2

Fuzzy Sets and Relations

The chapter provides an introduction to fuzzy sets, fuzzy relations and some elementary fuzzy operators such as t-norm, s-norm, max-min composition and max-product composition operators. The extension principle of fuzzy sets and the concept of projection and cylindrical extension have been outlined in the chapter with examples. A brief introduction to fuzzy linguistic variables and fuzzy hedges is also given at the end of the chapter.

2.1 Conventional Sets

Mathematicians define **sets** *as a collection of objects having one or more common characteristics*. The objects that belong to a set are called **members/ elements.** The characteristics used to define a set should be sufficient to identify its members. For example, persons enrolled for a course may together be called a STUDENT set. We call it a set as we can easily determine whether a person is a student by checking his/her name in the registrar book. LARGE RIVERS of a country, however, should not be called a set unless we clearly define their minimum length to be considered as LARGE.

Let A be a set and x be a member of A, we can denote this by the following notation:

$$x \in A \tag{2.1}$$

Also let y be a member of set B, such that for all y, y is also a member of A. Then B is called the subset of A, denoted by

$$B \subseteq A. \tag{2.2}$$

If B is a subset of A but every element of A are not present in B then B is called the **proper subset** of A, denoted by

$$B \subset A. \tag{2.3}$$

If every element contained in B is an element of A and every element contained in A is an element of B, then set A and set B are **equal**, given by

$$A = B. \tag{2.4}$$

For any 2 sets A and B if there exist at least one common element x of both A and B, then we say that

$$x \in (A \cap B), \tag{2.5}$$

where $\cap$ denotes a logical **intersection operation**.

For any 2 sets A and B if there exists at least one element x, such that x is a member of A or B, then we say that

$$x \in (A \cup B), \tag{2.6}$$

where $\cup$ denotes a **union operation**.

A **universal set U** in a particular domain is a set that includes all possible members of that domain. In other words, all sets in a given domain are subsets of the universal set U.

2.2 Fuzzy Sets

In a conventional set, the condition defining the set boundaries is very rigid. For example, consider a universal set AGE, OLD, VERY OLD, YOUNG, CHILD and BABY are subsets of the universal set AGE. The conventional approach to define these sets are illustrated below:

BABY=$\{age \in AGE\text{: } 0 \text{ year} \le age < 1\text{year}\}$,

CHILD=$\{age \in AGE\text{: } 1 \text{ year} \le age \le 10\text{years}\}$,

YOUNG= $\{age \in AGE\text{: } 19 \text{ years} \le age \le 40\text{years}\}$,

OLD= {age $\in$ AGE: 60 years $\leq$ age < 80years},

and VERY OLD= {age $\in$ AGE: 80 years $\leq$ age < 120 years}.

In the above definitions age is a variable that may presume any value in the range [0, 120] years. It is clear from the definitions that the boundary of each set is distinct. Thus an age=11 months 29 days is a member of the set BABY, but once it is 1 year it falls in the set CHILD. Thus there is a sharp demarcation in the boundary definition of the sets BABY and CHILD at age=1 year. Measurements in a real world system being highly imprecise, such a sharp demarcation of 2 set boundaries may cause a wrong allocation of the members to a given set.

Another characteristic of a conventional set includes assignment of a grade of membership 1 to all its members and 0 to all its non-members. The following connotation is used to describe that the membership of an element x in a set A is 1, and the membership of a non-element y in the set A is 0.

$$\mu_A (x)= 1 \tag{2.7}$$

$$\mu_A (y)= 0 \tag{2.8}$$

A **fuzzy set** extends the binary membership: {0,1} of a conventional set to a spectrum in the interval of [0, 1]. Further, unlike a conventional set, all elements of the universal set U are members of a given set A. Thus for each element $x \in U$,

$$0 \leq \mu_A (x) \leq 1. \tag{2.9}$$

It needs mention here that as all elements of a universal set U are members of a given fuzzy set A, therefore, 2 fuzzy sets A and B may have an overlap in the boundary definitions. For example, in contrast to the respective conventional sets: BABY, CHILD, YOUNG, OLD and VERY OLD, the corresponding fuzzy sets allow any age in the interval [0, 120] years as a member of each of the above sets but with different memberships in [0, 1]. As a specific instance, the age 22 is a member of all the fuzzy sets but the membership of age (=22) to belong to the sets BABY, CHILD, YOUNG, OLD and VERY OLD respectively are 0.001, 0.01,1.00, 0.60 and 0.20. The above example makes sense in the line of reasoning that an age of 22 corresponds to a young person, so the membership of age (=22) to be young is high (1.00). The relative grading of the other memberships thus can be easily understood from the usual meaning of the terms BABY, CHILD, OLD and VERY OLD.

A fuzzy set thus can be formally defined as follows.

Definition 2.1: *A **fuzzy set** A is a set of ordered pairs, given by*

$$A=\{(x, \mu_A (x)): x \in X\} \qquad (2.10)$$

where X is a universal set of objects (also called the universe of discourse) and $\mu_A(x)$ is the grade of membership of the object x in A. Usually, $\mu_A (x)$ lies in the closed interval of [0,1].

It may be added here that some authors [7] relax the range of membership from [0, 1] to [0, R_{max}] where R_{max} is a positive finite real number. One can easily convert [0, R_{max}] to [0, 1] by dividing the membership values in the range [0, R_{max}] by R_{max}.

There are other notations of fuzzy sets as well. For instance, the ordered pair (x, μ_A (x)) in the definition of fuzzy set is also written as x/ μ_A (x) or μ_A (x)/x as well. Let the elements of set X be x_1, x_2, …,x_n. Then the fuzzy set $A \subseteq X$ is denoted by any of the following nomenclature.

$$A = \{(x_1, \mu_A (x_1)), (x_2, \mu_A (x_2)), \ldots\ldots\ldots,(x_n, \mu_A (x_n))\}, \text{ or}$$

$$A= \{x_1 / \mu_A (x_1), x_2 / \mu_A (x_2), \ldots\ldots\ldots,x_n / \mu_A (x_n)\}, \text{ or}$$

$$A= \{x_1 / \mu_A (x_1) + x_2 / \mu_A (x_2) + \ldots\ldots\ldots+ x_n / \mu_A (x_n)\}, \text{ or}$$

$$A= \{\mu_A (x_1)/ x_1 + \mu_A (x_2)/ x_2 + \ldots\ldots\ldots+ \mu_A (x_n)/ x_n\}, \text{ or}$$

$$A= \{\mu_A (x_1)/ x_1, \mu_A (x_2)/ x_2, \ldots\ldots\ldots, \mu_A (x_n)/ x_n\}.$$

In this book we used the last option. The details of membership function $\mu_A(x)$ is formalized below.

2.3 Membership Functions

The grade of membership μ_A (x) maps the object or its attribute x to positive real numbers in the interval [0, 1]. Because of its mapping characteristics like a function, it is called **membership function**. A formal definition of the membership function is given below for the convenience of the readers.

Definition 2.2: *A **membership function** μ_A (x) is characterized by the following mapping:*

$$\mu_A: x \rightarrow [0, 1], \; x \in X \qquad (2.11)$$

where x is a real number describing an object or its attribute and X is the universe of discourse and A is a subset of X.

A question that naturally arises is: how to construct a membership function? The following examples provide a thorough insight to the selection of the membership functions.

