10

Dimensional analysis and law of similarity

The method of dimensional analysis is used in every field of engineering, especially in such fields as fluid dynamics and thermodynamics where problems with many variables are handled. This method derives from the condition that each term summed in an equation depicting a physical relationship must have same dimension. By constructing non-dimensional quantities expressing the relationship among the variables, it is possible to summarise the experimental results and to determine their functional relationship.

Next, in order to determine the characteristics of a full-scale device through model tests, besides geometrical similarity, similarity of dynamical conditions between the two is also necessary. When the above dimensional analysis is employed, if the appropriate non-dimensional quantities such as Reynolds number and Froude number are the same for both devices, the results of the model device tests are applicable to the full-scale device.

10.1 Dimensional analysis

When the dimensions of all terms of an equation are equal the equation is dimensionally correct. In this case, whatever unit system is used, that equation holds its physical meaning. If the dimensions of all terms of an equation are not equal, dimensions must be hidden in coefficients, so only the designated units can be used. Such an equation would be void of physical interpretation.

Utilising this principle that the terms of physically meaningful equations have equal dimensions, the method of obtaining dimensionless groups of which the physical phenomenon is a function is called dimensional analysis.

If a phenomenon is too complicated to derive a formula describing it, dimensional analysis can be employed to identify groups of variables which would appear in such a formula. By supplementing this knowledge with experimental data, an analytic relationship between the groups can be constructed allowing numerical calculations to be conducted.

10.2 Buckingham's π theorem

In order to perform the dimensional analysis, it is convenient to use the π theorem. Consider a physical phenomenon having n physical variables v_1, v_2, $v_3, \ldots, v_n$ and k basic dimensions[1] (L, M, T or L, F, T or such) used to describe them. The phenomenon can be expressed by the relationship among $n - k = m$ non-dimensional groups $\pi_1, \pi_2, \pi_3, \ldots, \pi_m$. In other words, the equation expressing the phenomenon as a function f of the physical variables

$$f(v_1, v_2, v_3, \ldots, v_n) = 0 \tag{10.1}$$

can be substituted by the following equation expressing it as a function ϕ of a smaller number of non-dimensional groups:

$$\phi(\pi_1, \pi_2, \pi_3, \ldots, \pi_m) = 0 \tag{10.2}$$

This is called Buckingham's π theorem. In order to produce $\pi_1, \pi_2, \pi_3, \ldots, \pi_m$, k core physical variables are selected which do not form a π themselves. Each π group will be a power product of these with each one of the m remaining variables. The powers of the physical variables in each π group are determined algebraically by the condition that the powers of each basic dimension must sum to zero.

By this means the non-dimensional quantities are found among which there is the functional relationship expressed by eqn (10.2). If the experimental results are arranged in these non-dimensional groups, this functional relationship can clearly be appreciated.

10.3 Application examples of dimensional analysis

10.3.1 Flow resistance of a sphere

Let us study the resistance of a sphere placed in a uniform flow as shown in Fig. 10.1. In this case the effect of gravitational and buoyancy forces will be neglected. First of all, as the physical quantities influencing the drag D of a sphere, sphere diameter d, flow velocity U, fluid density ρ and fluid viscosity μ, are candidates. In this case $n = 5$, $k = 3$ and $m = 5 - 3 = 2$, so the number of necessary non-dimensional groups is two. Select ρ, U and d as the k core physical quantities, and the first non-dimensional group π, formed with D, is

$$\begin{aligned}\pi_1 &= D\rho^x U^y d^z = [LMT^{-2}][L^{-3}M]^x[LT^{-1}]^y[L]^z \\ &= L^{1-3x+y+z}M^{1+x}T^{-2-y}\end{aligned} \tag{10.3}$$

[1] In general the basic dimensions in dynamics are three – length $[L]$, mass $[M]$ and time $[T]$ – but as the areas of study, e.g. heat and electricity, expand, the number of basic dimensions increases.

