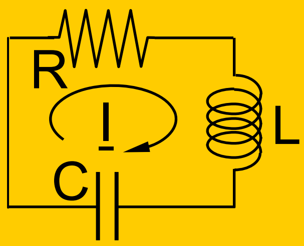
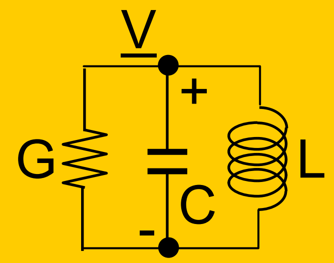
6.013 Electromagnetics and Applications
Spring 2009

RLC RESONATORS

Resonators trap energy:



Series RLC resonator



Parallel RLC resonator

Also:

terminated TEM lines, waveguides

Circuit equations, series resonator:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0 \;\Rightarrow\; j\omega L\underline{I} + R\underline{I} + \frac{\underline{I}}{j\omega C} = 0$$

$$\left[(j\omega)^2 + (j\omega)\frac{R}{L} + \frac{1}{LC}\right]\underline{I} = 0 \;\Rightarrow\; (j\omega - s_1)(j\omega - s_2) = 0^*$$

$$s_{1,2} = -\frac{R}{2L} \pm j\underbrace{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}_{\omega'}$$

Note: $s_2 = s_1^*$ $\quad \omega' = \frac{1}{\sqrt{LC}}$ for $R \rightarrow 0$

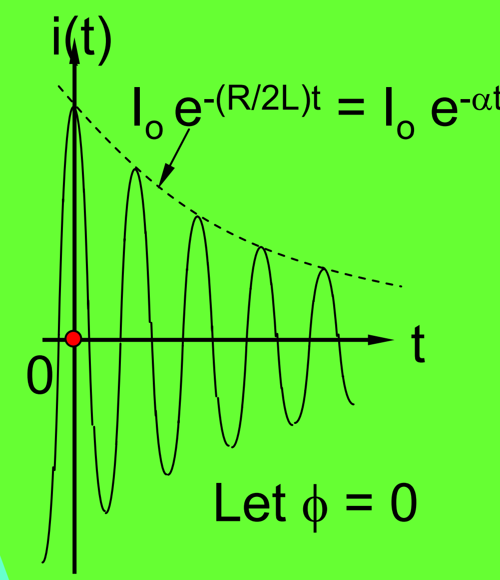
$$i(t) = R_e\{\underline{I}_o e^{j\omega' t}\}\, e^{-\frac{R}{2L}t}$$

*Let $j\omega = s_i$; recall: $as^2 + bs + c = 0$ $\quad s_i = \frac{(-b \pm \sqrt{b^2 - 4ac})}{2a}$

RLC RESONATOR WAVEFORMS

Series resonator current i(t):

$$i(t) = R_e\{\underline{I}_o e^{j\omega' t}\}\, e^{-\frac{R}{2L}t} = I_o \cos(\omega' t + \phi) e^{-\frac{R}{2L}t}$$



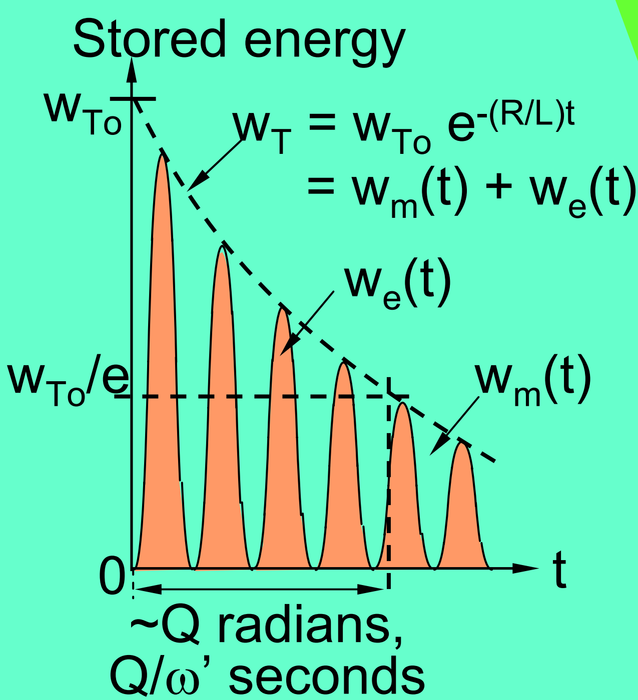
Energy w(t):