Example 2.1: Consider the problem of defining BABY, CHILD, YOUNG, OLD and VERY OLD by membership functions. The closer the age of a person to 0, the higher is his/her membership to be a BABY. So, if x is the age of the person, we can define BABY as follows:

$$BABY= \{(x, \mu_{BABY}(x)\} \text{ where } \mu_{BABY}(x)= \exp(-x). \qquad (2.12)$$

Thus as $x \to 0$, $\mu_{BABY}(x) \to 1$. Further, as x increases, $\mu_{BABY}(x)$ decreases exponentially. The membership function $\mu_{BABY}(x)$ can also be designed to have a controlled decrease with increasing x by including a factor α to x in exp(-x). Thus,

$$\mu_{BABY}(x)= \exp(-\alpha x) \text{ for } \alpha > 0. \qquad (2.13)$$

Larger the value of α, the higher is the falling rate of $\mu_{BABY}(x)$ over x. In a similar manner we can define the membership functions for CHILD, YOUNG, OLD and VERY OLD fuzzy sets. But before representing them mathematically let us take a look at them.

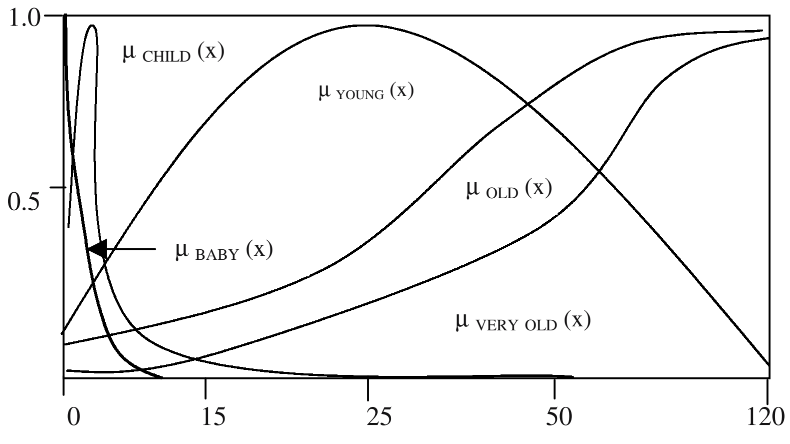


Fig. 2.1: Membership curves for the fuzzy sets: BABY, CHILD, YOUNG, OLD and VERY OLD. The x-axis denotes the age in years and the y-axis denotes the memberships of the given fuzzy sets at different ages.

The membership curves for the fuzzy sets: BABY, CHILD, YOUNG, OLD and VERY OLD are shown in Fig. 2.1. The curve for CHILD fuzzy set has the peak at some age slightly greater than 0 and has a sharp fall off around the peak. The logical interpretation of this directly follows from the meaning of the word child. The membership curve for the fuzzy set YOUNG has a peak at age x=25 and falls off very slowly on both sides around the peak. As youth is the most charming period of the human beings, we prefer to call people YOUNG even if they are away from 25 on either side. If the readers' view is different they can allow a sharp falloff of the curve around the age x=25. One interesting point to note about the OLD and VERY OLD membership curves is that OLD curve throughout has a higher membership than the VERY OLD curve until both saturate at age x = 120 years. This is meaningful because if someone is called VERY OLD then he must be OLD, but the converse may not be true.

There are many ways to represent the membership functions shown in Fig. 2.1 by mathematical functions. One such representation is given below:

$$\mu_{CHILD}(x) = ax^2/(1+bx^2 + cx), \ a, b, c>0. \quad (2.14)$$

$$\mu_{YOUNG}(x) = \exp[-(x - 25)^2 / 2\sigma^2], \quad \sigma > 0 \quad (2.15)$$

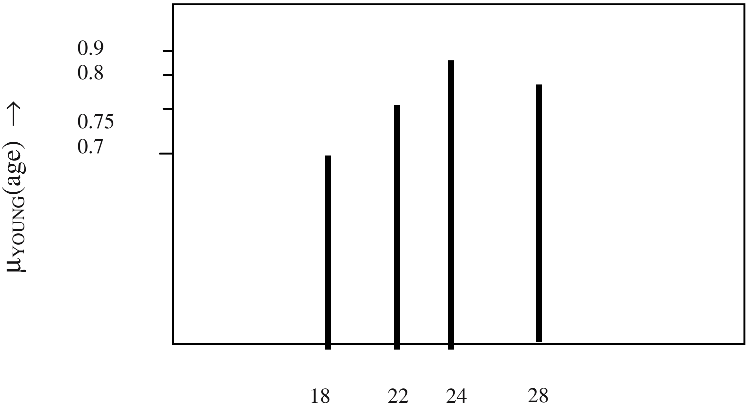
$$\mu_{OLD}(x) = 1 - \exp(-dx^2), \ d>0, \text{ and} \quad (2.16)$$

$$\mu_{VERY\ OLD}(x) = 1 - \exp(-dx), \ d>0. \quad (2.17)$$

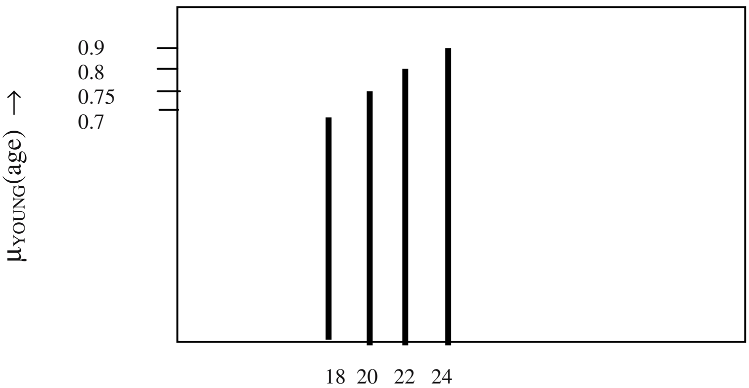
The parameters a, b, c and d in the above membership functions are selected intuitively by the experts based on their subjective judgement in the respective domains. Tuning of these parameters is needed to control the curvature and sharp changes on the curves around some selected x-values.

2.4 Continuous and Discrete Membership Functions

The **universe of discourse** (or simply the **universe)** of a fuzzy set may exist for both discrete and continuous spectrum. For example, the roll number of students in a class is a discrete universe. On the other hand the height of persons is a continuous universe as height may take up any values between 4' to 8'. It may be mentioned here that a continuous universe is sometimes sampled at regular or irregular intervals for using it as a discrete universe. The membership curve of YOUNG in Fig. 2.1 may be, for instance, discretized at age x= 18, 22, 24, 28. This is an example of **non-uniform/ irregular sampling** as the intervals of sampling 18-22, 22-24, 24-28 are unequal. The membership curve of YOUNG may alternatively be sampled at a regular interval of age x=18, 20, 22, 24, say. This is an example of **uniform/ regular** sampling. Fig. 2.2(a) and (b) describe the instances of the non-uniform and uniform sampling of the YOUNG membership curve.



(a)



(b)

Fig. 2.2: (a) Non-uniform and (b) uniform sampling of the YOUNG membership curve.

2.5 Typical Membership Functions

In theory, membership functions usually can take any form. But in most practical applications, triangular, Gaussian (bell-shaped), S-function and γ-functions are commonly used. In this section, these 4 functions are outlined.

2.5.1 The γ-Function

This function has 2 parameters α and β. It is formally defined by

$$\gamma(u; \alpha, \beta) = \begin{cases} 0, & u \le \alpha, \\ (u - \alpha)/(\beta - \alpha) & \alpha < u \le \beta, \\ 1 & u > \beta \end{cases} \quad (2.18)$$

Fig. 2.3 describes the graphical representation of the γ-function.