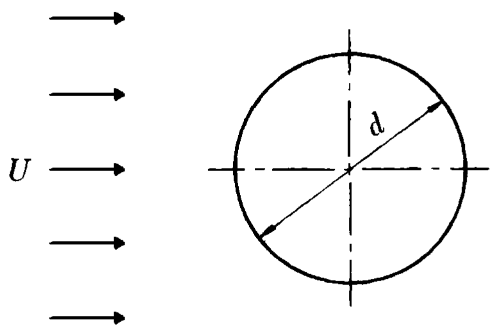


Fig. 10.1 Sphere in uniform flow

i.e.

$$\begin{aligned} L:&\quad 1-3x+y+z=0 \\ M:&\quad 1+x=0 \\ T:&\quad -2-y=0 \end{aligned}$$

Solving the above simultaneously gives

$$x=-1 \qquad y=-2 \qquad z=-2$$

Substituting these values into eqn (10.3), then

$$\pi_1 = \frac{D}{\rho U^2 d^2} \tag{10.4}$$

Next, select μ with the three core physical variables in another group, and

$$\begin{aligned} \pi_2 &= \mu\rho^x U^y d^z = [L^{-1}MT^{-1}][L^{-3}M]^x[LT^{-1}]^y[L]^z \\ &= L^{1-3x+y+z}M^{1+x}T^{-1-y} \end{aligned} \tag{10.5}$$

i.e.

$$\begin{aligned} L:&\quad -1-3x+y+z=0 \\ M:&\quad 1+x=0 \\ T:&\quad -1-y=0 \end{aligned}$$

Solving the above simultaneously gives

$$x=-1 \qquad y=-1 \qquad z=-1$$

Substituting these values into eqn (10.5), then

$$\pi_2 = \frac{D}{\rho U d} \tag{10.6}$$

Therefore, from the π theorem the following functional relationship is obtained:

$$\pi_1 = f(\pi_2) \tag{10.7}$$

Consequently

$$\frac{D}{\rho U^2 d^2} = f\left(\frac{D}{\rho U d}\right) \tag{10.8}$$

In eqn (10.8), since d^2 is proportional to the projected area of sphere $A = (\pi d^2/4)$, and $\rho U d/\mu = Ud/\nu = Re$ (Reynolds number), the following general expression is obtained:

$$D = C_D A \frac{\rho U^2}{2} \tag{10.9}$$

where $C_D = f(Re)$. Equation (10.9) is just the same as eqn (9.4). Since C_D is found to be dependent on Re, it can be obtained through experiment and plotted against Re. The relationship is that shown in Fig. 9.10. Even through this result is obtained through an experiment using, say, water, it can be applied to other fluids such as air or oil, and also used irrespective of the size of the sphere. Furthermore, the form of eqn (10.9) is always applicable, not only to the case of the sphere but also where the resistance of any body is studied.

10.3.2 Pressure loss due to pipe friction

As the quantities influencing pressure loss $\Delta p/l$ per unit length due to pipe friction, flow velocity v, pipe diameter d, fluid density ρ, fluid viscosity μ and pipe wall roughness ε, are candidates. In this case, $n = 6$, $k = 3$, $m = 6 - 3 = 3$.

Obtain π_1, π_2, π_3 by the same method as in the previous case, with ρ, v and d as core variables:

$$\pi_1 = \frac{\Delta p}{l}\rho^x v^y d^z = [L^{-3}F][L^{-4}FT^2]^x[LT^{-1}]^y[L]^z = \frac{\Delta p}{l}\frac{d}{\rho v^2} \tag{10.10}$$

$$\pi_2 = \mu\rho^x v^y d^z = [L^{-2}FT][L^{-4}FT^2]^x[LT^{-1}]^y[L]^z = \frac{\mu}{\rho v d} \tag{10.11}$$

$$\pi_3 = \varepsilon\rho^x v^y d^z = [L][L^{-4}FT^2]^x[LT^{-1}]^y[L]^z = \frac{\varepsilon}{d} \tag{10.12}$$

Therefore, from the π theorem, the following functional relationship is obtained:

$$\pi_1 = f(\pi_2, \pi_3) \tag{10.13}$$

and

$$\frac{\Delta p}{l}\frac{d}{\rho v^2} = f\left(\frac{\mu}{\rho v d}, \frac{\varepsilon}{d}\right)$$

