6.013 Electromagnetics and Applications
Spring 2009

REVIEW OF UPW BASICS

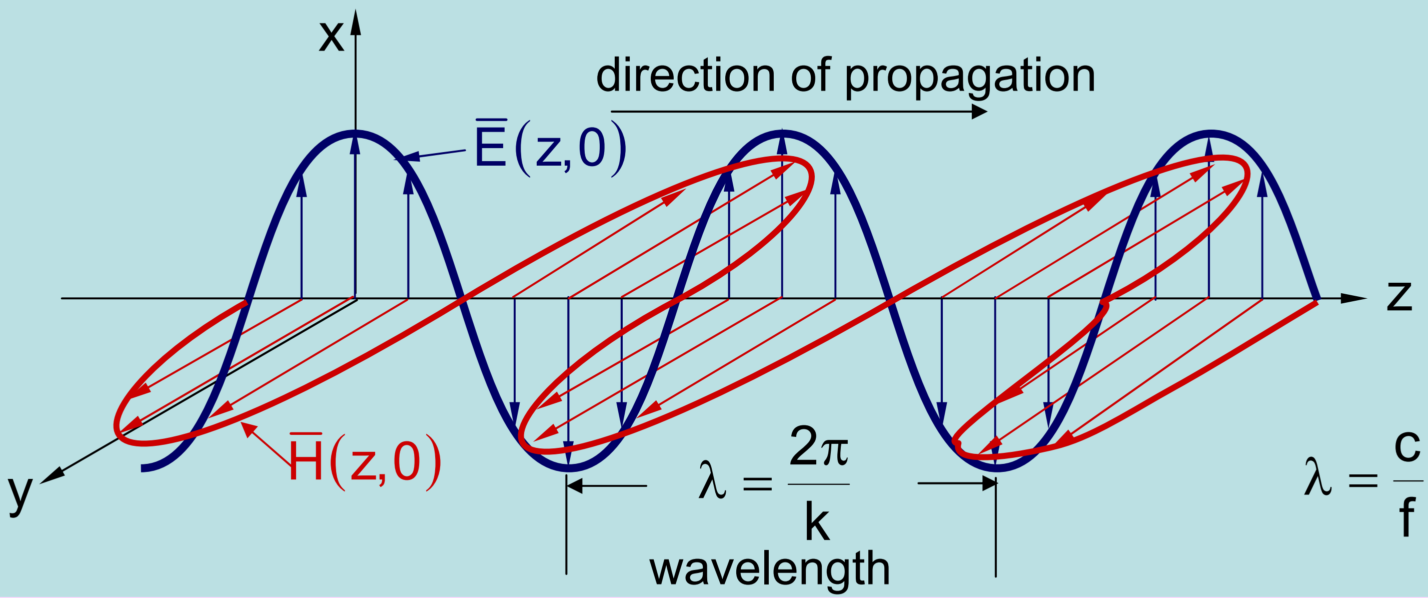
Example: $\hat{x}$-polarized UPW traveling in $+\hat{z}$ direction

$$\overline{E} = \hat{x}E_o\cos(\omega t - kz) \qquad \underline{\overline{E}}(z) = \hat{x}\underline{E}_o e^{-jkz}$$

$$\overline{H} = \hat{y}\frac{E_o}{\eta_o}\cos(\omega t - kz) \qquad \underline{\overline{H}}(z) = \hat{y}\frac{\underline{E}_o}{\eta_o}e^{-jkz}$$

$\overline{E}\times\overline{H}$: $\hat{z}$ direction of propagation

$$c = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = 3\text{x}10^8\,\text{m/s} \qquad \omega(\text{rads/s}) = 2\pi f \qquad k(\text{rads/m}) = \frac{2\pi}{\lambda} = \frac{\omega}{c} \qquad \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_0}}$$



HOW DO WAVES CONVEY POWER, ENERGY?

Recall: $\bar{E}\ [V/m] \bullet \bar{J}\ [A/m^2] = P_d\ [W/m^3]$ But $\bar{E} \perp \bar{H}$

Manipulate Ampere's law to get $\bar{E} \bullet \bar{J}$

$\bar{E} \cdot (\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t})$ For symmetry, compute $\bar{H} \cdot (\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t})$

$$\underbrace{\bar{H} \cdot \left(\nabla \times \bar{E}\right) - \bar{E} \cdot \left(\nabla \times \bar{H}\right)}_{\text{Vector Identity}} = -\bar{H} \cdot \left(\frac{\partial \bar{B}}{\partial t}\right) - \bar{E} \cdot \left(\bar{J} + \cdot \frac{\partial \bar{D}}{\partial t}\right)$$

$$\boxed{\nabla \cdot \left(\bar{E} \times \bar{H}\right) = -\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} - \bar{E} \cdot \bar{J} - \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} \quad (W/m^3)}$$

This is **Poynting's Theorem** What does it mean?

