6.013 Electromagnetics and Applications
Spring 2009

ELECTRIC FORCES ON CHARGES

Lorentz Force Law:

$$\overline{f} = q\left(\overline{E} + \overline{v} \times \mu_o \overline{H}\right) \text{ Newtons} \quad \Rightarrow \quad \overline{f} = q\overline{E} = m\overline{a}$$

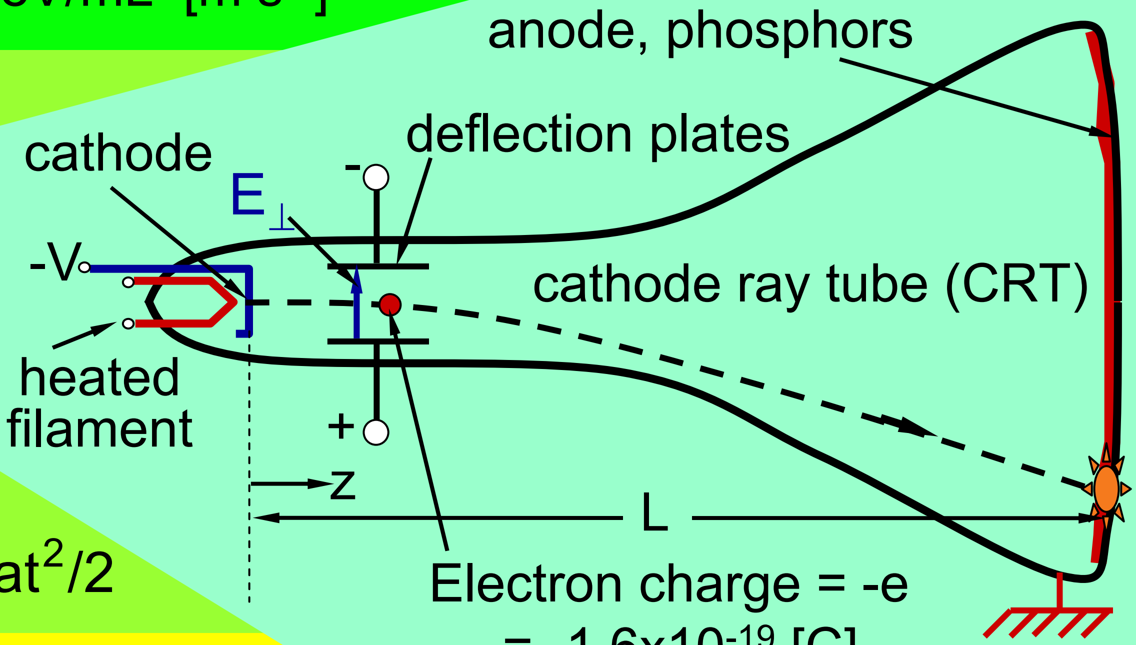
$$a = f/m = qE/m \approx eV/mL \quad [\text{m s}^{-2}]$$

Kinematics*:

$$\overline{v} = \int_0^t \overline{a}(t)dt$$

$$= \overline{v}_o + \hat{z}at$$

$$z = z_o + \hat{z} \bullet \overline{v}_o t + at^2/2$$



Electron charge = -e
= -1.6×10^{-19} [C]

Electron kinetic energy w_k:

$w_k = fs = (eV/L)L = eV$ [Joules]

Electron volt = energy of 1 electron moving 1 volt = e Joules

* For $\overline{E}$ in -z direction

ELECTRIC FORCES
ON CHARGED CONDUCTORS

Force on free charges:

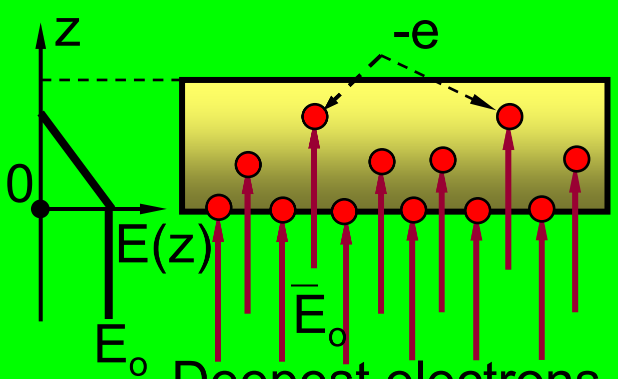
Force: $\bar{f} = q\bar{E} = -e\bar{E}$ [N]