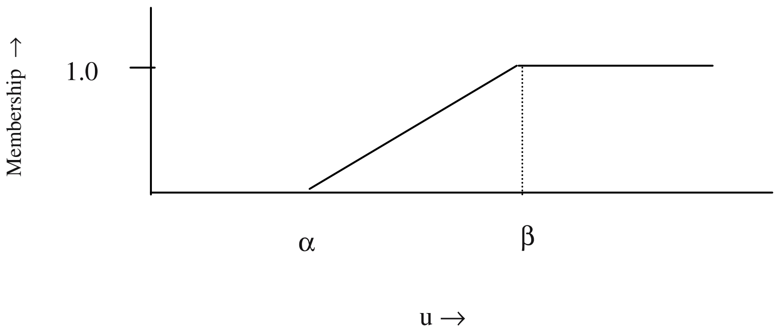


Fig. 2.3: The membership curve for the γ-function.

The membership function OLD in our previous example can be, for instance, described by the γ-function. Suppose we call someone OLD with some positive membership if his age exceeds 60 and call someone OLD with membership =1.0 when his age attains 90. So, the γ-function in the present context should be γ(age;60 ,90).

2.5.2 The s-Function

This function is a smooth version of the γ-function mentioned above. It is formally defined as follows.

$$S(u;\alpha,\beta,\gamma) = \begin{cases} 0, & u \le \alpha \\ 2\,[(u - \alpha)/(\gamma-\alpha)]^2, & \alpha < u \le \beta \\ 1 - 2[(u - \gamma)/((\gamma-\alpha)]^2, & \beta < u \le \gamma \\ 1, & u > \gamma. \end{cases} \quad (2.19)$$

Generally, $\beta=(\alpha + \gamma)/2$ is considered in most applications of fuzzy logic. One typical form of the S-function is presented in Fig. 2.4 below.

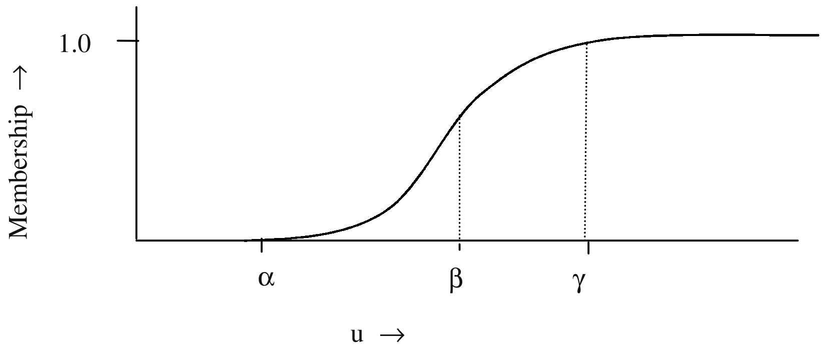


Fig. 2.4: One typical S-function.

The S-function also may be used to represent the OLD membership function. As the slope of the function at u =α is very small, we can select a smaller age u to represent the OLD membership function. S(age; 40,60,90) thus is one choice for the membership function OLD.

2.5.3 The L-Function

This function is the converse of the typical γ-function. It can be mathematically expressed as

$$\left.\begin{aligned} L(u;\alpha,\beta) &= 1, && u<\alpha \\ &= (\alpha - u)/(\beta - \alpha), && \alpha \le u \le \beta \\ &= 0, && u>\beta \end{aligned}\right\} \qquad (2.20)$$

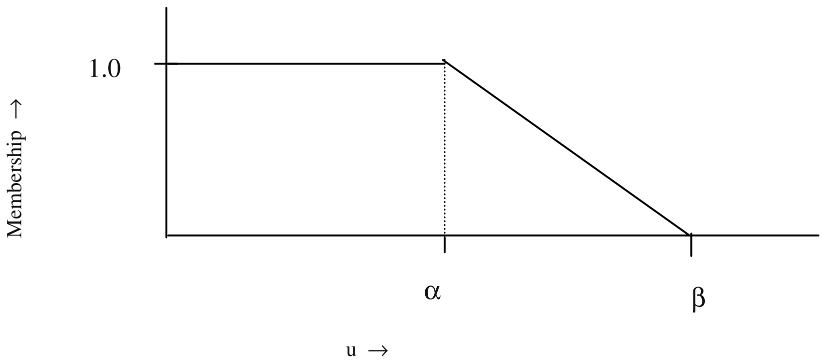


Fig. 2.5: One typical form of the L-function.

One typical form of the L-function is presented in Fig. 2.5. L-functions are generally used to represent the fuzzy linguistic *positive small*. Suppose u is a fuzzy variable which should essentially have a positive value. Now, as u increases its membership should decrease. As a second example, suppose we are interested to describe the average intensity of the pixels (points) in an image by a fuzzy linguistic: *not very dark*. So, until the average intensity exceeds α(=50, say), its membership of being not very dark is 1 and falls off if the average intensity exceeds α.

2.5.4 The Triangular Membership Function

The triangular membership function, also called the bell-shaped function with straight lines, can be formally defined as follows:

$$\Lambda(u;\alpha,\beta,\gamma) = \begin{cases} 0, & u \le \alpha \\ (u-\alpha)/(\beta-\alpha), & \alpha < u \le \beta \\ (\alpha-u)/(\beta-\alpha), & \beta < u \le \gamma \\ 0, & u > \gamma \end{cases} \qquad (2.21)$$

One typical plot of the triangular membership function is given in Fig. 2.6.

The YOUNG membership function, for instance, can be represented by the triangular membership function. We can set age α= 20, β= 25 and γ= 30 as the typical parameters for the YOUNG membership function in order to represent it by a triangular membership function.

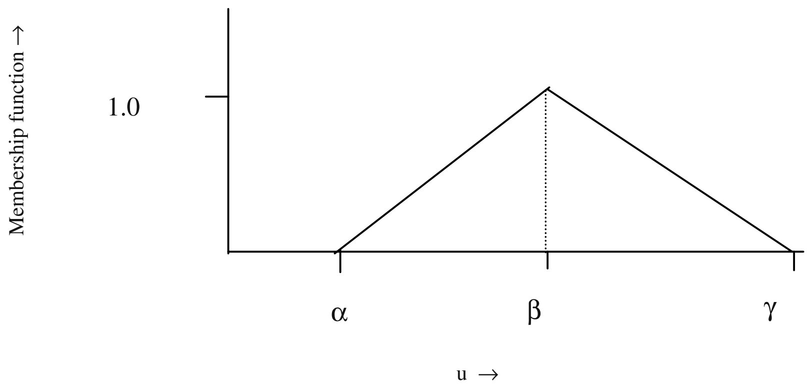


Fig. 2.6: One typical form of the triangular membership function.