POYNTING THEOREM

Poynting's Theorem: $\nabla \cdot (\bar{E} \times \bar{H}) = -\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} - \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} - \bar{E} \cdot \bar{J}$

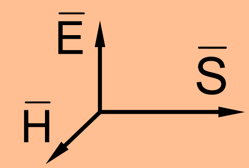
$\bar{B} = \mu \bar{H} \qquad \bar{D} = \varepsilon \bar{E}$

$$\underbrace{\nabla \cdot (\bar{E} \times \bar{H})}_{\substack{\text{Poynting vector,} \\ \bar{S}\ [\text{W/m}^2]}} = -\frac{d}{dt}\underbrace{\left(\frac{1}{2}\mu|\bar{H}|^2\right)}_{\substack{\text{Stored magnetic energy density,} \\ W_m}} - \frac{d}{dt}\underbrace{\left(\frac{1}{2}\varepsilon|\bar{E}|^2\right)}_{\substack{\text{Stored electric energy density,} \\ W_e}} - \underbrace{(\bar{E} \cdot \bar{J})}_{\substack{\text{Power density dissipated/m}^3, \\ W_d}} \ (\text{W/m}^3)$$

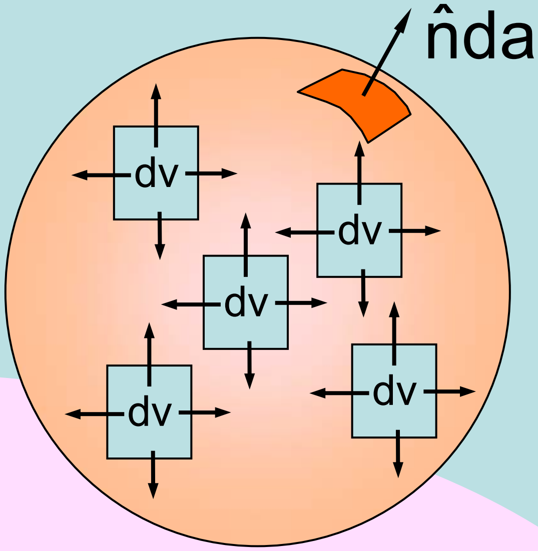
Energy is conserved!

Note: $\frac{d}{dt}\left(\frac{1}{2}\mu|\bar{H}|^2\right) = \mu|\bar{H}|\frac{d|\bar{H}|}{dt} = \bar{H} \bullet \frac{d\bar{B}}{dt}$

<u>Poynting Vector</u> $\bar{S} = \bar{E} \times \bar{H}$ (W/m^2) $\left(\frac{\text{volts}}{\text{m}} \cdot \frac{\text{amps}}{\text{m}} = \frac{\text{watts}}{\text{m}^2}\right)$



INTEGRAL POYNTING THEOREM



Use: $\oint_S \bar{A}\cdot\hat{n}\,da = \int_V \nabla\cdot\bar{A}\,dv$

Gauss's Theorem (not Gauss's Law)

Therefore:

$$\oint_S (\bar{E}\times\bar{H})\cdot\hat{n}\,da = \int_V \nabla\cdot(\bar{E}\times\bar{H})\,dv$$

$$= \int_V \left[-\frac{d}{dt}\left(\frac{1}{2}\mu|\bar{H}|^2\right) - \frac{d}{dt}\left(\frac{1}{2}\varepsilon|\bar{E}|^2\right) - (\bar{E}\cdot\bar{J}) \right] dv$$

$$\oint_S (\bar{E}\times\bar{H})\cdot\hat{n}\,da = -\int_V \frac{d}{dt}\left(\frac{1}{2}\varepsilon|\bar{E}|^2 + \frac{1}{2}\mu|\bar{H}|^2\right) dv - \int_V \bar{E}\cdot\bar{J}\,dv$$

Power emerging = released stored energy - dissipation [W]

The Poynting vector $\triangleq$ $\bar{S} = \bar{E}\times\bar{H}$ gives both the magnitude of the power density (intensity) and the direction of its flow.

