

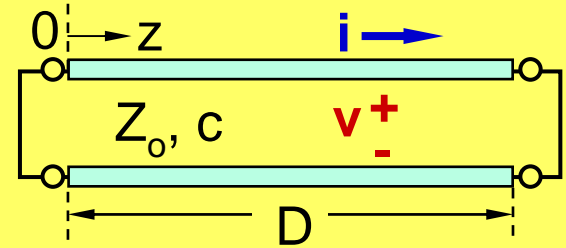
MIT OpenCourseWare
<http://ocw.mit.edu>

6.013 Electromagnetics and Applications
Spring 2009

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

TEM RESONATORS

Voltages and Currents:



$$v(t,z) = V_+ \cos(\omega t - kz) + V_- \cos(\omega t + kz)$$

$$\text{Boundary conditions: } v(t,z) = 0 \text{ at } z = 0 \Rightarrow V_- = -V_+$$

$$v(t,z) = \left(\frac{V_+}{Z_o}\right) V_+ [\cos(\omega t - kz) - \cos(\omega t + kz)] = 2V_+ \sin \omega t \sin kz *$$

$$i(t,z) = \left(\frac{V_+}{Z_o}\right) [\cos(\omega t - kz) + \cos(\omega t + kz)] = 2\left(\frac{V_+}{Z_o}\right) \cos \omega t \cos kz *$$

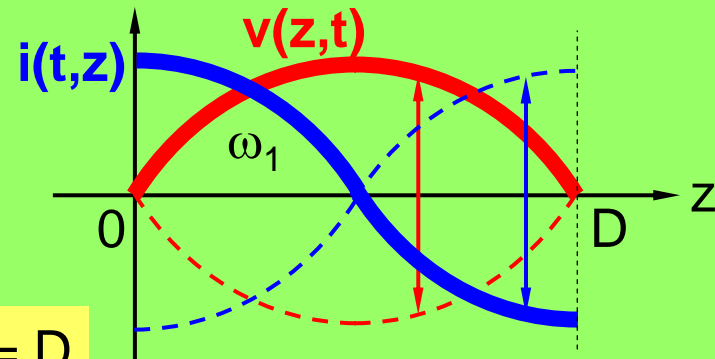
Resonant Frequencies ω_n :

$$2V_+ \sin(\omega t) \sin(kD) = 0 \Rightarrow$$

$$k_n D = n\pi, \quad n = 0, 1, 2, \dots$$

$$k_n D = (\omega_n/c)D = n\pi \Rightarrow$$

$$\omega_n = n\pi c/D = 2\pi c/\lambda_n \Rightarrow n\lambda_n/2 = D$$



* $\cos \alpha - \cos \beta = -2 \sin[(\alpha + \beta)/2] \sin [(\alpha - \beta)/2]$
 $\cos \alpha + \cos \beta = 2 \cos[(\alpha + \beta)/2] \cos [(\alpha - \beta)/2]$

MORE TEM RESONANCES

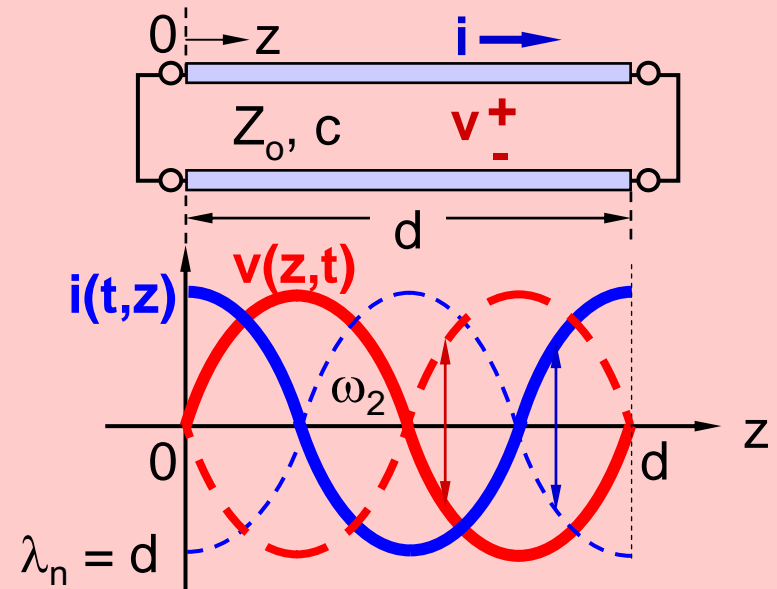
Resonant Frequencies ω_n :

$$\omega_n = \frac{n\pi c}{d} \quad \frac{n\lambda_n}{2} = d$$

DC Resonance ($\omega = 0$):

