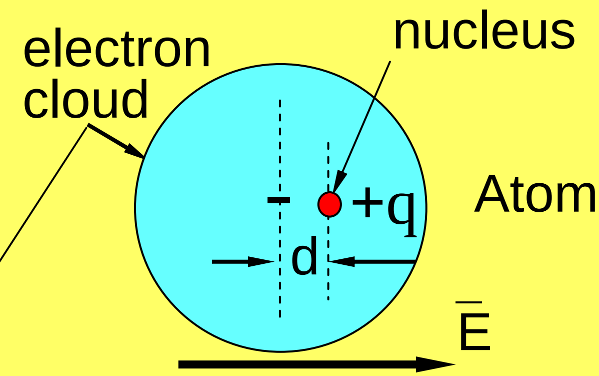
(t = time before collisions reset $v \cong 0$)

DIELECTRICS

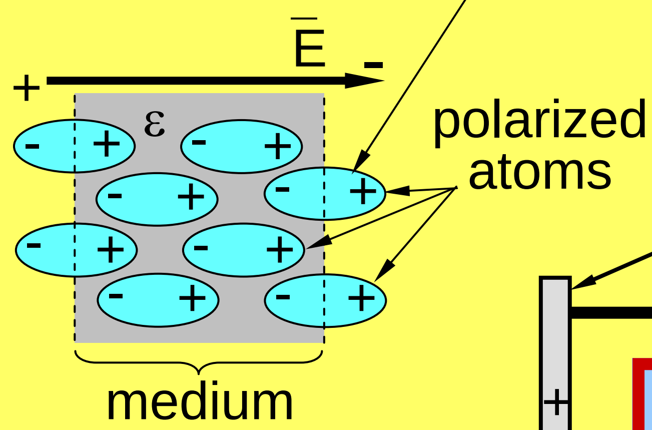
Vacuum:

$\overline{D} = \varepsilon_o \overline{E}$ $\quad\quad$ $\oint\!\!\oint_S \overline{D} \bullet \hat{n} da = \iiint_V \rho_f \; dv$

ρ_f = free charge density



Dielectric Materials:



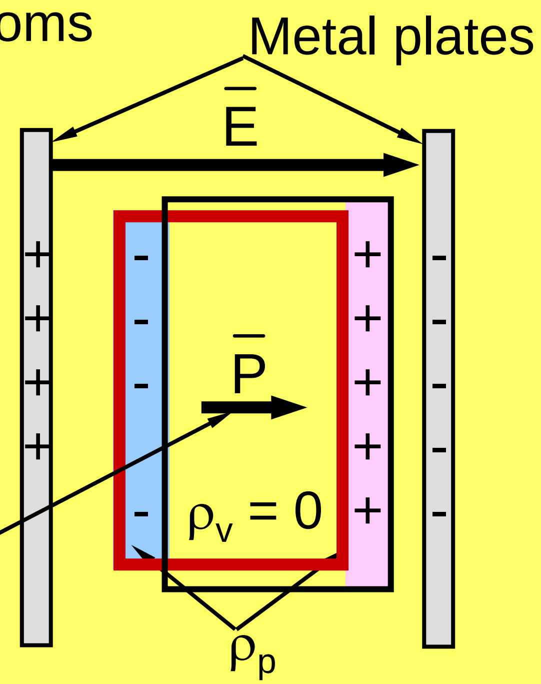
$\overline{D} = \varepsilon \overline{E} = \varepsilon_o \overline{E} + \overline{P}$

$\oint\!\!\oint_S \varepsilon_o \overline{E} \bullet \hat{n} da = \iiint_V (\rho_f + \rho_p) \; dv$

$\oint\!\!\oint_S \overline{P} \bullet \hat{n} da = -\iiint_V \rho_p dv$

ρ_p is polarization (surface) charge density

$\overline{P}$ = "Polarization Vector"



MAGNETIC MATERIALS

Basic Equations:

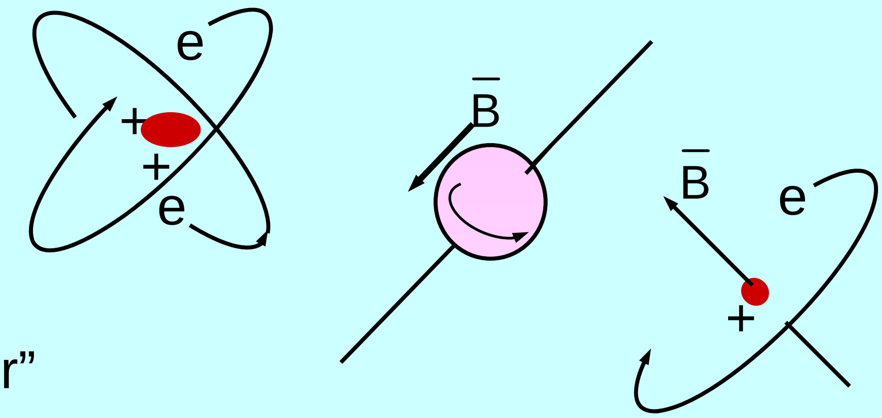
$$\oiint_S \overline{B} \bullet \hat{n} da = 0$$

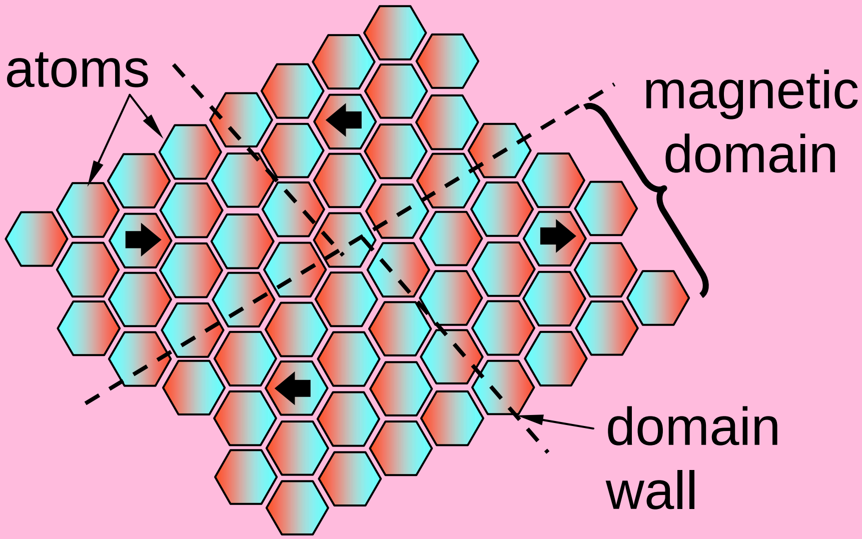
$$\overline{B} = \mu_o \overline{H} \text{ in vacuum}$$

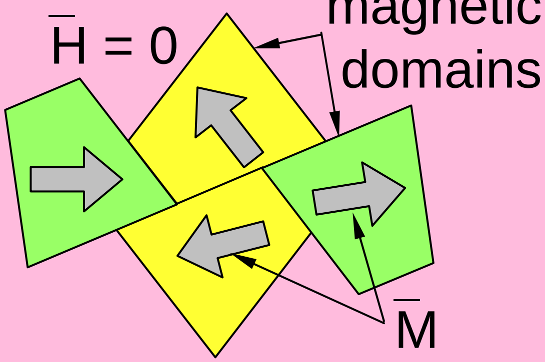
$$\overline{B} = \mu \overline{H} = \mu_o \left(\overline{H} + \overline{M}\right)$$

$\overline{M}$ = "Magnetization Vector"

