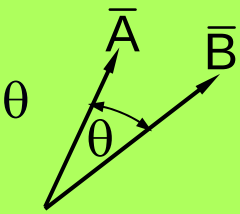
6.013 Electromagnetics and Applications
Spring 2009

VECTOR OPERATORS ∇, $\times$, $\bullet$

Vector: $$\overline{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$$

Vector Dot Product: $$\overline{A} \bullet \overline{B} = A_xB_x + A_yB_y + A_zB_z = |\overline{A}||\overline{B}|\cos\theta$$

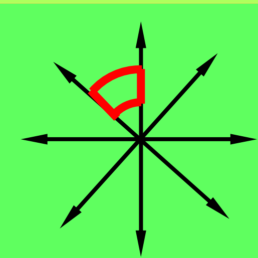
Vector Cross Product: $$\overline{A} \times \overline{B} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = |\overline{A}||\overline{B}|\sin\theta$$

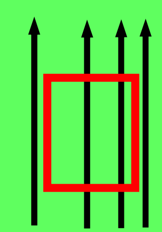


$$= \hat{x}\left(A_yB_z - A_zB_y\right) + \hat{y}\left(A_zB_x - A_xB_z\right) + \hat{z}\left(A_xB_y - A_yB_x\right)$$

"Del" (∇) Operator: $$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

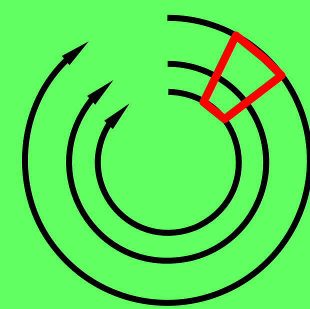
Gradient of ϕ: $$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$





Divergence of $\overline{A}$: $$\nabla \bullet \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

"Curl of $\overline{A}$": $$\nabla \times \overline{A} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$



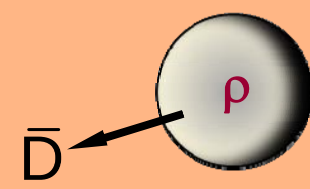
PHYSICAL SIGNIFICANCE OF $\nabla\bullet$, $\nabla\times$

$\nabla\bullet\overline{D}$ is the "divergence of the vector field $\overline{D}$"

Gauss's divergence theorem: $\int_V (\nabla\bullet\overline{A})dv = \oiint_S (\overline{A}\bullet\hat{n})da$

Gauss's Law, Differential Form: $\nabla\bullet\overline{D} = \rho$

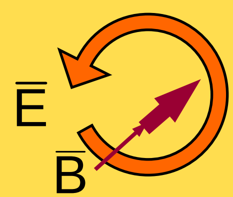
$$\int_V (\nabla\bullet\overline{D})dv = \boxed{\oiint_S (\overline{D}\bullet\hat{n})da = \int_V \rho\, dv}$$



$\nabla\times\overline{E}$ is the "curl of the vector field $\overline{E}$"

Stokes's theorem: $\oint_c \overline{E}\bullet d\overline{s} = \iint_A (\nabla\times\overline{E})\bullet\hat{n}\, da$

Faraday's Law, Differential Form: $\nabla\times\overline{E} = -\dfrac{\partial\overline{B}}{\partial t}$

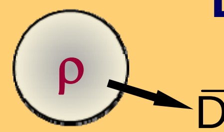


$$\oint_c \overline{E}\bullet d\overline{s} = \iint_A (\nabla\times\overline{E})\bullet\hat{n}\, da = \boxed{-\iint_A \frac{\partial\overline{B}}{\partial t}\bullet\hat{n}\, da = \oint_c \overline{E}\bullet d\overline{s}}$$

MAXWELL'S EQUATIONS

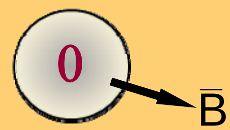
Integral Form: / **Differential Form:**

$$\oiint_S \bar{D}\bullet\hat{n}da = \iiint_V \rho\, dv \quad \Leftrightarrow \quad \nabla\bullet\bar{D} = \rho$$