That is,

$$\Delta p = \frac{l}{d}\rho v^2 f\left(\frac{\mu}{\rho v d}, \frac{\varepsilon}{d}\right) \tag{10.14}$$

The loss of head h is as follows:

$$h = \frac{\Delta p}{\rho g} = f\left(\frac{1}{Re}, \frac{\varepsilon}{d}\right)\frac{l}{d}\frac{v^2}{2g} = \lambda\frac{l}{d}\frac{v^2}{2g} \tag{10.15}$$

where $\lambda = f(Re, \varepsilon/d)$. Equation (10.15) is just the same as eqn (7.4), and λ can be summarised against Re and ε/d as shown in Figs 7.4 and 7.5.

10.4 Law of similarity

When the characteristics of a water wheel, pump, boat or aircraft are obtained by means of a model, unless the flow conditions are similar in addition to the shape, the characteristics of the prototype cannot be assumed from the model test result. In order to make the flow conditions similar, the respective ratios of the corresponding forces acting on the prototype and the model should be equal. The forces acting on the flow element are due to gravity F_G, pressure F_P, viscosity F_V, surface tension F_T (when the prototype model is on the boundary of water and air), inertia F_I and elasticity F_E.

The forces can be expressed as shown below.

gravity force	$F_G = mg = \rho L^3 g$
pressure force	$F_P = (\Delta p)A = (\Delta p)L^2$
viscous force	$F_V = \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)A = \mu\left(\frac{v}{L}\right)L^2 = \mu v L$
surface tension force	$F_T = TL$
inertial force	$F_I = m\alpha = \rho L^3\frac{L}{T^2} = \rho L^4 T^{-2} = \rho v^2 L^2$
elasticity force	$F_E = KL^2$

Since it is not feasible to have the ratios of all such corresponding forces simultaneously equal, it will suffice to identify those forces that are closely related to the respective flows and to have them equal. In this way, the relationship which gives the conditions under which the flow is similar to the actual flow in the course of a model test is called the law of similarity. In the following section, the more common force ratios which ensure the flow similarity under appropriate conditions are developed.

10.4.1 Non-dimensional groups which determine flow similarity

Reynolds number

Where the compressibility of the fluid may be neglected and in the absence of a free surface, e.g. where fluid is flowing in a pipe, an airship is flying in the air (Fig. 10.2) or a submarine is navigating under water, only the viscous force and inertia force are of importance:

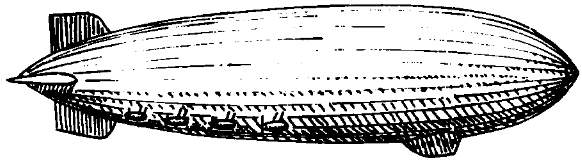


Fig. 10.2 Airship

$$\frac{\text{inertia force}}{\text{viscous force}} = \frac{F_I}{F_V} = \frac{\rho v^2 L^2}{\mu v L} = \frac{L v \rho}{\mu} = \frac{L v}{\nu} = Re$$

which defines the Reynolds number Re,

$$Re = Lv/\nu \tag{10.16}$$

Consequently, when the Reynolds numbers of the prototype and the model are equal the flow conditions are similar. Equations (10.16) and (4.5) are identical.

Froude number

When the resistance due to the waves produced by motion of a boat (gravity wave) is studied, the ratio of inertia force to gravity force is important:

$$\frac{\text{inertia force}}{\text{gravity force}} = \frac{F_I}{F_G} = \frac{\rho v^2 L^2}{\rho L^3 g} = \frac{v^2}{gL}$$

In general, in order to change v^2 above to v as in the case for Re, the square root of v^2/gL is used. This square root is defined as the Froude number Fr,

$$Fr = \frac{v}{\sqrt{gL}} \tag{10.17}$$

If a test is performed by making the Fr of the actual boat (Fig. 10.3) and of the model ship equal, the result is applicable to the actual boat so far as the wave resistance alone is concerned. This relationship is called Froude's law of similarity. For the total resistance, the frictional resistance must be taken into account in addition to the wave resistance.