2.5.5 The Π-function

The Π-function can be formally described as follows:

$$\Pi(u;\alpha,\beta,\gamma,\delta) = \left.\begin{array}{ll} 0, & u \le \alpha \\ (u-\alpha)/(\beta-\alpha), & \alpha < u \le \beta \\ 1, & \beta < u \le \gamma \\ (\gamma-u)/(\delta-\gamma), & \gamma < u \le \delta \\ 0, & u > \delta. \end{array}\right\} \qquad (2.22)$$

One typical plot of the Π-function is given in Fig. 2.7. The Π-function is used to represent the fuzzy linguistic: *neither so high nor so low*. For example suppose we want to express that today is *neither so hot nor so cold*. This can be represented by a fuzzy membership curve plotted against temperature. It may be noted that for temperature below a threshold th_1 and above a threshold th_2, the membership of the said curve should be close to one and it should have falloffs below th_1 and above th_2. Thus a Π-function is an ideal choice for the representation of the fuzzy linguistic *neither so hot nor so cold.*

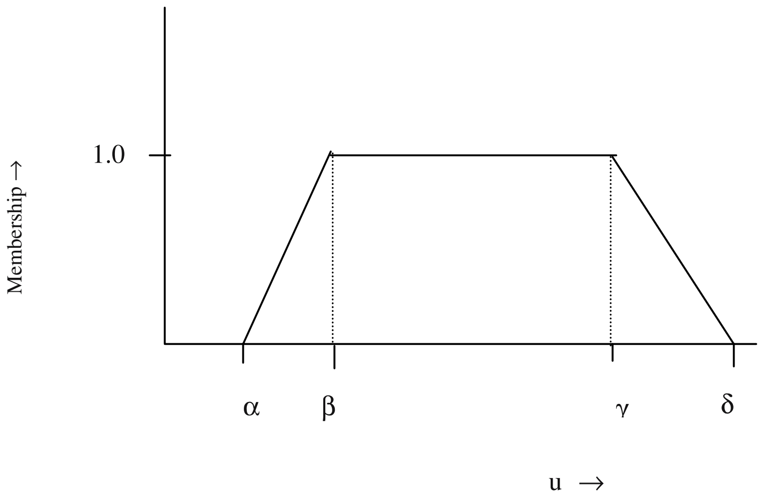


Fig. 2.7: One typical form of the Π-function.

2.5.6 The Gaussian Membership Function

A Gaussian membership function is defined by

$$G(u; m, \sigma) = \exp [- \{(u - m)/\sqrt{2}\sigma\}^2] \quad (2.23)$$

where the parameters m and σ control the center and width of the membership function. A plot of the Gaussian membership function is presented in Fig. 2.8.

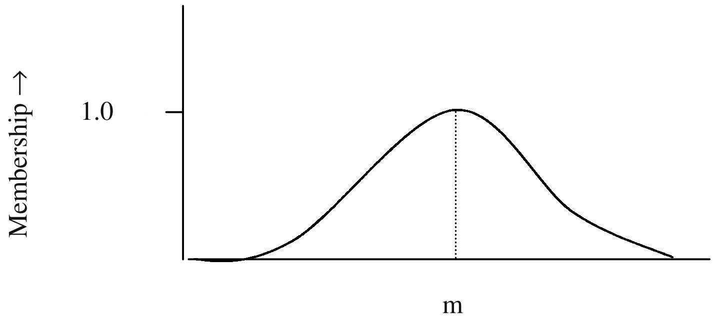


Fig. 2.8: One typical form of the Gaussian function.

The Gaussian membership function has a wide application in the literature on fuzzy sets and systems. The YOUNG membership function illustrated earlier, for instance, can also be described by a Gaussian membership function with mean m=22 years, say. Smaller the value of variance of the curve, higher is its sharpness around the mean.

2.6 Operation on Fuzzy Sets

Unlike conventional sets, the operations on fuzzy sets are usually described with reference to membership functions. Among the common operations on fuzzy sets fuzzy T-norm, fuzzy S-norm and fuzzy complementation need special mention. They are outlined below.

2.6.1 Fuzzy T-Norm

For any 2 fuzzy sets A and B under a common universe of discourse X, the intersection of the fuzzy sets, characterized by a T-norm operator, is given by

$$\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x)). \quad (2.24)$$

For any membership values a, b, c and d, the T-norm operator T can be formally defined as follows.

$T(0, 0)= 0, T(a, 1) =T(1, a)= a$ (boundary)

$T(a, b) \le T (c, d)$ if $a\le c$ and $b \le d$ (monotonicity)

$T(a, b) = T(b, a)$ (commutativity)

$T(a, T(b, c))= T(T(a, b), c)$ (associativity)

Definition 2.3: *A function T: [0, 1] × [0, 1]→ [0, 1] that satisfies the above 4 characteristics is called a T-norm.*

Example2.2: The following are the examples of the typical T-norm function.

a) Minimum: $T_{min} (a, b) = \min(a, b)$ (2.25)

b) Algebraic product: $T_{ap}(a, b) = a.b$ (2.26)

c) Einstein product: $T_{ep}(a, b) = ab/ \{2 - (a + b -ab)\}$ (2.27)

d) Drastic product: $T_{dp}(a, b)= a$ if $b=1$ (2.28)
$=b$ if $a=1$
$=0$ otherwise.

2.6.2 Fuzzy S-Norm

For any 2 fuzzy sets A and B under a common universe X, the union of fuzzy sets, characterized by a S-norm (T-co-norm) operator is given by

$$\mu_{A\cup B}(x) = S(\mu_A(x), \mu_B(x)). \quad (2.29)$$

For any membership values a, b, c and d, the S-norm operator S can be formally defined as follows.

$S(1, 1)= 1, S(a, 0) =S(0, a)= a$ (boundary)

$S(a, b) \leq S (c, d)$ if $a \leq c$ and $b\leq d$ (monotonicity)

$S(a, b) = S(b, a)$ (commutativity)

$S(a, S(b, c))= S(S(a, b), c)$ (associativity)

Definition 2.3: *A function S: [0, 1] × [0, 1]→ [0, 1] that satisfies the above 4 characteristics is called a S-norm.*

Example2.3: The following are the examples of the typical S-norm function.

a) Maximum: $S_{max} (a, b) = \max(a, b)$ (2.30)

b) Algebraic sum: $S_{as}(a, b) = a+ b- ab$ (2.31)

c) Einstein sum: $S_{es}(a, b) = (a + b)/ (1 + ab)$ (2.32)

d) Drastic sum: $S_{ds}(a, b) = a$ if $b=0$ (2.33)
$= b$ if $a=0$
$= 1$ otherwise.

2.6.3 Fuzzy Complement

Given a fuzzy set A under the universal set X, fuzzy complementation over set A is a mapping that transforms the membership function of A into the membership function of the complement of A, denoted by A^c. Mathematically, the complementation function c is given by

$$c[\mu_A(x)] = \mu_{A^c}(x) \qquad (2.34)$$

The fuzzy complementation function should essentially satisfy the following 2 criteria:

1. $c(0) = 1$ and $c(1) = 0$ (boundary condition)

2. For any 2 fuzzy memberships a and b,

if $a < b$ then $c(a) \geq c(b)$ (non-increasing condition)

Definition 2.4: *Any function c: [0, 1] →[0, 1] that satisfies the above 2 characteristics is called fuzzy complementation.*

Example 2.4: The following functions are typical examples of fuzzy complementation.

a) $c[\mu_A(x)] = 1 - \mu_A(x)$ (2.35)

b) $c_\lambda(a) = (1 - a)/ (1 + \lambda a)$ (Sugeno class of complements) (2.36)

where for each value of the parameter λ in $(-1, \infty)$ we obtain a particular fuzzy complement.

c) $c_w(a) = (1 - a^w)^{1/w}$ (Yager class of complements) (2.37)

where for each value of w in $(0, \infty)$ we obtain a particular fuzzy complement.

2.7 Basic Concepts Associated with Fuzzy Sets

This section introduces some elementary concepts associated with fuzzy sets. They are outlined below with examples.