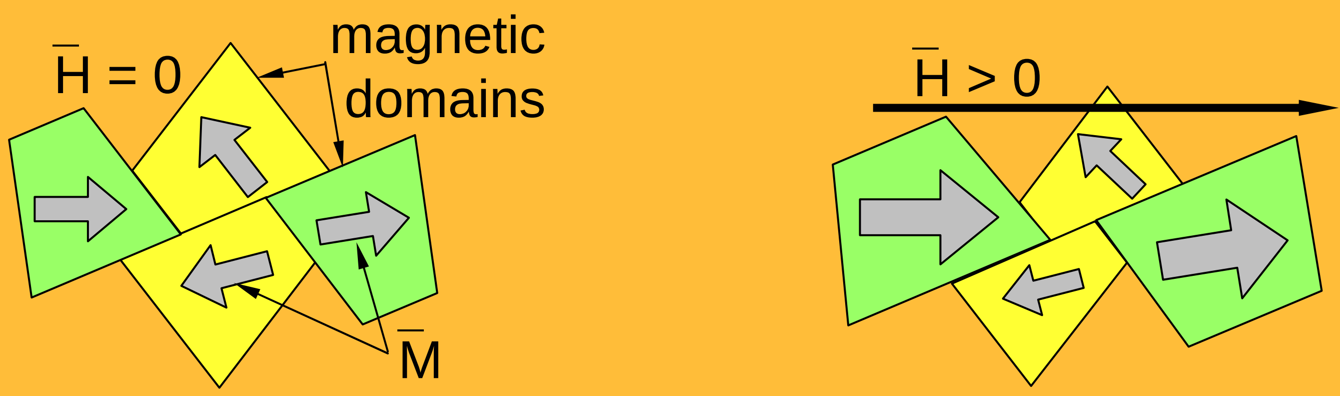
μ = permeability



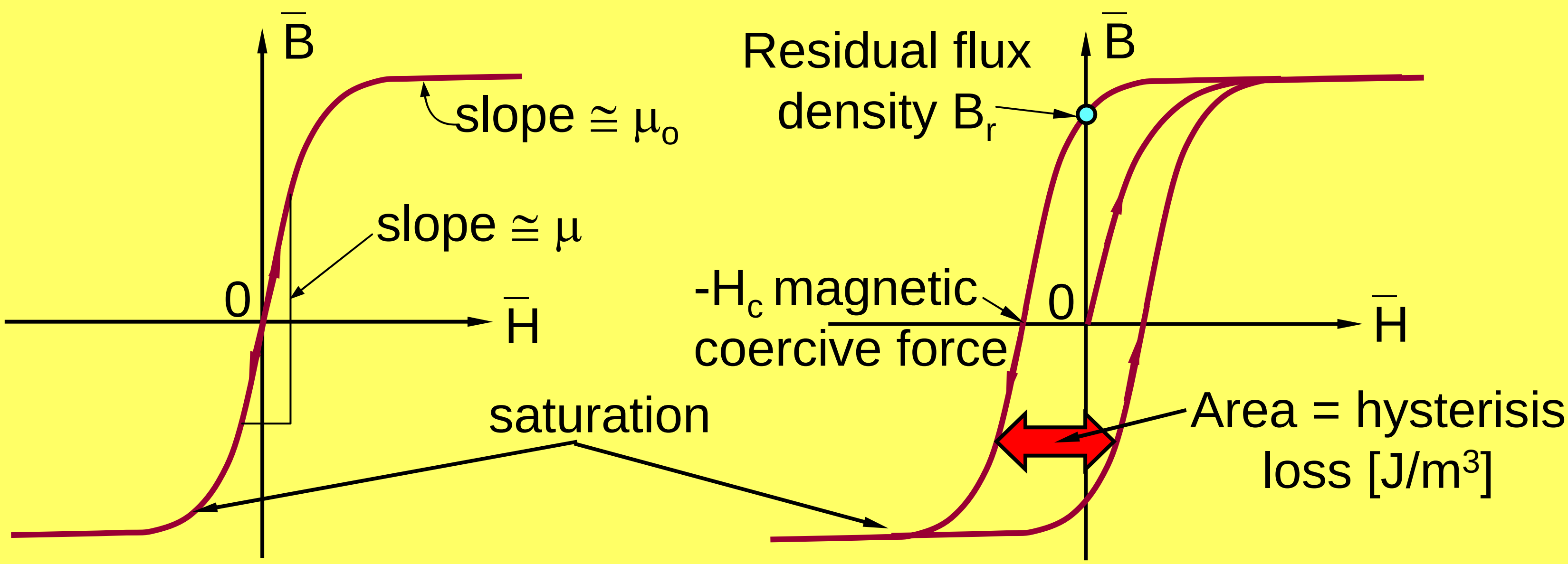




SATURATION AND HYSTERISIS



$\frac{1}{2}\bar{B} \bullet \bar{H} = W_m$ [J/m^3] Magnetic energy density



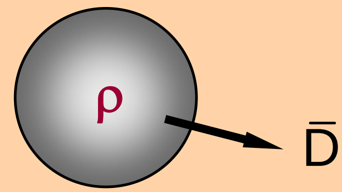
MEDIA PARAMETERS

Conductivity σ [Siemens/m]		Dielectric constant ($\varepsilon/\varepsilon_o$)		Relative permeabilities μ/μ_o	
Paraffin	$\sim 10^{-15}$	Vacuum	1.0	Bismuth	0.99983
Glass	10^{-12}	Wood (fir)	1.8-2.0	Silver	0.99998
Dry earth	10^{-4}-10^{-5}	Teflon, petroleum	2.1	Copper	0.999991
Distilled water	2×10^{-4}	Vaseline	2.2	Water	0.999991
Sea water	3-5	Paper	2-3	Vacuum	1.000000
Iron	10^{7}	Polystyrene	2.6	Air	1.0000004
Copper	5.8×10^{7}	Sandy soil	2.6	Aluminum	1.00002
Silver	6.1×10^{7}	Fused quartz	3.8	Cobalt	250
		Ice	4.15	Nickel	600
		Pyrex glass	5.1	Mild steel	2000
		Aluminum oxide	8.8	Iron	5000
		Ethyl alcohol	24.5	Mu metal	100,000