Electric pressure: $\bar{P}_e = \rho_s <\bar{E}>$ $[\mathrm{N\ m^{-2}}]$

Surface charge: $\rho_s = -\varepsilon_o E_o$ $[\mathrm{C\ m^{-2}}]$

Average E: $< E> = \dfrac{E_o}{2}$ $[\mathrm{V\ m^{-1}}]$

Electric pressure: $P_e = -\dfrac{1}{2}\varepsilon_o E_o^2$ $[\mathrm{N\ m^{-2}}]$



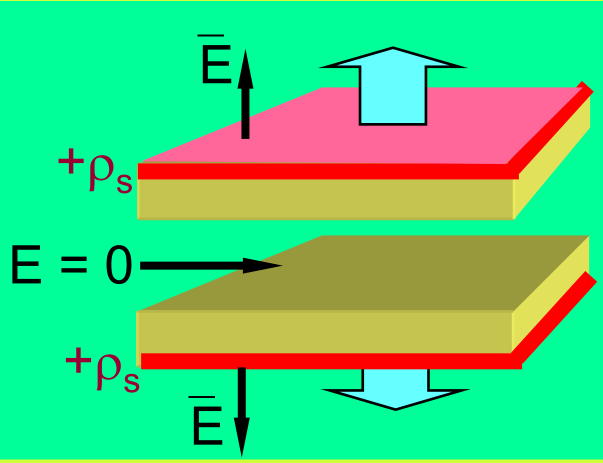
Deepest electrons experience zero force

Attractive pressure

Repulsive forces:

Like charges repel,

unlike charges attract



ENERGY METHOD
FOR FINDING FORCES

Force, work, and energy:

$$dw = f\,ds \quad \Rightarrow \quad f = \frac{dw}{ds} \quad [\text{N}]$$

$$C = \varepsilon_o A/s$$

$$w = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2 s}{\varepsilon_o A} \quad [\text{J}]$$

Q ≠ f(s) if C is open circuit $\bar{f}$

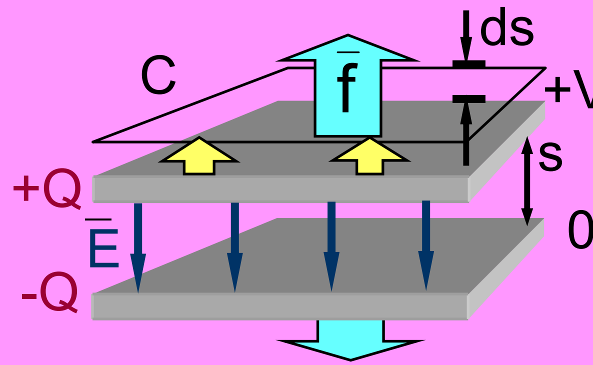
$$f = \frac{dw}{ds} = \frac{1}{2}\frac{Q^2}{\varepsilon_o A}$$

$$= \frac{(\varepsilon_o E A)^2}{2\varepsilon_o A} = \left(\frac{1}{2}\varepsilon_o E^2\right)A$$

$$= -P_e A \quad [\text{N}]$$

$$\Rightarrow P_e = \frac{1}{2}\varepsilon_o E^2 \quad [\text{N m}^{-2}] = [\text{J m}^{-3}]$$

$\bar{f}$ is the force externally applied to the upper plate



The static pressure P_e of the electric field on the upper plate is the same with a battery attached:

$$\bar{P}_e = \rho_s \langle \bar{E} \rangle = \frac{1}{2}\rho_s \bar{E}_o$$

Electric fields always pull on conductors → attractive force

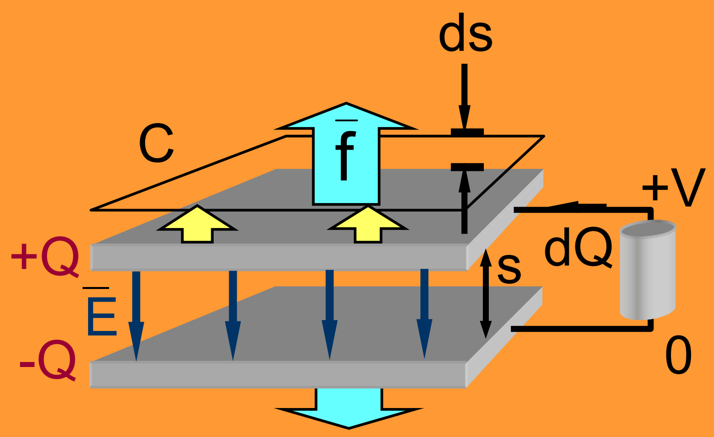
ENERGY METHOD WITH A BATTERY

Incremental work dw:

$$w = \frac{CV^2}{2}$$

$$dw = f\,ds = \underbrace{-V\,dQ}_{\text{To battery}} + \underbrace{(V^2/2)\,dC}_{\text{To capacitor}}$$

$$dQ = VdC \;\Rightarrow\; dw = -(V^2/2)dC$$



Externally applied force f

Force and pressure:

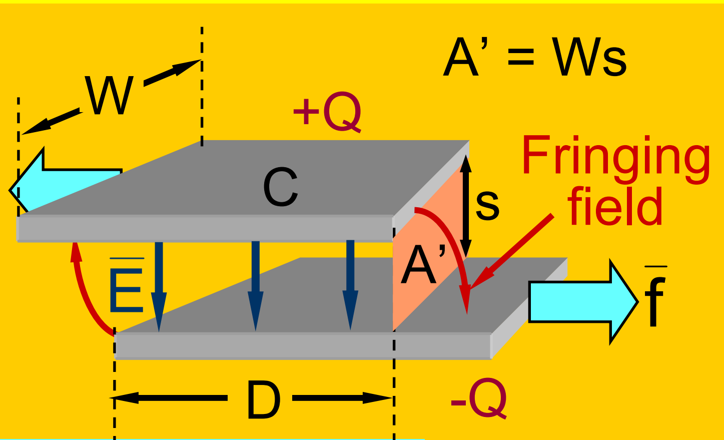
$$f = \frac{dw}{ds} = -\frac{V^2}{2}\frac{dC^*}{ds} = \frac{V^2}{2}\frac{\varepsilon_o A}{s^2} = \frac{\varepsilon_o E^2}{2}A = -P_e A$$

$\boxed{P_e = -\varepsilon_o E^2/2}$ (as before) Q.E.D.

*$C = \varepsilon A/s$

LATERAL FORCES – ENERGY METHOD

Energy derivative:

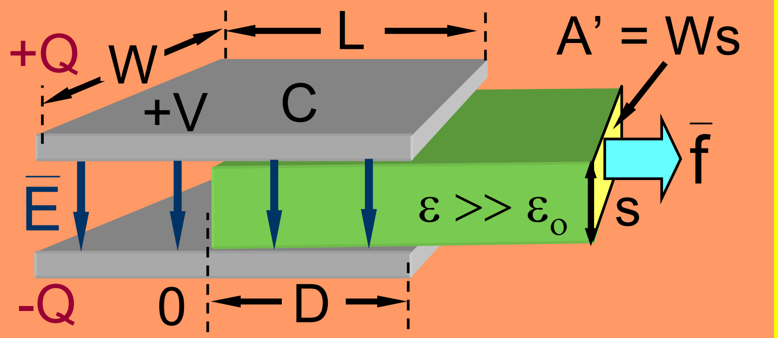


$$f = -\frac{dw}{dD} \quad \text{(externally applied)}$$

$$w = \frac{Q^2}{2C} = \frac{Q^2 s}{2\varepsilon_o WD}$$

$$f = \frac{Q^2 s}{2\varepsilon_o WD^2} = \frac{(\varepsilon_o EWD)^2 s}{2\varepsilon_o WD^2} = \boxed{(\frac{1}{2}\varepsilon_o E^2)Ws = P_e A' \quad [N]}$$

Energy derivative:



$$f = -\frac{dw}{dD} = -\frac{d(\frac{Q^2}{2C})}{dD}$$

$$C = C_D + C_o = \frac{\varepsilon DW}{s} + \frac{\varepsilon_o (L\text{-}D)W}{s}$$

$$= [D(\varepsilon - \varepsilon_o) + \varepsilon_o L]\frac{W}{s}$$

$$\boxed{f = \frac{(\varepsilon - \varepsilon_o)E^2}{2} A' \ [N] \ = \Delta P_e A';} \ A' = Ws \quad (E_{\parallel} \text{ pushes, } E_{\perp} \text{ pulls})$$

ROTARY ELECTROSTATIC MOTORS

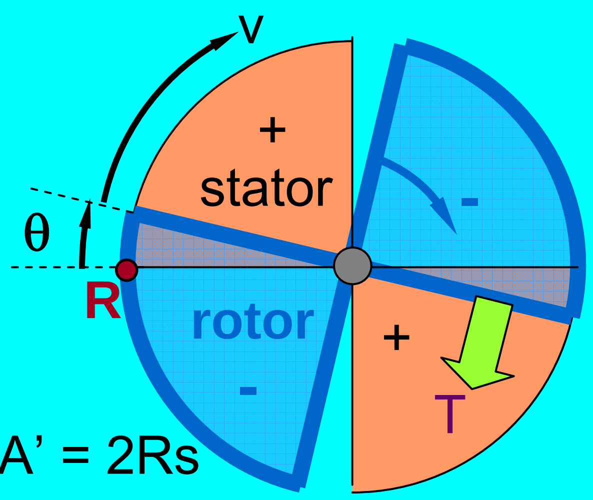
Energy derivative:

Torque: $$T[\text{Nm}] = -\frac{dw}{d\theta} = -\frac{d\left(\frac{Q^2}{2C}\right)}{d\theta}$$

$$= \frac{\varepsilon_o A}{s}, \quad A = 2\frac{R^2\theta}{2}$$

Therefore: $$T \overset{C}{\cong} \frac{Q^2 s}{2\varepsilon_o R^2 \theta^2} = \frac{\varepsilon_o E^2}{2} A' \frac{R}{2} \quad [\text{Nm}]$$

$$\cong \text{pressure} \times \text{gap-area } A' \times \frac{R}{2}$$



A’ = 2Rs

Motor power:

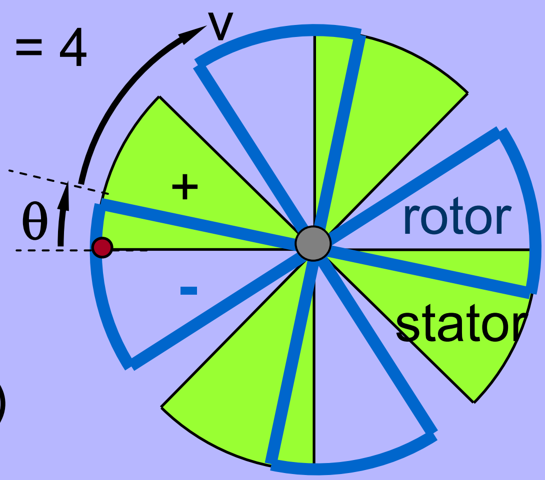
Peak power: $P = T\omega$ [W]

Average power: $P_{avg} = P/2$ (duty cycle = ½)

Segmentation advantage:

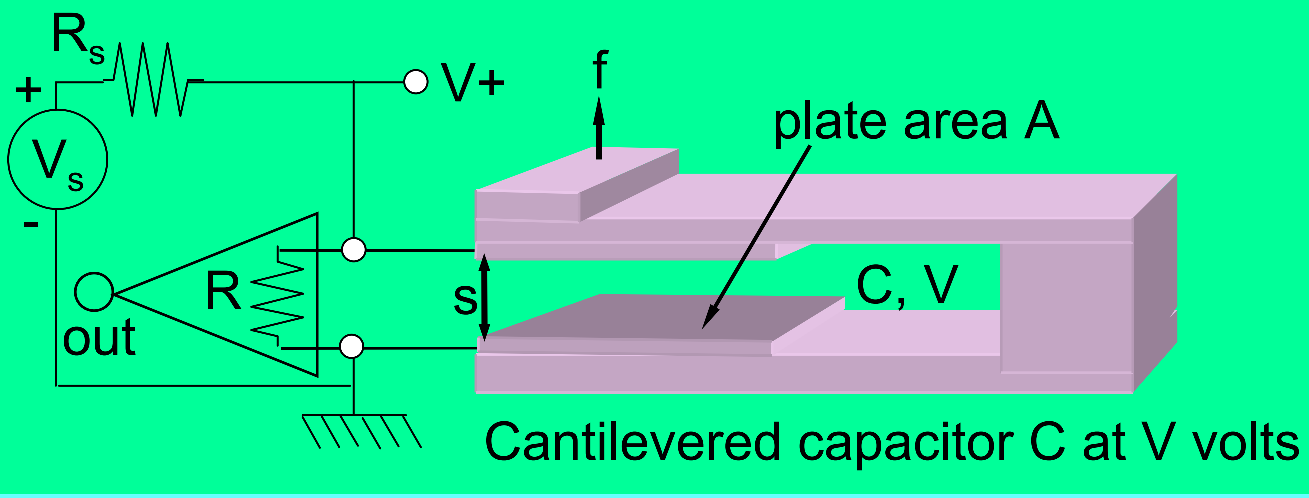
T [Nm] = -dw/dθ $\propto$ A’ $\propto$ nRs (n = # segments)

n = 4



ELECTROSTATIC SENSORS

Cantilevered microphone:



Cantilevered capacitor C at V volts

$s \rightarrow s + \delta \quad \Rightarrow \quad CV^2/2 \rightarrow C'V'^2/2$ initially, then:

Voltage decays to $V_o = V_sR/(R + R_s) \quad \Rightarrow$

Voltage pulse to amplifier, $\Delta w \approx w\delta/s > E_b > \sim 10^{-20}$ [J]

Minimum detectable $\delta \approx sE_b/w$ [m] (hears brownian motion)