Definition 2.5: *The **support** of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$. Formally,*

$$\text{Support}(A) = \{x \mid \mu_A(x) > 0\} \tag{2.38}$$

Definition 2.6: *The **core** of a fuzzy set A is the set of points x in X such that $\mu_A(x)=1$. Formally,*

$$\text{Core}(A) = \{x \mid \mu_A(x) = 1\} \tag{2.39}$$

Definition 2.7: *A fuzzy set with non-empty core is called **normal**. In other words, A is normal if $\exists x, \mu_A(x) = 1$.*

Definition 2.8: *A **crossover point** denotes a point x in X where $\mu_A(x) = 0.5$. Mathematically,*

$$\text{Crossover}(A) = \{x \mid \mu_A(x) = 0.5\}. \tag{2.40}$$

Definition 2.9: *The **α-cut**, also called **α-level**, of a fuzzy set A is a crisp set denoted by A_α is given by*

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}. \tag{2.41}$$

Definition 2.10: *The **strong α-cut,** also called **the strong α-level**, of a fuzzy set A is a crisp set denoted by A_α^+ is given by*

$$A_\alpha^+ = \{x \mid \mu_A(x) > \alpha\}. \tag{2.42}$$

Example 2.5: For the fuzzy set $A = \{0.1/x_1, 0.5/x_2, 0.7/x_3, 0.9/x_4, 1.0/x_5, 0.5/x_6\}$

Support (A)= $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ since for all the elements $x_1, x_2, x_3, x_4, x_5, x_6$ of set A the membership values are greater than 0.

Core (A) = $\{x_5\}$ since $\mu_A(x_5)=1$.

Crossover point (A) = $\{x_2, x_6\}$ since both at $x= x_2$ and x_6 the membership value is 0.5.

$A_\alpha\big|_{\alpha=0.7} = \{x_1, x_3, x_4, x_5\}$ because for all these elements the membership values are greater than or equal to 0.7.

$A_\alpha^+\big|_{\alpha=0.7} = \{x_1, x_4, x_5\}$ because for all these elements the membership values are greater than 0.7.

Definition 2.11: *A fuzzy set A is convex if and only for any 2 points x_1 and x_2 and a real scalar λ in [0, 1],*

$$\mu_A(\lambda x_1 + (1 - \lambda) x_2) \geq \min [\mu_A(x_1), \mu_A(x_2)] \tag{2.43}$$

Definition 2.12: *A **fuzzy number A** is a fuzzy set on the real line that satisfies the conditions for convexity and normality.*

Definition 2.13: *A fuzzy set A is **symmetric** around a point x= c if*

$\mu_A(c + x) = \mu_A(c - x)$ *for all x belonging to X.*

A Gaussian type membership function, for instance, is symmetric around the mean value of the function.

2.8 Extension Principle of Fuzzy Sets

Let f (.) be a mapping function from fuzzy universal set X to fuzzy universal set Y, and A and B are subsets of X and Y respectively. Let the fuzzy set A be given by

$$A= \{\mu_A (x_1)/ x_1, \mu_A (x_2)/ x_2, \ldots\ldots\ldots, \mu_A (x_n)/ x_n\}. \tag{2.44}$$

If there is a one to one mapping from x_i to $y_i=f(x_i)$ then B is given by

$$B= f(A) =\{\mu_A (x_1)/ f(x_1), \mu_A (x_2)/ f(x_2), \ldots\ldots\ldots, \mu_A (x_n)/ f(x_n)\}$$

$$=\{\mu_A (x_1)/ y_1, \mu_A (x_2)/ y_2, \ldots\ldots\ldots, \mu_A (x_n)/ y_n\}. \tag{2.45}$$

But if many to one mapping exists from set X to Y then a maximum of the memberships of $f(x_i)$, $f(x_j)$,…, $f(x_k)$, where $f(x_i)= f(x_j)=\ldots=f(x_k)$ should be taken. Formally, for many to one mapping from set X to Y,

$$\mu_B(y)= \max\ [\mu_A (x): x \in f^{-1}(y)] \tag{2.46}$$

Example 2.6: The computation of B= f(A) is illustrated in this example. Let A={0.2/ (-1), 0.4/ (-2), 0.6/(1), 0.8/(2), 0.9/ (3)}, and $f(x) = x^2$. Here, since f(-1)= f(1)= 1, and f(- 2) = f(2)=4,

$$\mu_B(f(-1))= \mu_B(f(1)) = \max\ [\mu_A (x)\,|_{x=1}, [\mu_A (x)\,|_{x=-1}]$$

$$= \max[0.6, 0.2]$$

$=0.6.$

Similarly, $\mu_B(f(-2))= \mu_B(f(2))= \max [0.4, 0.8]$

$=0.8.$

Consequently, $B = f(A) =\{0.6/(1^2), 0.8/(2^2), 0.9/(3^2)\}$

$=\{0.6/1, 0.8/4, 0.9/9\}.$

So far we discussed mapping from a one dimensional space X to another one dimensional space Y. In general the mapping can be defined from an n dimensional product space $X_1 \times X_2 \times X_3 \times \ldots \times X_n$ to a single universe Y. Here $X_1, X_2, \ldots, X_n$ denote fuzzy universes. The mapping function in the present context is denoted by $f(x_1, x_2, \ldots, x_n)$, where $x_1 \in X_1$, $x_2 \in X_2$, …., and $x_n \in X_n$. If $A_1, A_2, \ldots, A_n$ are n fuzzy sets in $X_1, X_2, \ldots$ and X_n respectively, then extension principle asserts that the fuzzy set B induced by the mapping from $A_1, A_2, \ldots, A_n$ is given by

$B=\{\mu_B(y)/ y$, where $y = f(x_1, x_2, \ldots, x_n)\}$ with

$$\mu_B(y)= \max_{(x_1,x_2,\ldots,x_n)\in f^{-1}(y)} [\min \{ \mu_{A1}(x_1), \mu_{A2}(x_2),\ldots, \mu_{An}(x_n),], \text{ if } f^{-1}(y) \neq \text{null set},$$

$$= 0, \text{ otherwise.} \qquad (2.47)$$

Example 2.7: In this example we illustrate the extension principle for a function f of 2 variables x_1 and x_2. Let $f(x_1, x_2) = x_1+x_2$, and

$A_1=\{0.2/-1, 0.4/0, 0.6/1\}$ and $A_2=\{0.8/-1, 0.6/0, 0.7/1\}$.

Here, $X_1= \{-1, 0, 1\}$ and $X_2= \{-1, 0, 1\}$ as well.

Thus $X_1 \times X_2 =((-1,-1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1,-1), (1,0), (1,1)\}$

Consequently, $f(-1,0)= f(0, -1)= 0+(-1) =-1,$

$f(1, -1)= f(-1, 1)= -1 +1 =0,$

$f(1,0)= f(0, 1) =0 +1=1,$

$f(-1,-1) =-2, f(0, 0) =0, f(1,1) =2.$

So, $\mu_B(y)$ at $y= f(-1,0)$

$=\mu_B(y)$ at y= f(0, -1)

=max{min(0.2, 0.6), min (0.4, 0.8)}=0.4.

Similarly, $\mu_B(y)$ at y= f(1,-1)

$=\mu_B(y)$ at y= f(-1, 1)

=max{min(0.6, 0.8), min (0.2, 0.7)}=0.6, and

$\mu_B(y)$ at y= f(1,0)

$=\mu_B(y)$ at y= f(0, 1)

=max{min(0.6, 0.6), min (0.4, 0.7)}=0.6, and

$\mu_B(y)$ at y= f(-1,-1)

= min (0.2, 0.8)= 0.2,

$\mu_B(y)$ at y= f(0,0)

=min(0.4, 0.6) =0.4,

and $\mu_B(y)$ at y= f(1,1)

=min (0.6, 0.7) =0.6.