Current flows in loop,
Voltages are zero

$$w_e = 0, \quad w_m \neq 0 \text{ [J]}$$



Open-Circuit Resonator:

$$n\lambda_n/2 = d, \quad n = 0, 1, 2, \dots \Rightarrow$$

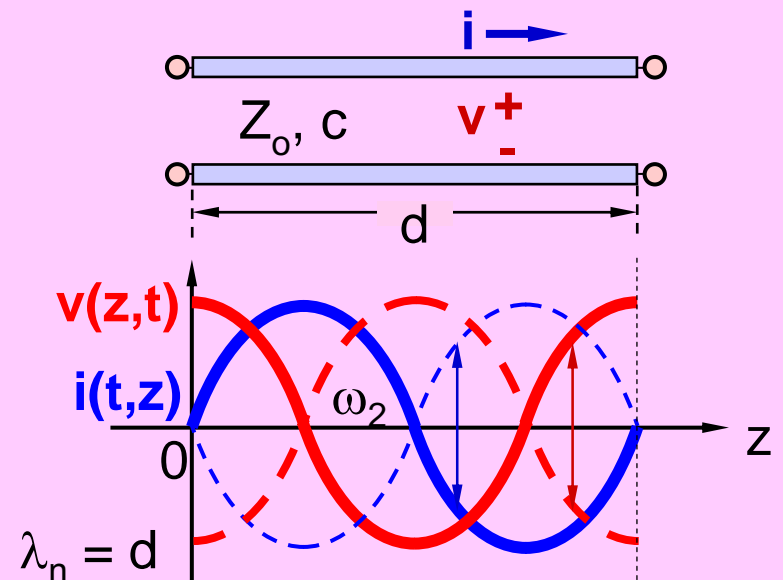
$$\lambda_n = 2d/n \quad \omega_n = n\pi c/d = 2\pi c/\lambda_n$$

$$\lambda_0 = \infty \quad \rightarrow \quad \omega_0 = 0$$

$$\lambda_1 = 2d \quad \rightarrow \quad \omega_1 = \pi c/d$$

$$\lambda_2 = d \quad \rightarrow \quad \omega_2 = 2\pi c/d$$

$$\lambda_3 = 2d/3 \quad \rightarrow \quad \omega_3 = 3\pi c/d$$



ANOTHER TEM RESONANCE

Quarter-Wave Resonator:

$$D = \lambda/4, 3\lambda/4, 5\lambda/4, \dots \Rightarrow$$

$$D = (\lambda_n/4)(2n - 1) \quad [n = 1, 2, 3, \dots]$$

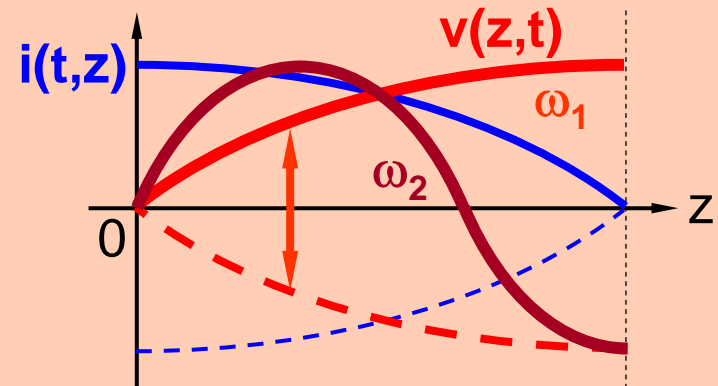
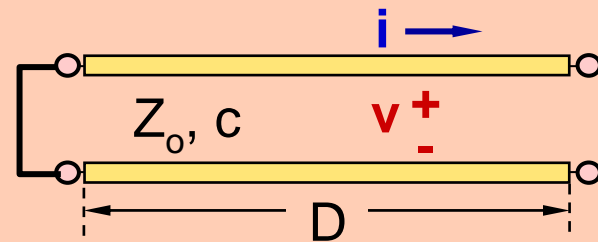
$$\lambda_n = 4D/(2n - 1)$$

$$\lambda_1 = 4D$$

$$\lambda_2 = 4D/3$$

$$\lambda_3 = 4D/5$$

$$\lambda_4 = 4D/7$$



ENERGY IN TEM RESONATORS

Electric Energy Density $W_e = Cv^2/2$ [J/m]:

$$v(t,z) = V_o \sin \omega_n t \sin k_n z$$

$$W_e(t,z) = \frac{1}{2} CV_o^2 (\sin^2 \omega_n t) (\sin^2 k_n z) \quad [\text{J/m}]$$