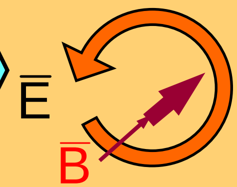


$$\bar{D} = \varepsilon\bar{E},\ \bar{B} = \mu\bar{H}$$

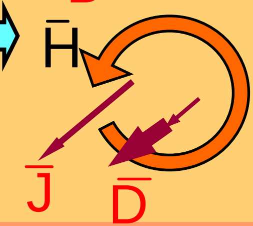
$$\oiint_S \bar{B}\bullet\hat{n}da = 0 \quad \Leftrightarrow \quad \nabla\bullet\bar{B} = 0$$



$$\oint_c \bar{E}\bullet d\bar{s} = -\cancelto{0}{\frac{\partial}{\partial t}}\iint_A \bar{B}\bullet\hat{n}da \quad \Leftrightarrow \quad \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$$



$$\oint_c \bar{H}\bullet d\bar{s} = \iint_A \bar{J}\bullet\hat{n}da + \cancelto{0}{\frac{\partial}{\partial t}}\iint_A \bar{D}\bullet\hat{n}da \quad \Leftrightarrow \quad \nabla\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t}$$



$\bar{E}$	Electric field	[volts/meter, V m^{-1}]
$\bar{H}$	Magnetic field	[amperes/meter, A m^{-1}]
$\bar{B}$	Magnetic flux density	[Tesla, T]
$\bar{D}$	Electric displacement	[ampere sec/m^2, A s m^{-2}]
$\bar{J}$	Electric current density	[amperes/m^2, A m^{-2}]
ρ	Electric charge density	[coulombs/m^3, C m^{-3}]

MAXWELL'S EQUATIONS: VACUUM SOLUTION

Faraday's Law: $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$

Ampere's Law: $\nabla \times \overline{H} = \cancel{\overline{J}} + \frac{\partial \overline{D}}{\partial t}$ ($\overline{J} \to 0$)

Gauss's Law

$\nabla \bullet \overline{D} = \cancel{\rho}$ ($\rho \to 0$)

$\nabla \bullet \overline{B} = 0$

Constitutive Relations

$\overline{D} = \varepsilon_o \overline{E}$

$\overline{B} = \mu_o \overline{H}$

EM Wave Equation:

Eliminate $\overline{H}$: $\nabla \times (\nabla \times \overline{E}) = -\mu_o \frac{\partial}{\partial t}(\nabla \times \overline{H})$

Use identity: $\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \bullet \overline{A}) - \nabla^2 \overline{A}$

Yields: $\underbrace{\nabla(\nabla \bullet \overline{E})}_{0} - \nabla^2 \overline{E} = -\mu_o \frac{\partial}{\partial t}(\nabla \times \overline{H}) = -\mu_o \varepsilon_o \frac{\partial^2 \overline{E}}{\partial t^2}$

EM Wave Equation[1] $\boxed{\nabla^2 \overline{E} - \mu_o \varepsilon_o \frac{\partial^2 \overline{E}}{\partial t^2} = 0}$

Second derivative in space $\propto$ second derivative in time, therefore solution is any f(r,t) with identical dependencies on r,t

[1]**Laplacian Operator:** $\nabla \bullet (\nabla \phi) = \nabla^2 \phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\phi$

WAVE EQUATION SOLUTION

Many are possible $\Rightarrow$ Try Uniform Plane Wave (UPW), $\neq f(x,y)$

Example: Try: $E = \hat{y}\, E_y(z)$ in $\nabla^2 \overline{E} - \mu_o \varepsilon_o \dfrac{\partial^2 \overline{E}}{\partial t^2} = 0$

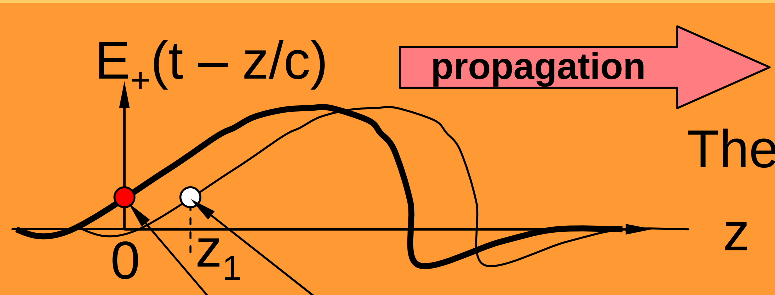
$$\Rightarrow \nabla^2 E_y = \Big(\underbrace{\cancel{\frac{\partial^2}{\partial x^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2}{\partial y^2}}}_{0} + \frac{\partial^2}{\partial z^2}\Big) E_y$$