Also included in the circumstances where gravity inertia forces are

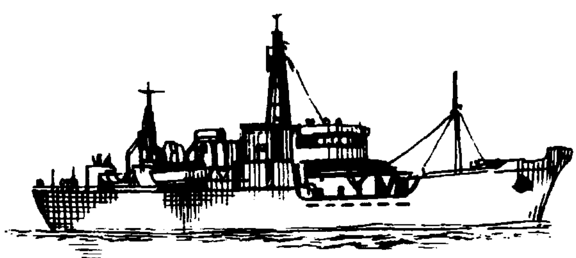


Fig. 10.3 Ship

important are flow in an open ditch, the force of water acting on a bridge pier, and flow running out of a water gate.

Weber number

When a moving liquid has its face in contact with another fluid or a solid, the inertia and surface tension forces are important:

$$\frac{\text{inertia force}}{\text{surface tension}} = \frac{F_I}{F_T} = \frac{\rho v^2 L^2}{TL} = \frac{\rho v^2 L}{T}$$

In this case, also, the square root is selected to be defined as the Weber number We,

$$We = v\sqrt{\rho L/T} \tag{10.18}$$

We is applicable to the development of surface tension waves and to a poured liquid.

Mach number

When a fluid flows at high velocity, or when a solid moves at high velocity in a fluid at rest, the compressibility of the fluid can dominate so that the ratio of the inertia force to the elasticity force is then important (Fig. 10.4):

$$\frac{\text{inertia force}}{\text{elastic force}} = \frac{F_I}{F_E} = \frac{\rho v^2 L^2}{KL} = \frac{v^2}{K/\rho} = \frac{v^2}{a^2}$$

Again, in this case, the square root is selected to be defined as the Mach number M,

$$M = v/a \tag{10.19}$$

$M < 1$, $M = 1$ and $M > 1$ are respectively called subsonic flow, sonic flow and supersonic flow. When $M = 1$ and $M < 1$ and $M > 1$ zones are coexistent, the flow is called transonic flow.

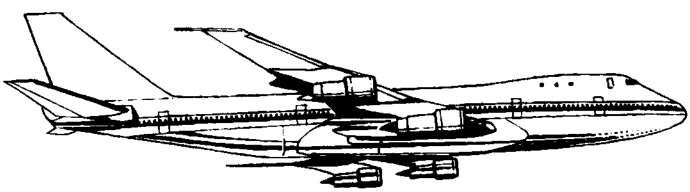
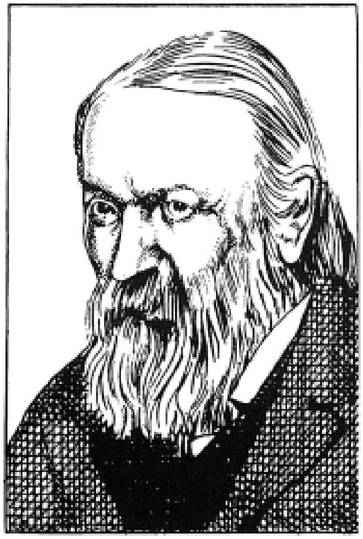


Fig. 10.4 Boeing 747: full length, 70.5 m; full width, 59.6 m; passenger capacity, 498 persons; turbofan engine and cruising speed of 891 km/h ($M = 0.82$)

10.4.2 Model testing

From such external flows as over cars, trains, aircraft, boats, high-rise buildings and bridges to such internal flows as in tunnels and various machines like pumps, water wheels, etc., the prediction of characteristics



Ernst Mach (1838–1916)
Austrian physicist/philosopher. After being professor at Graz and Prague Universities became professor at Vienna University. Studied high-velocity flow of air and introduced the concept of Mach number. Criticised Newtonian dynamics and took initiatives on the theory of relativity. Also made significant achievements in thermodynamics and optical science.

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through model testing is widely employed. Suppose that the drag D on a car is going to be measured on a 1 : 10 model (scale ratio $S = 10$). Assume that the full length l of the car is 3 m and the running speed v is 60 km/h. In this case, the following three methods are conceivable. Subscript m refers to the model.