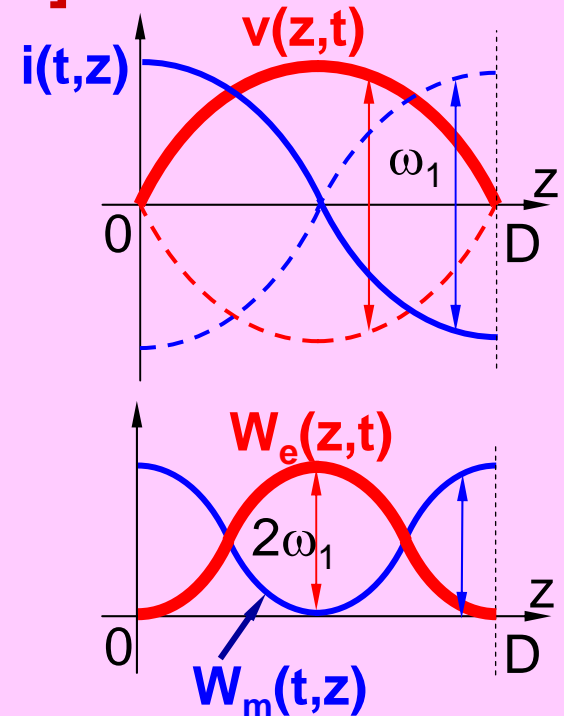
$$= \frac{1}{2} CV_o^2 (1 - \cos 2\omega_n t) (1 - \cos 2k_n z) \geq 0$$

Magnetic Energy Density $W_m = Li^2/2$:

$$i(t,z) = \frac{V_o}{Z_o} \cos \omega_n t \cos k_n z$$

$$W_m(t,z) = \frac{1}{2} L \left(\frac{V_o}{Z_o} \right)^2 \cos^2 \omega_n t \cos^2 k_n z \quad [\text{J/m}]$$

$$= \frac{1}{2} L \left(\frac{V_o}{Z_o} \right)^2 (1 + \cos 2\omega_n t) (1 + \cos 2k_n z)$$



Total Electric and Magnetic Energies w_e and w_m :

$$\langle W_e \rangle = \frac{1}{4} CV_o^2 (1 - \cos 2k_n z) \quad \langle W_m \rangle = \frac{1}{4} L \left(\frac{V_o}{Z_o} \right)^2 (1 + \cos 2k_n z)$$

Over $n\frac{\lambda}{4}$ we have $\langle w_e \rangle = \langle w_m \rangle$ because $CV_o^2 = L \left(\frac{V_o}{Z_o} \right)^2$ (recall $Z_o^2 = \frac{L}{C}$)

* $\langle \bullet \rangle$ signifies time average of \bullet .

LUMPED LOSSES IN TEM RESONATORS

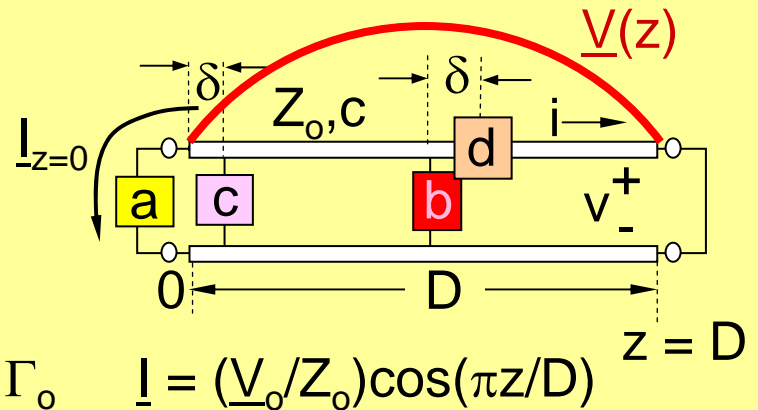
Calculation of Q:

$$Q = \frac{\omega_0 W_T}{P_d}$$

Perturbation Technique:

Find P_d using unperturbed v or i

Need $(\Delta v/v_0) \ll 1, (\Delta i/i_0) \ll 1, \Gamma \cong \Gamma_0$



Loss Computation Examples:

Use unperturbed $\underline{V}, \underline{I}$

a $P_d \cong \frac{1}{2} |I_{z=0}|^2 R_a$

c $P_d \cong \frac{1}{2R_c} |V_0 \text{sinc} \delta|^2$

b $P_d \cong \frac{1}{2R_b} |V_{z=D/2}|^2$

d $P_d \cong \frac{1}{2} |(\underline{V}_0/Z_0) \text{sinc} \delta|^2 R_d$

When is a perturbation too large?