		Water	81	Supermalloy	1,000,000
		Titanium dioxide	100		

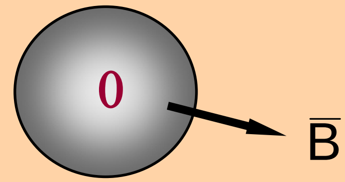
INTEGRAL MAXWELL'S EQUATIONS

Graphical Equations:

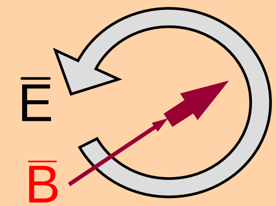
$$\oiint_S \overline{D} \bullet \hat{n}\,da = \iiint_V \rho \; dv$$ Gauss $\Leftrightarrow$



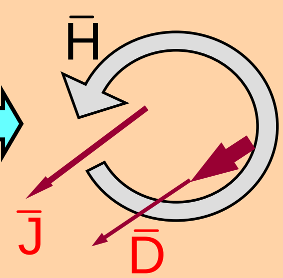
$$\oiint_S \overline{B} \bullet \hat{n}\,da = 0$$ Gauss $\Leftrightarrow$



$$\oint_c \overline{E} \bullet d\overline{s} = -\frac{\partial}{\partial t} \iint_A \overline{B} \bullet \hat{n}\,da$$ Faraday $\Leftrightarrow$



$$\oint_c \overline{H} \bullet d\overline{s} = \iint_A \overline{J} \bullet \hat{n}\,da + \frac{\partial}{\partial t} \iint_A \overline{D} \bullet \hat{n}\,da$$ Ampere $\Leftrightarrow$



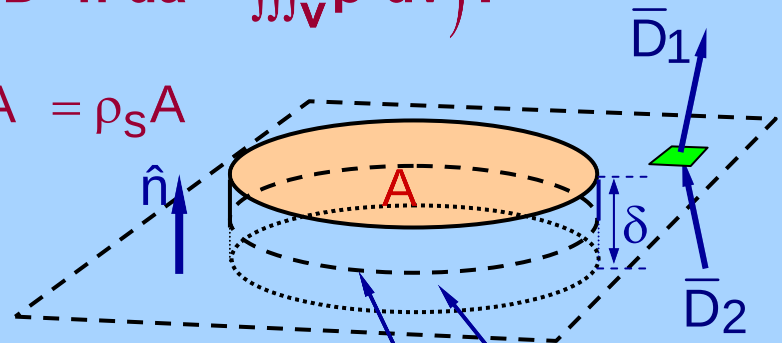
$\overline{D} = \varepsilon \overline{E}, \quad \overline{B} = \mu \overline{H}$ Constitutive relations