Test in a wind tunnel In order to make the Reynolds numbers equal, the velocity should be $v_m = 167\,\mathrm{m/s}$, but the Mach number is 0.49 including compressibility. Assuming that the maximum tolerable value M of incompressibility is 0.3, $v_\mathrm{m} = 102\,\mathrm{m/s}$ and $Re_\mathrm{m}/Re = v_\mathrm{m}/Sv = 0.61$. In this case, since the flows on both the car and model are turbulent, the difference in C_D due to the Reynolds numbers is modest. Assuming the drag coefficients for both $D/(\rho v^2 l^2/2)$ are equal, then the drag is obtainable from the following equation:

$$D = D_\mathrm{m}\left(\frac{v}{v_\mathrm{m}}\frac{l}{l_\mathrm{m}}\right)^2 = D_\mathrm{m}\left(\frac{Sv}{v_\mathrm{m}}\right)^2 \tag{10.20}$$

This method is widely used.

Test in a circulating flume or towing tank In order to make the Reynolds numbers for the car and the model equal, $v_\mathrm{m} = vSv_\mathrm{m}/v = 11.1\,\mathrm{m/s}$. If water is made to flow at this velocity, or the model is moved under calm water at this velocity, conditions of dynamical similarity can be realised. The conversion formula is

$$D = D_\mathrm{m}\frac{\rho}{\rho_\mathrm{m}}\left(\frac{Sv}{v_\mathrm{m}}\right)^2 = D_\mathrm{m}\frac{\rho v^2}{\rho_\mathrm{m} v_\mathrm{m}^2} \tag{10.21}$$

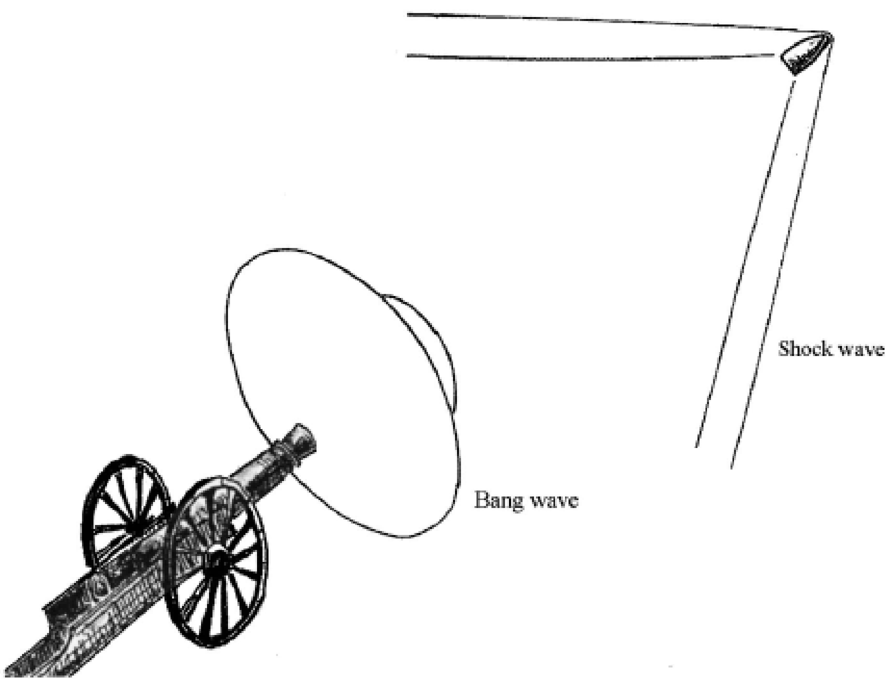
Test in a variable density wind tunnel If the density is increased, the Reynolds numbers can be equalised without increasing the air flow velocity. Assume that the test is made at the same velocity; it is then necessary to increase the wind tunnel pressure to 10 atm assuming the temperatures are equal. The conversion formula is