a Want $\Gamma_{z=0} \cong -1 \cong \frac{Z_n - 1}{Z_n + 1}$

$\Rightarrow R_a \ll Z_0 \quad (Z_{an} \ll 1)$

b Want $\Gamma_{(z=D/2)} \cong 1$

$\Rightarrow R_b \gg Z_0 \quad (Z_{bn} \gg 1)$

c Want $R_c \gg |-jZ_0 \sin k\delta|$

$\Rightarrow R_c > \sim Z_0 \quad (\text{if } 2\pi\delta/\lambda \ll 1)$

COUPLED TEM RESONATORS

Example: computation of Q:

$$Q_{\text{int,ext,or loaded}} = \frac{\omega_0 W_T}{P_{\text{d:int,ext,or loaded}}}$$

$$\omega_0 = \frac{\pi C}{D} \quad W_T = 2 \langle w_e \rangle = 2D \frac{C |V_0|^2}{8}$$

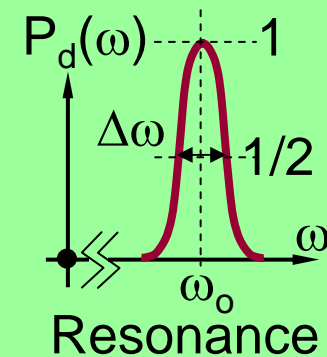
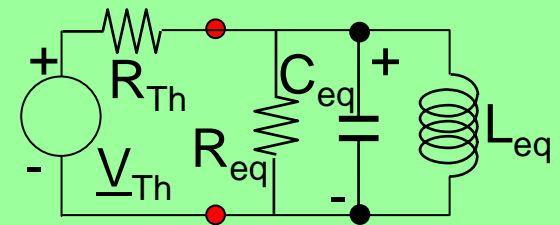
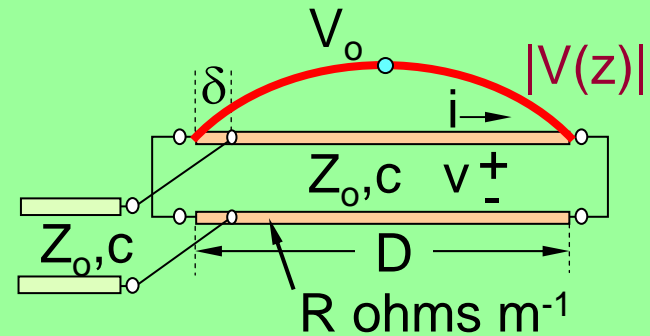
$$P_{\text{d;ext}} = \frac{|V_0 \sin(\frac{\pi \delta}{D})|^2}{2Z_0} \Rightarrow Q_{\text{ext}} \cong \frac{(\frac{D}{\delta})^2}{2\pi}$$

$$P_{\text{d;int}} = \frac{RD}{4} \left| \frac{V_0}{Z_0} \right|^2 \Rightarrow Q_{\text{int}} = \frac{\pi Z_0}{RD}$$

Example, perfect coupling:

$$Q_{\text{int}} = Q_{\text{ext}} \Rightarrow \delta = \sqrt{\frac{RD^3}{2\pi^2 Z_0}}$$

$$\Delta\omega = \frac{\omega_0}{Q_L} = \frac{2\omega_0}{Q_1} = 2 \frac{R}{L} = \frac{R_{\text{eq}}}{L_{\text{eq}}} [\text{r/s}]$$



COUPLED TEM RESONATORS

Resonator equivalent circuit

$$\Delta\omega = \frac{\omega_0}{Q_L} = \frac{2\omega_0}{Q_I} = \frac{R_{eq}}{L_{eq}} \quad [\text{r/s}]$$

$$\omega_0 = \frac{\pi C}{D} = \frac{1}{\sqrt{L_{eq} C_{eq}}}$$

$$Z_n(\omega = \omega_0) = \frac{R_{eq}}{R_{Th}} = \frac{P_{d;ext}}{P_{d;int}} = \frac{Q_I}{Q_E}$$

$$\Rightarrow \Gamma \text{ at } \omega_0 \quad \left(\Gamma = \frac{R_{eq} - R_{Th}}{R_{eq} + R_{Th}} \right)$$

3 Eqn
3 Unk

Imperfect coupling if $Q_E \neq Q_I$

Lossless bandpass filter

If both lines are Z_0 and $\delta_1 = \delta_2$,

Then $Q_I = Q_E \Rightarrow$ perfect coupling
and perfect match at ω_0