UNIFORM PLANE WAVE EXAMPLE

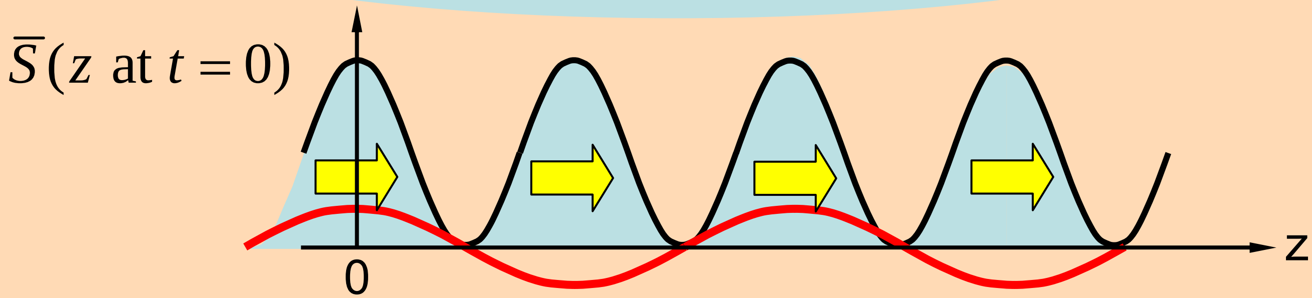
$\bar{E} = \hat{x} E_o \cos(\omega t - kz)$

$W_e = \frac{1}{2} \varepsilon_o E_o^2 \cos^2(\omega t - kz)$

$\bar{H} = \hat{y}\left(\frac{E_o}{\eta_o}\right)\cos(\omega t - kz)$

$W_m = \frac{1}{2} \frac{\mu_o}{\eta_o^2} E_o^2 \cos^2(\omega t - kz)$

$$\bar{S}(t) = \bar{E} \times \bar{H} = \hat{z}\left(\frac{E_o^2}{\eta_o}\right)\cos^2(\omega t - kz) \quad (W/m^2)$$



$$\bar{S} = \hat{z}\frac{E_o^2}{\eta_0}\cos^2(\omega t - kz) \Rightarrow \langle \bar{S} \rangle = \hat{z}\frac{1}{2}\frac{E_o^2}{\eta_o} = I(\theta, \phi, r) \ \ [w/m^2]$$

The time average $\langle \bar{S}(r, \theta, \phi) \rangle$ is "intensity" [W/m^2]

COMPLEX NOTATION – POYNTING VECTOR

Defining a meaningful $\underline{\overline{S}}$ and relating it to $\overline{S}$ is not obvious.
Let's work backwards to find the time average $\langle\overline{S}\rangle$ and then $\underline{\overline{S}}$

$$\overline{S}(t) = \overline{E} \times \overline{H} = \mathrm{Re}\left[\underline{\overline{E}} \cdot e^{j\omega t}\right] \times \mathrm{Re}\left[\underline{\overline{H}} \cdot e^{j\omega t}\right]$$

$$= [\overline{E}_r \cos(\omega t) - \overline{E}_i \sin(\omega t)] \times [\overline{H}_r \cos(\omega t) - \overline{H}_i \sin(\omega t)]$$

$$\Rightarrow \langle\overline{S}(t)\rangle = \frac{1}{2}\left[\left(\overline{E}_r \times \overline{H}_r\right) + \left(\overline{E}_i \times \overline{H}_i\right)\right]$$

$$= \frac{1}{2} R_e \left(\underbrace{\underline{\overline{E}} \times \underline{\overline{H}}^*}_{\underline{\overline{S}}}\right) \quad \left[= \frac{1}{2} R_e \{(E_r + jE_i) \times (H_r - jH_i)\}\right]$$

$\underline{\overline{S}}$ (by definition

Thus, we can define $\langle\overline{S}\rangle = \frac{1}{2}\mathrm{Re}\left(\underline{\overline{E}} \times \underline{\overline{H}}^*\right)$ and $\underline{\overline{S}} = \underline{\overline{E}} \times \underline{\overline{H}}^*$

Recall: $\underline{\overline{E}} = \overline{E}_r + j\overline{E}_i \qquad \underline{\overline{H}} = \overline{H}_r + j\overline{H}_i \qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$

UPW REFLECTED BY PERFECT CONDUCTOR

$$\bar{E} = \hat{x}E_+ \cos(\omega t - kz) + \hat{x}E_- \cos(\omega t + kz)$$
$$= 0 \text{ at } z = 0 \text{ (perfect conductor)}$$