Yields: $\dfrac{\partial^2 E_y}{\partial z^2} - \mu_o \varepsilon_o \dfrac{\partial^2 E_y}{\partial t^2} = 0$

Trial solution: $E_y(z,t) = E_+(t - z/c)$; $E_+(\text{arg})$ = arb. function of (arg)

Test solution: $c^{-2}\, E''_+(t - z/c) - \mu_o \varepsilon_o E''_+(t - z/c) = 0$ iff:

$c = \dfrac{1}{\sqrt{\mu_o \varepsilon_o}} \approx 3\times 10^8$ [m s^{-1}] in vacuum (velocity of light)



The position • where arg = 0 moves at velocity c

$z = t = 0 \Rightarrow \text{arg} = 0$ arg = 0 at $t_1 = z_1/c$

UNIFORM PLANE WAVE IN Z-DIRECTION

Example: $E_y(z,t) = \underbrace{E_+}(\underbrace{t - z/c})$ [V/m]

$\text{Func(arg)} = \text{Func*}[(-c)(\text{arg})] = \text{Func*}(z - ct)$

E.G.: $E_y(z,t) = E_+ \cos[\omega(t - z/c)] = E_+ \cos(\omega t - kz)$,

where $k = \omega/c = \omega\sqrt{\mu_o \varepsilon_o}$

To find magnetic fields:

Faraday's Law: $\nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t} \quad \Rightarrow \overline{H} = -\int(\nabla \times \overline{E})\mu_o^{-1}\, dt$

$$\nabla \times \overline{E} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cancel{\partial/\partial x}_{\,0} & \cancel{\partial/\partial y}_{\,0} & \partial/\partial z \\ \cancel{E_x}_{\,0} & E_y & \cancel{E_z}_{\,0} \end{vmatrix} = -\hat{x}\partial E_+ \cos(\omega t - kz)/\partial z$$

$$= -\hat{x}\, kE_+ \sin(\omega t - kz)$$

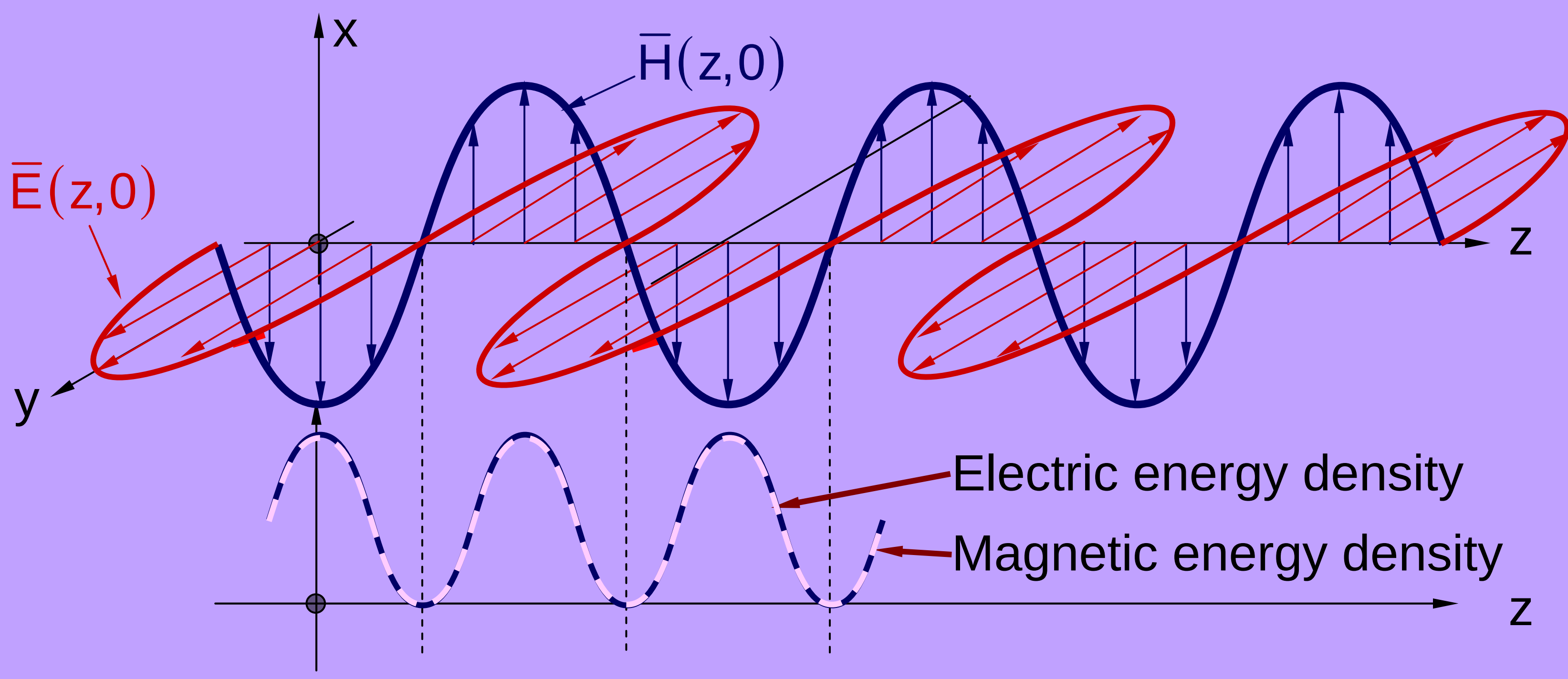
$$\overline{H} = \hat{x}\int(k/\mu_o)E_+ \sin(\omega t - kz)\, dt = -\hat{x}(E_+/\eta_o)\cos(\omega t - kz)$$