FIELDS PERPENDICULAR TO BOUNDARIES

Using Gauss's Law: $\left(\oiint_S \overline{D} \bullet \hat{n}\, da = \iiint_V \rho\, dv\right)$:

$$\oiint_S \overline{D} \bullet \hat{n} da \rightarrow \left(D_{1\perp} - D_{2\perp}\right) A = \rho_S A$$

(Lim $A \rightarrow 0$, $\delta^2 << A$)



Therefore:

$D_{1\perp} - D_{2\perp} = \rho_S$ yields: $\hat{n} \bullet \left(\overline{D}_1 - \overline{D}_2\right) = \rho_S$

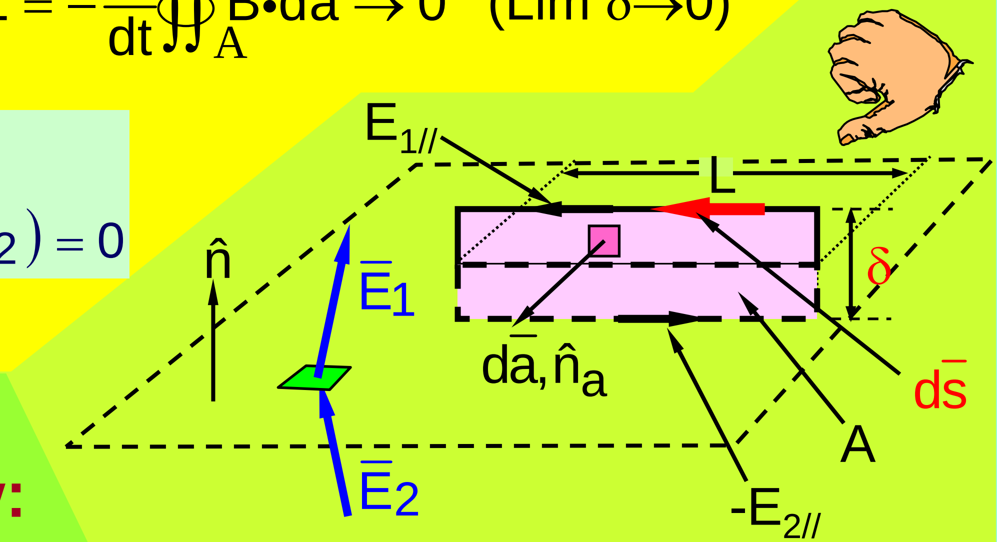
$\oiint_A \overline{B} \bullet \hat{n} da = \left(B_{1\perp} - B_{2\perp}\right) A = 0$ yields: $\hat{n} \bullet \left(\overline{B}_1 - \overline{B}_2\right) = 0$

FIELDS PARALLEL TO BOUNDARIES

Using Faraday's Law: $\oint_c \overline{\mathbf{E}} \bullet d\overline{\mathbf{s}} = -\frac{\partial}{\partial t}\iint_A \overline{\mathbf{B}} \bullet \hat{\mathbf{n}}\mathbf{da}$

$$\oint_C \overline{E} \bullet d\overline{s} \rightarrow \left(E_{1//} - E_{2//}\right)L = -\frac{d}{dt}\oiint_A \overline{B} \bullet d\overline{a} \rightarrow 0 \quad (\text{Lim } \delta \rightarrow 0)$$

Therefore: $\begin{cases} \overline{E}_{1//} = \overline{E}_{2//} \\ \hat{n} \times \left(\overline{E}_1 - \overline{E}_2\right) = 0 \end{cases}$



