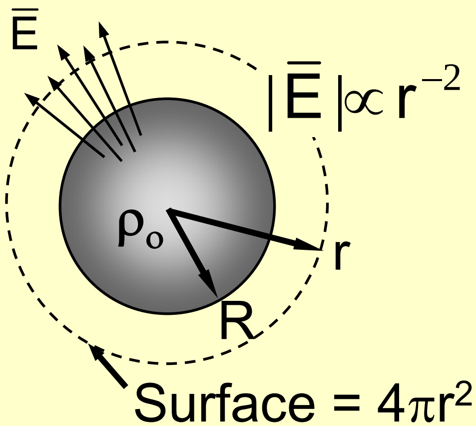
6.013 Electromagnetics and Applications
Spring 2009

Estimating Static Fields

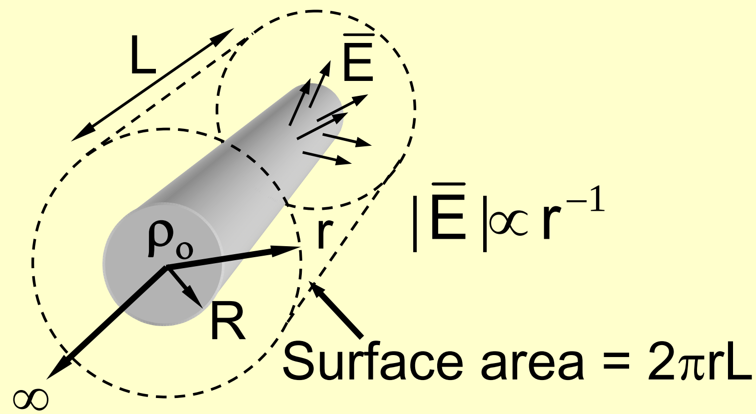
Examples of point and line field emission

Images of lightning rod, power line corona, and 0.1 micron integrated circuits removed due to copyright restrictions.

$$\Phi_{ab} = \int_a^b \overline{E}\bullet d\overline{z} \propto r^{-1} \text{ (spherical case), or } \propto \ln r \text{ (cylindrical case)}$$



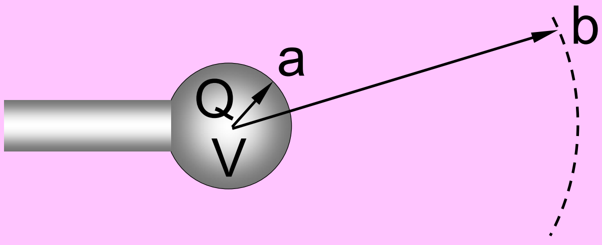
Spherical geometry



Cylindrical geometry

Simple Static Example

Spherical breakdown



$$E_r(a) = \frac{V_a}{a}$$

$$\Uparrow$$

$$Q = \int_A \bar{D} \bullet \hat{n}\, da = 4\pi r^2 \varepsilon E_r(r) \quad \Rightarrow \quad E_r(r) = \frac{Q}{4\pi\varepsilon r^2} = \frac{V_a 4\pi\varepsilon a}{4\pi\varepsilon r^2} = \frac{V_a a}{r^2}$$

$$\Uparrow$$

$$V_a = \int_a^b E_r(r) dr = \frac{Q}{4\pi\varepsilon}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\varepsilon}\left(\frac{1}{b} - \frac{1}{a}\right) \cong \frac{Q}{4\pi\varepsilon a} \quad \Rightarrow \quad Q = V_a 4\pi\varepsilon a$$

(b >> a)

$$\Downarrow$$

$$V(r) = \frac{Q}{4\pi\varepsilon r}$$

Lightning rod: $a \cong 1$ mm, $V \tilde{>} 10^4$ volts $\Rightarrow 10^7$ V/m breakdown/corona

Power line: $a \cong 1$ cm, $V \tilde{>} 10^5$ volts $\Rightarrow 10^7$ V/m breakdown/corona

Integrated Circuit: $a \cong 0.1\mu$m, V = 1 volt $\Rightarrow 10^7$ V/m breakdown/corona

SOLVING FOR STATIC FIELDS

Approaches when given ρ, $\bar{J}$:

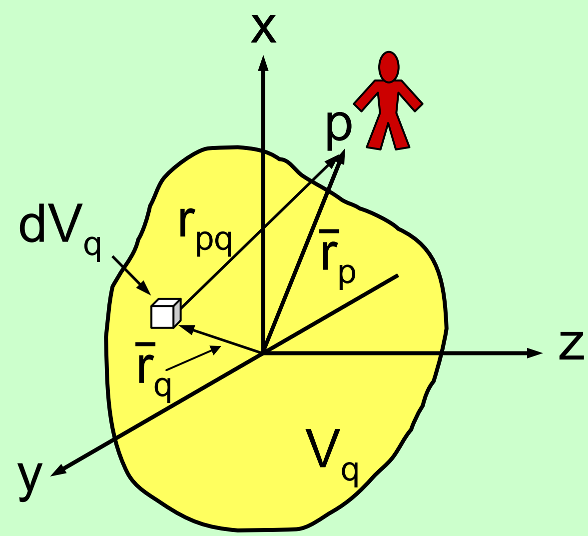
Use $\Phi_p = \int_{V_q} \frac{\rho_q}{4\pi\varepsilon r_{pq}} dv$, $\bar{E} = -\nabla\Phi$

$$\bar{E}_p = \int_{V_q} \hat{r}_{pq} \frac{\rho_q}{4\pi\varepsilon r_{pq}^2} dv$$