Consequently, B ={0.4/ f(-1, 0), 0.6/ f(-1, 1), 0.6/ f(1, 0),0.2/ f(-1, -1), 0.4/ f(0,0), 0.6/f(1,1)}= {0.4/-1, 0.6/0, 0.6/1, 0.2/-2, 0.4/0, 0.6/2}

2.9 Fuzzy Relations

Let X and Y be two arbitrary universal sets in the real plane. A fuzzy relation between sets X and Y is given by

$$R(x, y)= \{\mu_R(x, y)/(x, y) \mid (x, y) \in X \times Y\} \tag{2.48}$$

where $\mu_R(x, y)$ denotes the membership of relation R(x, y). The following example illustrates a fuzzy relation.

Example 2.8: Let X={x_1, x_2, x_3} and Y= {y_1, y_2} and suppose we are interested in determining a fuzzy relation R(x, y), where the distance between x and y is close to zero for all $x \in X$ and for all $y \in Y$. One typical function that can describe this is

$$R(x, y) = \exp [- (x-y)^2]. \qquad (2.49)$$

How does the above expression describe the fuzzy relation 'x is close to y'? The answer to this follows from the explanation below.

When x =y, R(x, y)=1; and when the difference between x and y is very large R(x, y) approaches 0. Thus for a small difference between x and y the R(x, y) is large and close to 1, and for a large difference between x and y the R(x, y) is close to 0. To be specific, let X= {1, 2, 3} and Y = {1, 2}. Then we can express R(x, y) by

$$R(x, y)= \{\exp[-(1-1)^2]/ (1, 1),\ \exp[-(1-2)^2]/ (1,2), \exp[-(2-1)^2]/(2,1),$$

$$\exp[-(2 -2)^2]/ (2,2), \exp[-(3 -1)^2/ (3,1), \exp[-(3 -2)^2/(3,2)\}$$

$$=\{1.0/(1,1), 0.43/(1,2), 0.43/ (2,1), 1.0/ (2,2), 0.16/(3, 1), 0.43/(3,2)\}.$$

Generally, a relational matrix is used to describe a fuzzy relation. For instance, the fuzzy relation: 'x is close to y' can be described as

y →

x ↓

R(x, y)=

	1	**2**
1	1.0	0.43
2	0.43	1.0
3	0.16	0.43

The membership values $\mu_R(x, y)$ in the matrix is shown by faint numbers and the x- and y- values are denoted by bold numbers. Representation of a fuzzy relation by matrices has many advantages to be explored gradually.

The fuzzy relation introduced above represents relationship between 2 fuzzy sets. So, it is called *binary fuzzy relation.* A generalized fuzzy relation on the other hand represents relationship among many fuzzy sets. A formal definition of a (generalized) fuzzy relation is presented below.

Definition 2.14: A **fuzzy relation** is a fuzzy set defined in the Cartesian product of crisp sets $X_1, X_2,\ldots, X_n$. A fuzzy relation $R(x_1,x_2,\ldots,x_n)$ thus is defined as

$$R(x_1,x_2,,\ldots,x_n)$$

$$= \{\mu_R(x_1,x_2,,\ldots,x_n)/(x_1,x_2,,\ldots,x_n) \mid (x_1,x_2,,\ldots,x_n) \in X_1 \times X_2 \times \ldots \times X_n\} \quad (2.50)$$

where μ_R: $X_1 \times X_2 \times \ldots \times X_n \rightarrow [0, 1]$.

A binary fuzzy relation is a special case of the generalized fuzzy relation where in stead of n universes we need only 2 universes.

2.10 Projection of Fuzzy Relations

Let us first consider a binary fuzzy relation R(x, y) defined on the Cartesian product of the universes X and Y. The projection of R(x, y) on X, denoted by R1 is given by

$$\mu_{R1}(x) = \max_{y \in Y} [\mu_R(x, y)] \quad (2.51)$$

Similarly, the projection of R(x, y) on Y, denoted by R2 is given by

$$\mu_{R2}(y) = \max_{x \in X} [\mu_R(x, y)] \quad (2.52)$$

Example 2.9: This example illustrates the principle of projection of a fuzzy relation R(x, y) on 2 fuzzy universes X and Y. We take the fuzzy relation constructed in the last section. The projection of R(x, y) on X universe is computed by determining the maximum element in each row, and its dimension will be same as the number of columns in R. The projection of R(x,y) on Y universe is also computed similarly by determining the maximum element in each column of R and its dimension will be same as the number of rows in R.

x↓ \ y→	1	2
1	1.0	0.43
2	0.43	1.0
3	0.16	0.43

1.0
1.0
0.43

X-projection

1.0	1.0

Y-projection

Fig. 2.9: Illustrating X- and Y- projection of a fuzzy relation.

Until now we considered projection of a binary fuzzy relation. Now, we extend the principle of projection for generalized fuzzy relations.

Definition 2.15: ***Projection of a fuzzy relation*** *$R(x_1, x_2,\ldots, x_n)$ on to $X_i \times X_j \times_{\ldots} \times X_k$ for any i, j and k in[1, n] is defined as a fuzzy relation R_p where*

$$R_p(x_i, x_j,\ldots,x_k) = \{ \max_{x_i \in X_i,\, x_j \in X_j,\ldots,x_k \in X_k} \mu_{Rp}(x_i, x_j,\ldots,x_k)/ (x_i, x_j,\ldots,x_k)\}. \quad (2.53)$$

2.11 Cylindrical Extension of Fuzzy Relations

Informally, cylindrical extension from a X-projection means filling all the columns of the relational matrix by the X-projection. Similarly cylindrical extension from a Y-projection means filling all the rows of the relational matrix by the Y-projection. Examples illustrating the construction of cylindrical extension from the X- and the Y-projections are given in Fig. 2.10 and Fig. 2.11 respectively.

1.0	1.0
1.0	1.0
0.43	0.43

Cylindrical extension

1.0
1.0
0.43

X-projection

Fig. 2.10: Construction of cylindrical extension from X-projection.

Cylindrical extension

1.0	1.0
1.0	1.0
1.0	1.0

1.0	1.0

Y-projection

Fig. 2.11: Construction of cylindrical extension from the Y-projection.

A cylindrical extension of a fuzzy relation R1(x) thus can be defined on X ×Y as a binary fuzzy relation given by

$$R1_c(x, y)= \{\mu_{R1}(x)/ (x, y)\}. \quad (2.54)$$

The suffix c of R1 in the above expression denotes its cylindrical extension. A cylindrical extension from a projected fuzzy relation of m dimension, m > 1 is presented below.