$$D = D_m \frac{\rho}{\rho_m} S^2 \tag{10.22}$$

Two mysteries solved by Mach

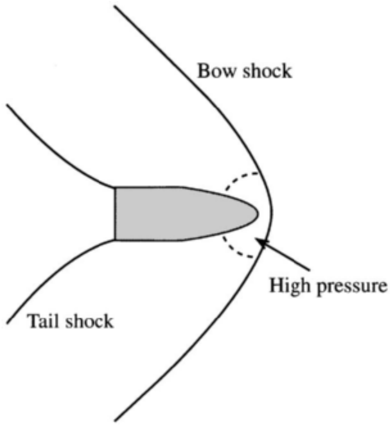
[No. 1] The early Artillerymen knew that two bangs could be heard downrange from a gun when a high-speed projectile was fired, but only one from a low-speed projectile. But they did not know the reason and were mystified by these phenomena. Following Mach's research, it was realised that in addition to the bang from the muzzle of the gun, an observer downrange would first hear the arrival of the bow shock which was generated from the head of the projectile when its speed exceeded the velocity of sound.

By this reasoning, this mystery was solved.



[No. 2] This is a story of the Franco-Prussian war of 1870–71. It was found that the novel French Chassepôt high-speed bullets caused large crater-shaped wounds. The French were suspected of using explosive projectiles and therefore violating the International Treaty of Petersburg prohibiting the use of explosive projectiles. Mach then gave the complete and correct explanation that the explosive type wounds were caused by the highly pressurised air caused by the bullet's bow wave and the bullet itself.

So it was clear that the French did not use explosive projectiles and the mystery was solved.



10.5 Problems

1. Derive Torricelli's principle by dimensional analysis.
2. Obtain the drag on a sphere of diameter d placed in a slow flow of velocity U.
3. Assuming that the travelling velocity a of a pressure wave in liquid depends upon the density ρ and the bulk modulus k of the liquid, derive a relationship for a by dimensional analysis.
4. Assuming that the wave resistance D of a boat is determined by the velocity v of the boat, the density ρ of fluid and the acceleration of gravity g, derive the relationship between them by dimensional analysis.
5. When fluid of viscosity μ is flowing in a laminar state in a circular pipe of length l and diameter d with a pressure drop Δp, obtain by dimensional analysis a relationship between the discharge Q and d, $\Delta p/l$ and μ.
6. Obtain by dimensional analysis the thickness δ of the boundary layer distance x along a flat plane placed in a uniform flow of velocity U (density ρ, viscosity μ).
7. Fluid of density ρ and viscosity μ is flowing through an orifice of diameter d bringing about a pressure difference Δp. For discharge Q, the

discharge coefficient $C = Q/[(\pi d^2/4)\sqrt{2\Delta p/\rho}]$, and $Re = d\sqrt{2\rho\Delta p}/\mu$, show by dimensional analysis that there is a relationship $C = f(Re)$.

8. An aircraft wing, chord length 1.2 m, is moving through calm air at 20°C and 1 bar at a velocity of 200 km/h. If a model wing of scale 1:3 is placed in a wind tunnel, assuming that the dynamical similarity conditions are satisfied by Re, then:

   (a) If the temperature and the pressure in the wind tunnel are respectively equal to the above, what is the correct wind velocity in the tunnel?
   (b) If the air temperature in the tunnel is the same but the pressure is increased by five times, what is the correct wind velocity? Assume that the viscosity μ is constant.
   (c) If the model is tested in a water tank of the same temperature, what is the correct velocity of the model?

9. Obtain the Froude number when a container ship of length 245 m is sailing at 28 knots. Also, when a model of scale 1:25 is tested under similarity conditions where the Froude numbers are equal, what is the proper towing velocity for the model in the water tank? Take 1 knot = 0.514 m/s.

10. For a pump of head H, representative size l and discharge Q, assume that the following similarity rule is appropriate:

$$\frac{l}{l_m} = \left(\frac{Q}{Q_m}\right)^{1/2}\left(\frac{H}{H_m}\right)^{-1/4}$$

where, for the model, subscript m is used.

If a pump of $Q = 0.1\,\mathrm{m^3/s}$ and $H = 40\,\mathrm{m}$ is model tested using this relationship in the situation $Q_m = 0.02\,\mathrm{m^3/s}$ and $H_m = 50\,\mathrm{m}$, what is the model scale necessary for dynamical similarity?