Superposition integrals

$$\bar{H}_p = \int_{V_q} \frac{\bar{J} \times (\bar{r}_p - \bar{r}_q)}{4\pi |\bar{r}_p - \bar{r}_q|^2} dv_q$$

Biot-Savart Law



Approaches when given Φ, Ψ on boundaries:

Use Laplace's Equation. Derivation:

$$\text{Electric}: \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = 0 \text{ (statics)} \Rightarrow \bar{E} = -\nabla\Phi. \quad \rho = 0 \Rightarrow \nabla \bullet \bar{E} = 0 \Rightarrow \nabla^2\Phi = 0$$

$$\text{Magnetic}: \nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = 0 \text{ (statics; } \bar{J} = 0) \Rightarrow \bar{H} = -\nabla\Psi. \quad \nabla \bullet \bar{H} = 0 \Rightarrow \nabla^2\Psi = 0$$

SEPARATION OF VARIABLES

Static charge-free regions obey Laplace's equation:

Electric potential $\nabla^2\Phi(\overline{r}) = 0$ Magnetic potential $\nabla^2\Psi(\overline{r}) = 0$

Assume Φ(x,y) = X(x)Y(y) $\Rightarrow$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = Y(y)\frac{d^2X}{dx^2} + X(x)\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 \text{ "separation constant"}$$

Solutions:

$$\frac{d^2X}{dx^2} = -k^2X \Rightarrow X(x) = A\cos(kx) + B\sin(kx)$$

$$\frac{d^2Y}{dy^2} = k^2Y \Rightarrow Y(y) = C\cosh(ky) + D\sinh(ky)$$

For $k^2 > 0$; (swap x,y If $k^2 < 0$)

or: $Y(y) = C'e^{ky} + D'e^{-ky}$ (equivalent to above)

SEPARATION OF VARIABLES

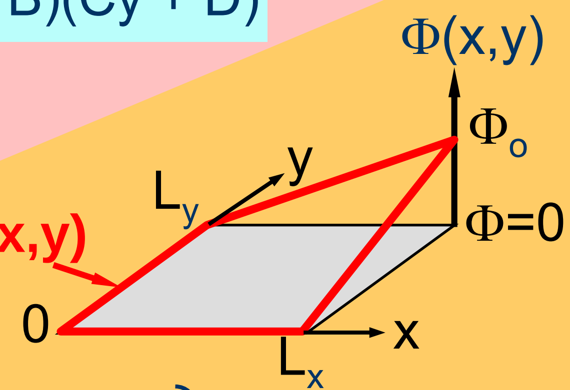
Solution to Laplace's equation when $k^2 = 0$:

$$\Phi(x,y) = X(x)Y(y) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

$$\frac{d^2X}{dx^2} = 0 \;\Rightarrow\; X(x) = Ax + B$$

$$\frac{d^2Y}{dy^2} = 0 \;\Rightarrow\; Y(y) = Cy + D$$

$$\Phi(x,y) = (Ax + B)(Cy + D)$$



Example, Cartesian coordinates:

$\{\Phi = (Ax+B)(Cy+D) = 0$ at $x=0$ for all $y\} \Rightarrow B = 0$

$\{\Phi = (Ax+B)(Cy+D) = 0$ at $y=0$ for all $x\} \Rightarrow D = 0$

$\{\Phi = \Phi_o$ at $x=L_x, y=L_y\} \Rightarrow AC = \Phi_o/L_xL_y$

$$\Phi = xy\Phi_o/L_xL_y$$

Φ is matched at all 4 boundaries

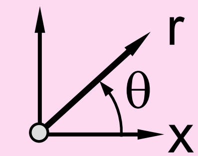
CIRCULAR COORDINATES

Separation of Laplace's equation:

Only in cartesian, cylindrical, spherical, and elliptical coordinates

Circular coordinates:

$$\nabla^2\Phi(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\Phi}{\partial\theta^2}\right) = 0$$



$$\Phi = R(r)\Theta(\theta) \;\Rightarrow\; \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2}\right) = m^2$$

Solutions (pick the one matching boundary condition):

$$\Phi(r,\theta) = (A + B\theta)(C + D\ln r) \qquad \text{for } m^2 = 0$$

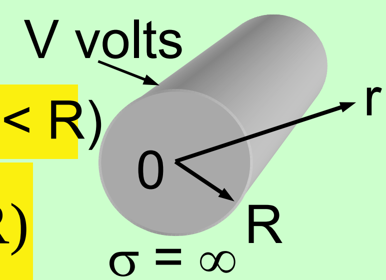
$$\Phi(r,\theta) = (A\sin m\theta + B\cos m\theta)(Cr^{m} + Dr^{-m}) \qquad \text{for } m^2 > 0$$

$$\Phi(r,\theta) = [A\sinh m\theta + B\cosh m\theta)[C\cos(m\ln r) + D\sin(m\ln r)] \quad \text{for } m^2 < 0$$