Definition 2.16: *Let R1 be a fuzzy relation in $X_i \times X_j \times_{....} \times X_k$ for any (i, j, …,k) in [1, n]. The cylindrical extension of R1 to $X_1 \times X_2 \times_{....} \times X_n$ is a fuzzy relation $R1_c$ given by*

$$R1_c(x_1, x_2, \ldots, x_n) =\{\mu_{R1}(x_1, x_2,\ldots,x_n)/ (x_1, x_2,\ldots,x_n)\} \quad (2.55)$$

2.12 Fuzzy Max-Min and Max-Product Composition Operation

Let us consider 2 fuzzy relations R_1 and R_2 defined on X × Y and Y × Z respectively. The *max-min composition* of R_1 and R_2 is a fuzzy set defined by

$R_3 = R_1 \text{ o } R_2$

$=\{\mu_{R3}(x, z)/ (x, z)\}$

where $\mu_{R3}(x, z)= \max_y \{\min (\mu_{R1}(x, y), \mu_{R2}(y, z)) \mid x\in X, y\in Y, z,\in Z\}$. (2.56)

When expressed as relational matrices, the computation of R_1 o R_2 is straightforward like matrix multiplication with the replacement of sum by max and product by min operators. The following example illustrates the computation process of the max-min composition operation.

Example 2.10: Let X={x_1, x_2, x_3}, Y={y_1, y_2} and Z={z_1, z_2} be three universe of discourses and R_1 (x, y) and R_2 (y, z) are two fuzzy relations on X × Y and Y × Z respectively. Suppose R_1 and R_2 denote 'x is close to y' and 'y is close to z' respectively. Given R_1 and R_2 as follows we are interested to compute the relation R_3 that denotes ' x is close to z'.

Let $R_1 = \begin{pmatrix} 0.1 & 0.2 \\ 0.4 & 0.5 \\ 0.7 & 0.8 \end{pmatrix}$ and $R_2 = \begin{pmatrix} 0.9 & 0.8 \\ 0.7 & 0.6 \end{pmatrix}$

Then $R_3 = R_1 o R_2$

$$= \begin{pmatrix} \max\{\min(0.1, 0.9), \min(0.2,0.7)\} & \max\{\min(0.1, 0.8), \min(0.2,0.6)\} \\ \max\{\min(0.4, 0.9), \min(0.5, 0.7)\} & \max\{\min(0.4, 0.8), \min(0.5, 0.6)\} \\ \max\{\min(0.7, 0.9), \min(0.8, 0.7)\} & \max\{\min(0.7, 0.8), \min(0.8, 0.6)\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 & 0.2 \\ 0.5 & 0.5 \\ 0.7 & 0.7 \end{pmatrix}$$

It may be added that the (i, j)th element of the relational matrix R_3 indicates the membership of closeness of x_i and z_j. By notation R_3 (i, j)= $\mu_{R3}(x_i, z_j)$.

It is interesting to note that the max-min composition operation satisfies the following properties. Let P, Q and R be 3 relational matrices defined on $X \times Y$, $Y \times Z$ and $Z \times W$ respectively. Also assume that the matrices composed by o operator have dimensional compatibility.

$P \text{ o } (Q \text{ o } R) = (P \text{ o } Q) \text{ o } R$ (*associative*) (2.57)

$P \text{ o } (Q \cup R) = (P \text{ o } Q) \cup (P \text{ o } R)$ (*distributive over union*) (2.58)

$P \text{ o } (Q \cap R) \subseteq (P \text{ o } Q) \cap (P \text{ o } R)$ (*weakly distributive over intersection*) (2.59)

$Q \subseteq R \Rightarrow P \text{ o } Q \subseteq P \text{ o } R$ (*monotonic*) (2.60)

Besides max-min composition, max-product composition operator is also prevalent in the literature of fuzzy sets. A formal definition of max-product composition operation is introduced below.

Given 2 fuzzy relations R_1 (x, y) and R_2 (y, z) defined on $X \times Y$ and $Y \times Z$ respectively. The fuzzy relation R_3 (x, z) defined on $X \times Z$, can be obtained by max-product composition operator as outlined below:

$R_3(x, z) = \{\mu_{R3}(x, z)/ (x, z)\}$

$$\text{where } \mu_{R3}(x, z) = \max_{y}[\mu_{R1}(x, y) * \mu_{R2}(y, z)] \tag{2.61}$$

The asterisk (*) in the last expression denotes algebraic multiplication. The max-product composition between R_1 and R_2 is symbolically denoted in this book by an asterisk. So, R_3 can be written as $R_1 * R_2$. The computation of R_3 is illustrated below in example 2.9.

Example 2.11: Assuming same R_1 and R_2 as in example 2.10, we evaluate R_3 in this example by max-product composition. Replacing the min by product in the relational matrix R_3 (vide example 2.10) we find:

$R_3 = R_1 * R_2$

$$= \begin{pmatrix} \max\{(0.1 * 0.9), (0.2 * 0.7)\} & \max\{(0.1 * 0.8), (0.2 * 0.6)\} \\ \max\{(0.4 * 0.9), (0.5 * 0.7)\} & \max\{(0.4 * 0.8), (0.5 * 0.6)\} \\ \max\{(0.7 * 0.9), (0.8 * 0.7)\} & \max\{(0.7 * 0.8), (0.8, * 0.6)\} \end{pmatrix}$$

$$= \begin{pmatrix} 0.14 & 0.12 \\ 0.36 & 0.32 \\ 0.63 & 0.56 \end{pmatrix}.$$

This is all about the computation of R_3.

2.13 Fuzzy Linguistic Hedges

Fuzzy systems are capable of representing sentences containing terms 'more or less', 'about to', 'very slow', ' a little way' and the like. These are called *linguistic hedges*. Representation of the linguistic hedges by membership functions is necessary for modeling fuzzy systems. In this section we first define *linguistic variables* and then introduce linguistic hedges by membership functions.

Definition 2.17: *A **linguistic variable** is characterized by a 4-tuple: <x, LV, DR, μ > where*

x is the name of the linguistic variable,

LV is the linguistic values that x can take,

DR is the dynamic range of the linguistic variable,

and μ is the membership function of the linguistic variable x for all linguistic values supplied in LV.

Example 2.12: The linguistic variable defined above is illustrated in this example. Let *age* be a linguistic variable. So, by our definition x = age. Let us assume that age can assume the following linguistic values: YOUNG, OLD, VERY-OLD etc. So, LV={YOUNG, OLD, VERY-OLD}. The dynamic range DR for age is [0, 120] years. We also need to consider the membership functions of age in the fuzzy sets YOUNG, OLD and VERY-OLD. In other words, the membership functions μ_{YOUNG}(age), μ_{OLD}(age) and $\mu_{VERY\text{-}OLD}$(age) should be defined for the range of age in [0, 120] years.

Given a linguistic variable x= age say. We can qualify YOUNG or OLD further by VERY-YOUNG, VERY-VERY-YOUNG, MORE-OR-LESS-YOUNG, VERY-OLD, NOT-SO-OLD, NOT-SO-YOUNG etc. How can we represent these linguistic hedges by membership functions?

Suppose we know the membership function μ_{YOUNG} (age), the membership function VERY-YOUNG then can be defined as follows:

$$\mu_{VERY\text{-}YOUNG}\,(age) = [\mu_{YOUNG}\,(age)]^2. \tag{2.62}$$

Similarly, we can define the membership functions VERY-VERY-YOUNG as

$$\mu_{VERY\text{-}VERY\text{-}YOUNG}\,(age) = [\mu_{YOUNG}\,(age)]^3 \tag{2.63}$$

and MORE-OR-LESS-YOUNG as

$$\mu_{MORE\text{-}OR\text{-}LESS\text{-}YOUNG}\,(age) = [\mu_{YOUNG}\,(age)]^{0.5}. \tag{2.64}$$

The following definitions are generally used to represent linguistic hedges by membership functions.