$$w_m(t) = \frac{1}{2} L i^2 \;\propto\; \cos^2(\omega' t)\, e^{-\frac{R}{L}t}$$

$$w_e(t) = \frac{1}{2} C v^2 \;\propto\; \sin^2(\omega' t)\, e^{-\frac{R}{L}t}$$

$$w_{emax} = w_{mmax}$$

$$\Rightarrow v_{max} = i_{max}\sqrt{\frac{L}{C}}$$



$$e^{-2\alpha t} \cong e^{-\omega' t/Q}$$

$$= e^{-(R/L)t}$$

$$Q \cong \omega'/2\alpha = L\omega'/R$$

$$\cong \sqrt{L/C}\,/R$$

(series resonance)

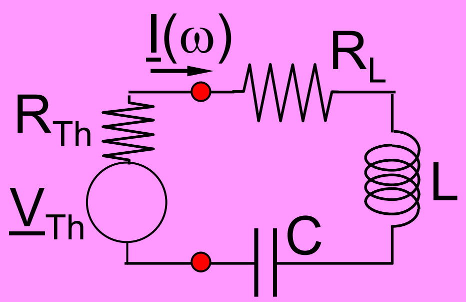
$$Q \cong \omega'/2\alpha = C\omega'/G$$

$$\cong \sqrt{C/L}\,/G$$

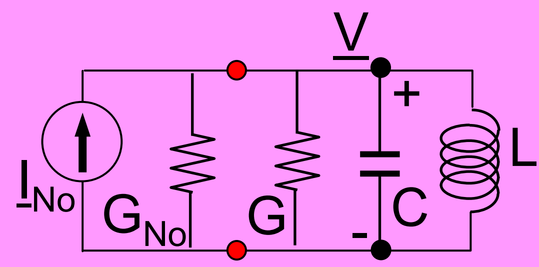
(parallel)

COUPLING TO RLC RESONATORS

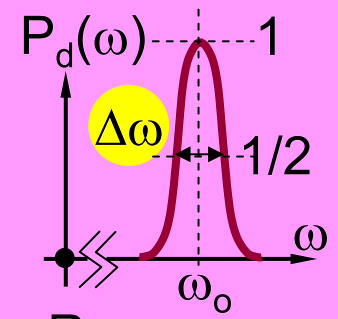
Thevenin and Norton Equivalent Sources:



Thevenin equivalent source



Norton equivalent



Resonance

Power dissipated $P_d(f)$ in $R = R_L + R_{Th}$:

$$P_d(\omega) = \frac{1}{2}\frac{|\underline{V}_{Th}|^2}{|\underline{Z}|^2}R = \frac{1}{2}\frac{|\underline{V}_{Th}|^2}{\left|R + j\omega L + \frac{1}{j\omega C}\right|^2}R$$

Dominant factor near ω_o

$$= \frac{|\underline{V}_{Th}|^2 R\omega^2}{2L^2}\left|\left(\omega - \frac{1}{\sqrt{LC}} - j\frac{R}{2L}\right)\left(\omega + \frac{1}{\sqrt{LC}} - j\frac{R}{2L}\right)\right|^{-2}$$

Half-power frequencies: $\omega \cong \omega_o \pm R/2L = \omega_o \pm \alpha$, where $\omega_o = 1/\sqrt{LC}$

so: $\Delta\omega = 2\alpha = \omega_o/Q$ and $Q = \omega_o/\Delta\omega$

RESONATOR Q

General derivation of Q (all resonators):

$w_T \cong w_{To} e^{-\omega' t/Q}$ (total stored energy [J])

$P_d = -dw_T/dt \cong (\omega'/Q) w_T$ (power dissipated [W])