Example – conducting cylinder (m = 0):

$\Phi(r,\theta)$ = C+D ln r = V[2-(lnr/lnR)](r $\geq$ R); $\Phi(r,\theta)$ = V (r < R)

$$\overline{E}(r) = -\nabla\Phi = -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\Phi = \frac{V}{r\ln R} \;\; [V/m]\;(r > R)$$



INHOMOGENEOUS MATERIALS

Governing Equations:

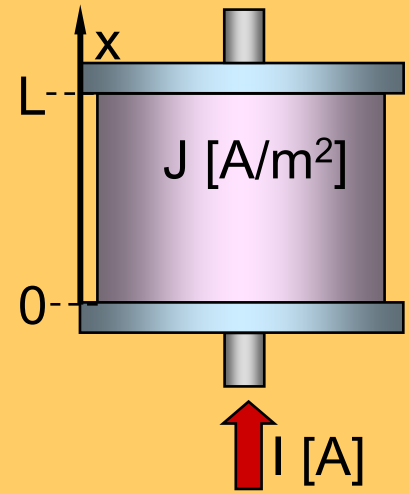
$$\bar{J} = \sigma \bar{E} \qquad \bar{D} = \varepsilon \bar{E} = \varepsilon_o \bar{E} + \bar{P} \qquad \nabla \bullet \bar{D} = \rho_f \qquad \nabla \bullet \bar{P} = -\rho_p$$

Non-uniform Conductivity $\sigma(x)$ (e.g., doping gradients in pn junctions):

Assume $\sigma = \dfrac{\sigma_o}{1 + \frac{x}{L}}$ [S/m]

$$\bar{E} = \frac{\bar{J}}{\sigma} = \hat{x}\frac{J_o(1+\frac{x}{L})}{\sigma_o} \text{ [V/m]}$$

Note: Non-uniform conductors have free charge density ρ_f and polarization charge density ρ_p throughout.



$$\rho_f = \nabla \bullet \bar{D} = \nabla \bullet \varepsilon \bar{E} = \varepsilon \frac{d}{dx}\left[\frac{J_o(1+\frac{x}{L})}{\sigma_o}\right] = \frac{\varepsilon J_o}{\sigma_o L} \text{ [C/m}^3\text{]}$$

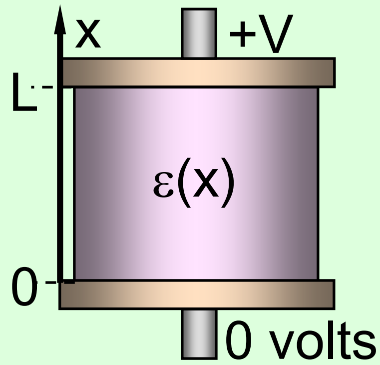
$$\rho_p = -\nabla \bullet \bar{P} = -\nabla \bullet (\varepsilon - \varepsilon_o)\bar{E} = -(\varepsilon - \varepsilon_o)\frac{d}{dx}\left[\frac{J_o(1+\frac{x}{L})}{\sigma_o}\right] = -\frac{(\varepsilon - \varepsilon_o)J_o}{\sigma_o L} \text{ [C/m}^3\text{]}$$

INHOMOGENEOUS PERMITTIVITY

Governing Equations:

$$\bar{J} = \sigma\bar{E} \qquad \bar{D} = \varepsilon\bar{E} = \varepsilon_o\bar{E} + \bar{P} \qquad \nabla \bullet \bar{D} = \rho_f \qquad \nabla \bullet \bar{P} = -\rho_p$$

Example, Non-uniform Permittivity ε(x):



Assume: $\varepsilon = \varepsilon_o(1+\frac{x}{L})$ [F/m]

$$\bar{D} = \varepsilon\bar{E} \neq f(x) \Rightarrow E = \frac{D_o}{\varepsilon} = \frac{D_o}{\varepsilon_o(1+\frac{x}{L})} = \frac{E_o}{(1+\frac{x}{L})}$$

$$V = \int_0^L E_x dx = \int_0^L \frac{E_o}{1+\frac{x}{L}} dx = E_o L \ln 2 \text{ [V]} \Rightarrow E_o = \frac{V}{L \ln 2} \text{ [V/m]}$$

Therefore: $\bar{E} = \hat{x}\frac{V}{L\ln 2(1+\frac{x}{L})}$ [V/m] Note: Non-uniform E(x)

$$\rho_p = -\nabla \bullet \bar{P} = -\nabla \bullet (\bar{D}-\varepsilon_o\bar{E}) = \varepsilon_o\nabla\bullet\bar{E} = (V/L\ln 2)\frac{d}{dx}(1+\frac{x}{L})^{-1} = \frac{-V}{L^2\ln 2(1+\frac{x}{L})^2}$$

Non-uniform dielectrics have polarization charge density ρ_p throughout.