Definition 2.18: *Let $\mu_A(x)$ be membership function of a linguistic variable x in a fuzzy set A. The operation* ***concentration*** *(**CON**) and* ***dilation*** *(**DIL**) are then defined by the following membership functions:*

$$\mu_{CON\text{-}A}(x) = [\mu_A(x)]^2 \tag{2.65}$$

$$\mu_{DIL\text{-}A}(x) = [\mu_A(x)]^{0.5}. \tag{2.66}$$

Fuzzy linguistic hedges such as VERY or VERY-VERY can be best described by the concentration operation. On the other hand, linguistic hedges MORE-OR-LESS, AROUND etc. are usually denoted by the dilation operation.

Another interesting operation, well-known as **contrast intensification** increases the values of $\mu_A(x)$, when it is above 0.5 and diminishes those which are below 0.5. The following definition presents one way of contrast intensification.

Definition 2.19: *The operation* ***contrast intensification*** *on a linguistic value A is defined by the following membership function.*

$$\left.\begin{aligned} \mu_{INT\text{-}A}(x) &= 2\,[\mu_A(x)]^2 \text{ for } 0 \le \mu_A(x) < 0.5, \\ &= 1 - 2(1 - \mu_A(x))^2 \text{ for } 0.5 \le \mu_A(x) \le 1, \end{aligned}\right\} \tag{2.67}$$

Membership functions of the linguistic variables in a fuzzy set are generally constructed in a manner so as to satisfy the criteria of orthogonality, presented below.

Definition 2.20: *Let $T=\{t_1, t_2, \ldots., t_n\}$ be a term set of a linguistic variable x on the universe X. The term set T is called orthogonal if it satisfies the following criterion:*

$$\sum_{i=1}^{n} \mu_{ti}(x) = 1, \ \forall x \in X, \tag{2.68}$$

where t_i 's are convex and normal fuzzy sets defined on X.

Example 2.13: Consider a fuzzy set AGE. Let CLOSE-TO-18 and CLOSE-TO-20 and CLOSE-TO-22 be 3 fuzzy linguistic values of the variable age in the range [18, 22] years. We now construct the membership functions of these fuzzy sets in a manner such that sum of the membership of these fuzzy sets over the age variable is 1 and the fuzzy sets are convex. One way of constructing the above fuzzy sets is illustrated below.

$\mu_{CLOSE\text{-}TO\text{-}18}$(age) ={1.0/18, 0.0/20, 0.0/22}

$\mu_{CLOSE\text{-}TO\text{-}20}$(age)= { 0.0/18, 1.0/20, 0.0/22} and

$\mu_{CLOSE\text{-}TO\text{-}22}$(age)={ 0.0/18, 0.0/20, 1.0/22}

It may be noted that all the above fuzzy sets are convex since each membership function has a single rising peak. Further, sum of the memberships at any age is always 1.

2.14 Summary

The chapter presented an introduction to fuzzy sets, fuzzy relations and a few important fuzzy operators and concepts. Among the fuzzy operators, t-norm and s-norm have massive applications in fuzzy reasoning systems. The concept of projection of fuzzy relations also has significant applications in the logic of fuzzy sets. The chapter also introduced different types of membership functions. Selection of these functions in a particular application calls for a domain specific knowledge of the users. For example, the YOUNG fuzzy set can be represented by a Gaussian type membership function, whereas the fuzzy set MODERATE-SPEED of a mechanical moving system can be described by a trapezoidal membership function.

Linguistic variables are of a great concern in designing a fuzzy system. Design of an orthogonal set of linguistic terms is usually very difficult. So, in most cases a near orthogonal linguistic set of terms, where the sum of membership at all values of the dynamic range of the linguistic variables is approximately equal to one, is preferred.

Exercise

1. According to Ohm's law the current passing through a resistive device causes a potential drop across the device, and the drop thus obtained is proportional to the amplitude of the current.

   Assume that we have 3 ammeters and 2 voltmeters, which are used in 6 possible combinations to measure the resistance. Let the readings obtained be denoted by a (I, V) pair in mA and volts respectively, where

   $(I_1, V_1) = (1,10)$, $(I_2, V_2) = (1.1,10.1)$, $(I_3, V_3)= (0.9, 9.1)$,

(I_4, V_4)= (0.95, 9.52), (I_5, V_5)= (1.15, 11.6) and (I_6, V_6)= (1.2, 11,2).

Let the absolute value of the resistance identified from its color code be 10K-ohm. Design a fuzzy set that describes GOOD-MEASUREMENT.

[**Hints:** Compute $R_i = V_i/I_i$ for i = 1 to 6. Determine the absolute value of the deviations abs(R_i- $R_{theoretical}$). Then the measurement is good for those i where the absolute value of deviation is small, and those measurements should have a membership close to 1. When the absolute value of deviation is large, the membership should be small. One membership function is given below.

$$\mu_{GOOD\text{-}MEASUREMENT}\,(R_i)$$

$$= 1 - \{abs(R_i - R_{theoretical}) \,/\, \{\max_i abs(R_i - R_{theoretical})\}\}]$$

2. Apply dilation and concentration on the constructed fuzzy set in Exercise 1 to determine the fuzzy sets MORE-OR-LESS-GOOD-MEASUREMENT and VERY-GOOD-MEASUREMENT.

3. Determine the membership distribution of the fuzzy set NEITHER-VERY-GOOD-NOR-VERY-POOR-MEASUREMENT for the problem given in Exercise 1.

   [**Hints:** Try with max{(1-$\mu_{VERY\text{-}GOOD}(R_i)$), (1-$\mu_{VERY\text{-}POOR}(R_i)$)}.]

4. Given a fuzzy set A={0.1/1, 0.2/-1, 0.2/2, 0.4/-2, 0.3/3}. Also given a function f(x) = x^2. Determine B = f(A) by usng the extension principle.

   [**Hints:** B=f(A)= { max(0.1, 0.2)/ 1^2 , max(0.2, 0.4)/ 2^2, 0.3/3^2}
   ={0.2/1, 0.4/4, 0.3/9}]

5. Given f(x)= x^2 +4, find f(A) by extension principle for the following fuzzy set A= {0.1/2, 0.3/-2, 0.6/-3}.

   [**Answer:** f(A)= {0.3/ 8, 0.6/ 13}]

6. Find the X- and Y-projections of the following relational matrix.

x ↓ y→

R=		1	2	3
	5	0.8	0.9	0.6
	6	0.2	0.4	0.7
	7	0.1	0.2	0.5

[**Answer:** X-projection = {0.9/5, 0.7/6, 0.5/7} and
Y-projection ={0.8/1, 0.9/2, 0.7/3}.]

7. Verify with an example of two (3 × 3) fuzzy relational matrices that DeMorgan's theorem presented below holds good for the matrices:

$$(R_1 \text{ o } R_2)^c = (R_1^c \; \Phi \; R_2^c)^c,$$

where R_1 and R_2 are 2 relational matrices of compatible order, c denotes the one's complementation operation over the elements of the matrix; o and **Φ** denote max-min composition and min-max composition operator respectively. The min-max composition operator is applied like max-min composition with the replacement of max by min and min by max.

8. x and y are two fuzzy linguistic variables in the same universe U. We define two fuzzy relations to represent that 'x is close to y'. Which of these two relations can represent a sharp estimation of CLOSENESS of x and y?

a) $R1_{\text{CLOSE-TO}}(x, y) = \exp[-(x - y)^2]$

b) $R2_{\text{CLOSE-TO}}(x, y)=\exp[-\text{abs}(x - y)]$

[**Answer:** (a) because the square term helps falling off the exponential function for a slightly large difference between x and y.]

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