Using Ampere's Law:

$$\oint_c \overline{\mathbf{H}} \bullet d\overline{\mathbf{s}} = \iint_A \left(\overline{\mathbf{J}} + \partial\overline{\mathbf{D}}/\partial t\right) \bullet d\overline{\mathbf{a}}$$

$$\oint_c \overline{H} \bullet d\overline{s} \rightarrow \left(H_{1//} - H_{2//}\right)L = \underbrace{\oiint_A \overline{J} \bullet d\overline{a}}_{\rightarrow \left(\overline{J}_s \bullet \hat{n}_a\right)L} - \underbrace{\frac{\partial}{\partial t}\oiint_A \overline{D} \bullet d\overline{a}}_{\rightarrow 0}$$

Therefore: $\hat{n} \times \left(\overline{H}_1 - \overline{H}_2\right) = \overline{J}_s$

PERFECT CONDUCTORS

Electric Fields:

$$\{\text{if } \sigma\to\infty \text{ and } \overline{E}\neq 0\} \Rightarrow \{\overline{J}=\sigma\overline{E}\to\infty\} \Rightarrow \{\overline{H}\to\infty \text{ since}$$

$$\oint_c \overline{H}\bullet d\overline{s} = \iint_A (\overline{J}+\partial\overline{D}/\partial t)\bullet d\overline{a}\,\} \Rightarrow \{W_m = \mu H^2/2\ [J/m^3] \to\infty, \text{ and } w_m\to\infty\}$$

Therefore:
- $\overline{E} = 0$ inside perfect conductors
- $\rho = 0$ (since $\int_V \rho\, dv = \oint\!\!\oint_S \varepsilon\overline{E}\bullet\hat{n}\, da$

Magnetic Fields:

$$\{\text{If } \overline{E}=0 \text{ and } \frac{\partial}{\partial t}\iint_A \overline{B}\bullet\hat{n}da = -\oint_c \overline{E}\bullet d\overline{s}\} \Rightarrow \{\partial\overline{B}/\partial t = 0\}$$

Therefore:
- $\overline{H} = 0$ inside perfect conductors
- (if $\sigma = \infty$, and $H(t=0) = 0$)

Superconductors (Cooper pairs don't impact lattice):

$\overline{B} \cong 0$ inside because $\sigma = \infty$

Cooper pairs of electrons disassociate and superconductivity fails when the external B(T) is above a critical threshold

SUMMARY: BOUNDARY CONDITIONS

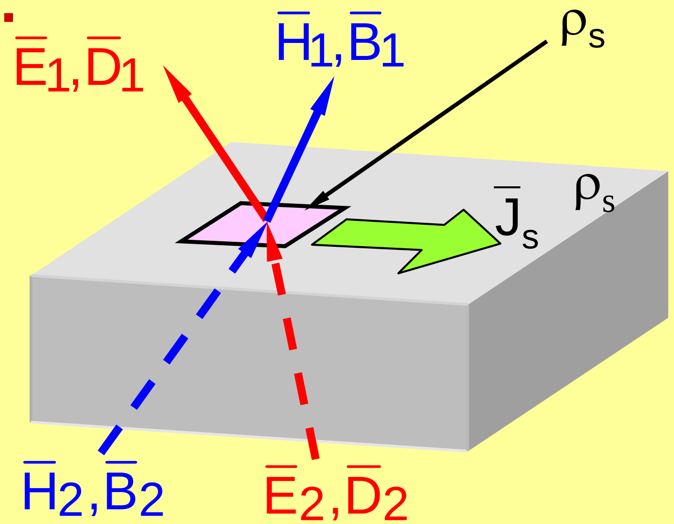
General Boundary Conditions:

$$\hat{n} \bullet (\overline{D}_1 - \overline{D}_2) = \rho_s$$

$$\hat{n} \bullet (\overline{B}_1 - \overline{B}_2) = 0$$

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = 0$$

$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s$$



Inside Perfect Conductors:

$$\overline{D}_2 = \overline{B}_2 = \overline{E}_2 = 0$$

$$\hat{n} \bullet \overline{D}_1 = \rho_s$$

$$\hat{n} \bullet \overline{B}_1 = 0$$ $\Rightarrow$ $\overline{B}$ is parallel to perfect conductors

$$\hat{n} \times \overline{E}_1 = 0$$ $\Rightarrow$ $\overline{E}$ is perpendicular to perfect conductors

$$\hat{n} \times \overline{H}_1 = \overline{J}_s$$