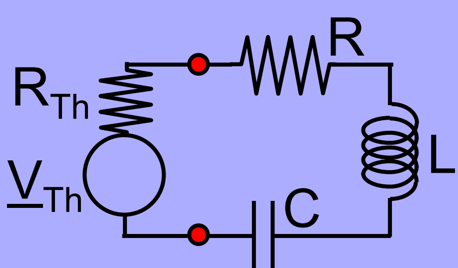
$Q \cong \omega' w_T / P_d$ * (resonator Q ["radians" is dimensionless])

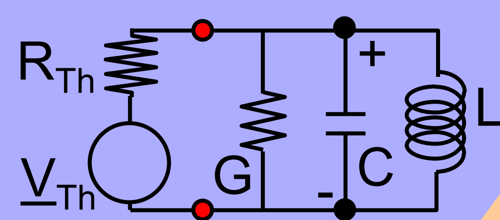
Internal, external, and loaded Q (Q_I, Q_E, Q_L):

$Q_I = \omega' w_T / P_{dI}$ (P_{dI} is power dissipated internally, in R)

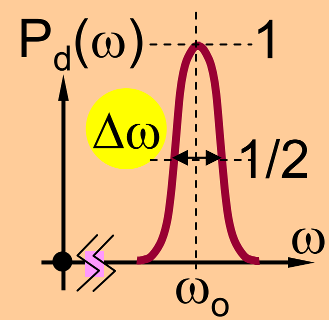
$Q_E = \omega' w_T / P_{dE}$ (P_{dE} is power dissipated externally, in R_{Th})

$Q_L = \omega' w_T / P_{dT}$ (P_{dT} is the total power dissipated, in R and R_{Th})





$P_{dT} = P_{dI} + P_{dE} \Rightarrow$ $Q_L^{-1} = Q_I^{-1} + Q_E^{-1}$



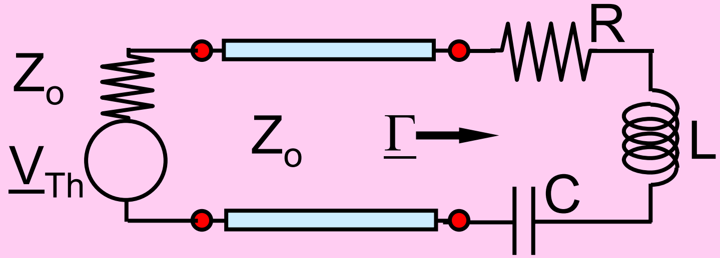
$Q_L \approx \omega_o / \Delta\omega$

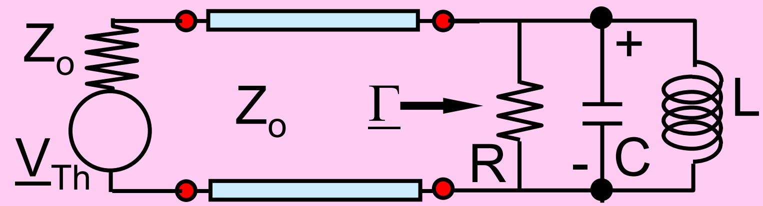
Perfect Match: $Q_I = Q_E$

*IEEE definition: $Q = \omega_o w_T / P_d$

MATCHING TO RESONATORS

Transmission line feed:





At ω_o: $$|\underline{\Gamma}|^2 = \left|\frac{R - Z_o}{R + Z_o}\right|^2$$

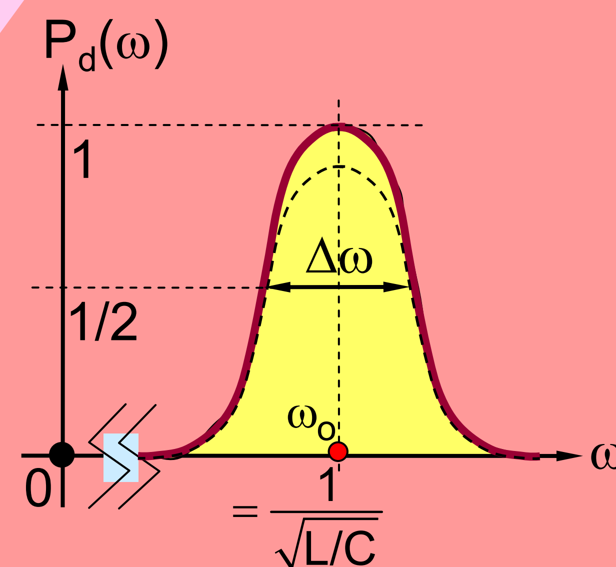
$= 0$ if matched, $R = Z_o$

$= 1/9$ if $R = Z_o/2$ or $2Z_o$

Behavior away from resonance:

Series resonance: Open circuit

Parallel resonance: Short circuit



EXAMPLE #1 – CELL PHONE FILTER

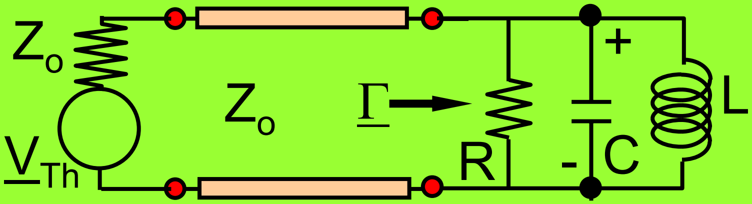
Bandpass filter specifications:

Looks like a short circuit far from ω_o

At ω_o: reflect 1/9 of the incident power and let $\underline{\Gamma} < 0$

$\omega_o = 5 \times 10^9$ and $\Delta\omega = 5 \times 10^7$

Z_o = 100-ohm line



Filter solution:

Parallel resonators look like short circuits far from ω_o

$|\underline{\Gamma}|^2 = 1/9$ and $\underline{\Gamma} < 0 \Rightarrow \underline{\Gamma} = -1/3$ at ω_o. $\underline{Z} = Z_o \dfrac{1+\underline{\Gamma}}{1-\underline{\Gamma}} \Rightarrow$ **R = 50Ω**

$\sqrt{LC} = 1/\omega_o = (5x10^9)^{-1}$

$Q_L = R'/\sqrt{L/C}$ (parallel) $\Rightarrow \sqrt{L/C} = R'/Q_L = 33/100 = 0.33$ ($R' = R \,//\, Z_o$)

$L = \sqrt{LC}\sqrt{L/C} = (5x10^9)^{-1} \times 0.33 =$ **6.67×10^{-12} [Hy]**

$C = \sqrt{LC}\,/\,\sqrt{L/C} = (5x10^9)^{-1}/0.33 =$ **6×10^{-10} [F]**

} Small, hard to build, use TEM?

EXAMPLE #2 – BAND-STOP FILTER

Filter specifications:

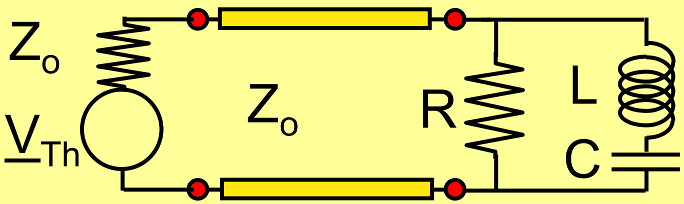
Far from ω_o the load is matched (signal goes to amplifier R)

At ω_o reflect all incident power; let $\underline{\Gamma} = -1$ (short circuit)

$\omega_o = 5 \times 10^6$

$\Delta\omega = 5 \times 10^4$ (rejected band, notch filter) $\Rightarrow$ Q = 100

Z_o = 100-ohm line



Filter solution:

Lossless series resonators look like short circuits at ω_o

$R = Z_o = 100\Omega \quad \Rightarrow \quad |\underline{\Gamma}|^2 = 0$ at ω far from ω_o

$\sqrt{LC} = 1/\omega_o = (5\text{x}10^6)^{-1}$

$Q_L = \sqrt{L/C}\,/R_L$ (series) $\quad \Rightarrow \quad \sqrt{L/C} = R_L Q_L = 50\text{x}100 = 5000$

$L = \sqrt{LC}\sqrt{L/C} = (5\text{x}10^6)^{-1} \text{ x } 5000 = 10^{-3}$ [Hy]

$C = \sqrt{LC}\,/\,\sqrt{L/C} = (5\text{x}10^6)^{-1}/5000 = 4 \text{ x } 10^{-11}$ [F]