$$k = \omega\sqrt{\mu_o \varepsilon_o}\ , \qquad \eta_o = \sqrt{\mu_o / \varepsilon_o}$$

UNIFORM PLANE WAVE: EM FIELDS

EM Wave in z direction:

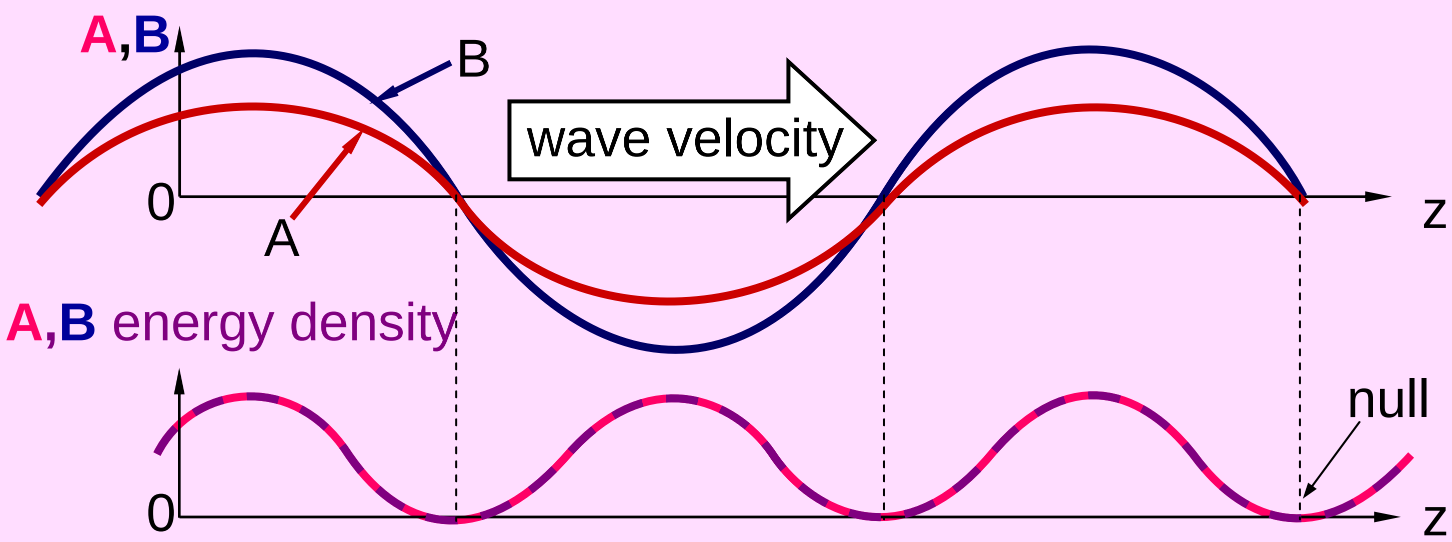
$$\bar{E}(z,t) = \hat{y}E_+\cos(\omega t - kz)\ ,\ \ \bar{H}(z,t) = -\hat{x}\left(E_+/\eta_o\right)\cos(\omega t - kz)$$