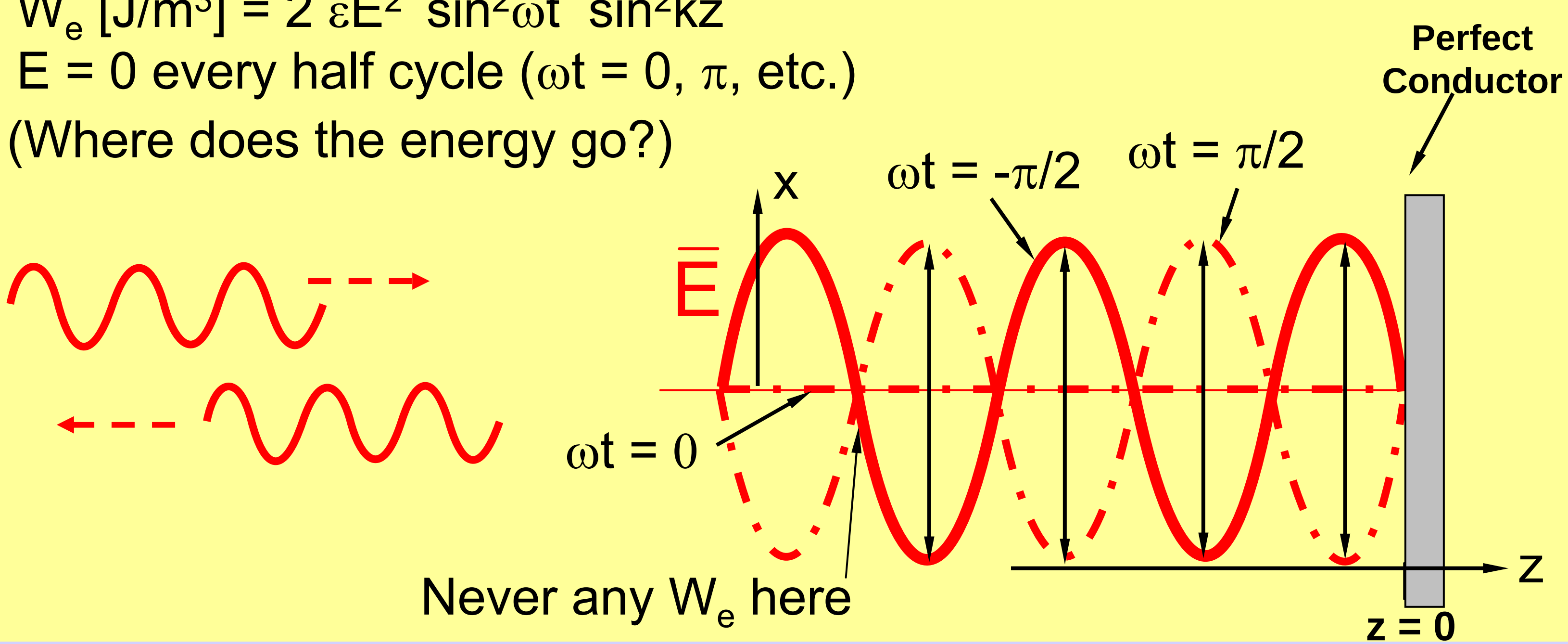
Forward plus reflected wave

$\Rightarrow E_- = -E_+$ (Solving for unknown reflection)

$\Rightarrow \bar{E} = \hat{x}2E_+ \sin\omega t \cdot \sin kz$ Standing waves, oscillate without moving

(recall: $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2} \bullet \sin\frac{\alpha-\beta}{2}$)

W_e [J/m^3] = $2\,\varepsilon E^2 \sin^2\omega t \sin^2 kz$
E = 0 every half cycle ($\omega t = 0, \pi$, etc.)
(Where does the energy go?)



STANDING WAVE EXAMPLE - CONTINUED

(It's in the H field!) $\bar{E} = \hat{x}\,[E_+ \cos(\omega t - kz) + E_- \cos(\omega t + kz)]$

$$\bar{H} = \hat{y}\,[\frac{E_+}{\eta_o}\cos(\omega t - kz) - \frac{E_-}{\eta_o} y \cos(\omega t + kz)]$$

Note

$$E_- = -E_+ \Rightarrow \quad \bar{H} = \hat{y}\frac{2E_+}{\eta_o}\cos\omega t \cdot \cos kz$$

$$W_m[J/m^3] = \frac{1}{2}\mu_o\left(\frac{2E_+}{\eta_o}\right)^2 \cos^2\omega t \cos^2 kz = 2\varepsilon E_+^2 \cos^2\omega t \cos^2 kz$$

$= 0$ when $\omega t = \pi/2, 3\pi/2$, etc.)

$$\bar{S} = \bar{E}\times\bar{H} = \hat{z}\;4\frac{E_+^2}{\eta_o}\cos\omega t \sin\omega t \cdot \cos kz \sin kz$$

$$= \hat{z}\frac{E_+^2}{\eta_o}\cdot \sin 2kz \cdot \sin 2\omega t$$

$$\Rightarrow \langle \bar{S}\rangle = 0 = \frac{1}{2}\mathrm{Re}\left(\underline{\bar{E}}\times\underline{\bar{H}}^*\right)$$