Linearity implies superposition of n→∞ waves, all θ,φ

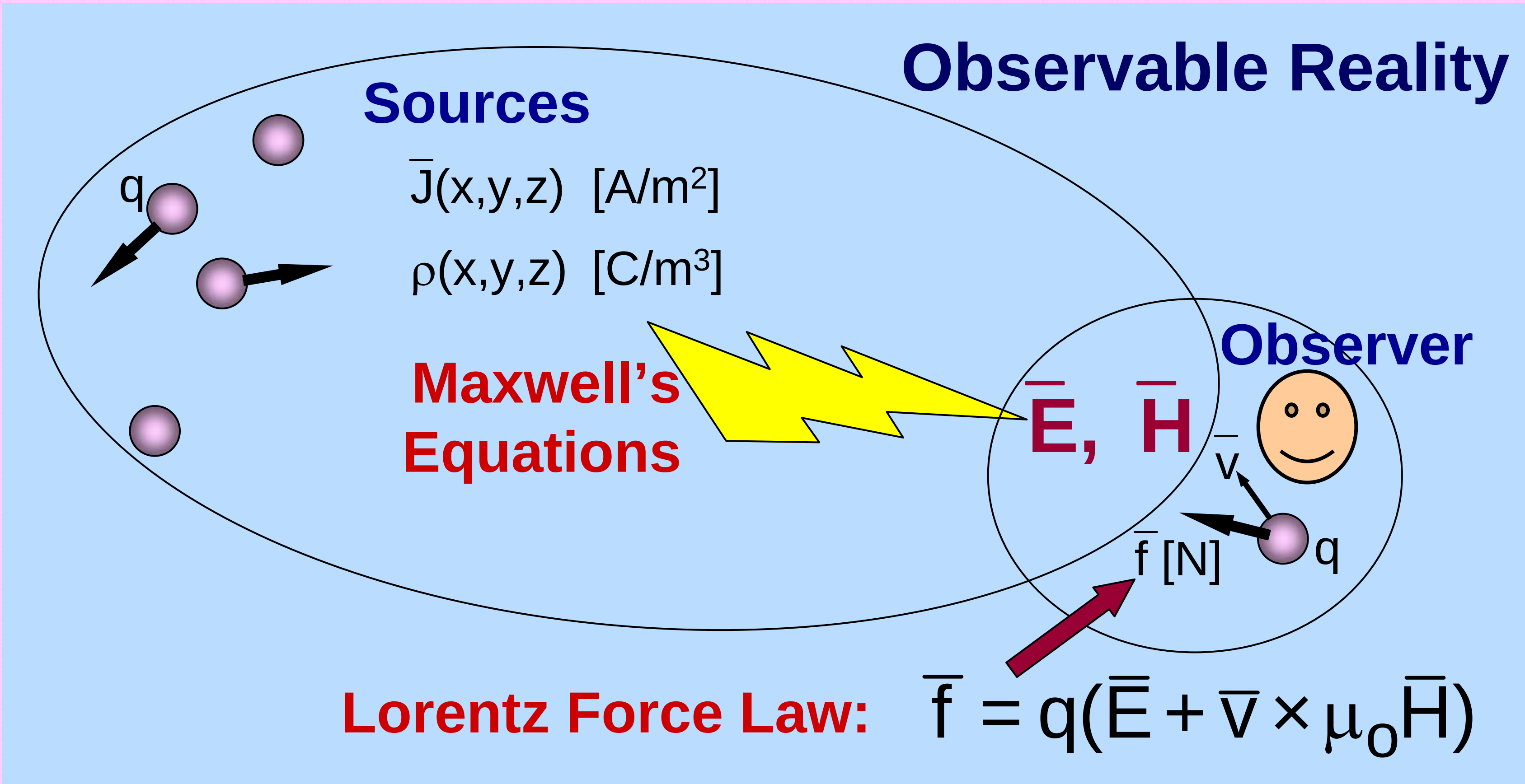
ELECTROMAGNETIC AND OTHER WAVES

A "wave" is a fixed disturbance propagating through a medium



Medium	**A**	**B**	**A energy**	**B energy**
String	stretch	velocity	potential	kinetic
Acoustic	pressure	velocity	potential	kinetic
Ocean	height	velocity	potential	kinetic
Electromagnetic	**H**	**E**	magnetic	electric

Role of Maxwell's Equations and Fields



The fields $\bar{E}$, $\bar{H}$ and the displacement and flux densities $\bar{D}$, $\bar{B}$ permit division of electromagnetics into the Maxwell and Lorentz equations